**Mathematical Logic**

Roberto Maieli

# A new correctness criterion for multiplicative non-commutative proof nets

**Abstract.** We introduce a new correctness criterion for multiplicative non commutative proof nets which can be considered as the non-commutative counterpart to the Danos-Regnier criterion for proof nets of linear logic. The main intuition relies on the fact that any switching for a proof net (obtained by mutilating one premise of each disjunction link) can be naturally viewed as a *series-parallel order variety* (a cyclic relation) on the conclusions of the proof net.

## 1. Introduction

Non-commutative logic [8], shortly NL, is a conservative extension of both Linear [4] and Cyclic Logic [9], the latter being a classical extension of Lambek calculus [6]. The main mathematical novelty of the NL sequent calculus is given by the structure of *order variety*, a ternary cyclic relation which can be presented as partial orders once we focus on a point in the support of the structure. A special role in NL is played by the class of *series-parallel* order varieties, i.e. structures which can be represented as rootless planar graphs, called *seaweeds*. Typically, a sequent of formulas $\vdash A_1, ..., A_n$ is represented as a series-parallel order variety $\alpha$ on $A_1, ..., A_n$.

Proof nets for the multiplicative fragment of non commutative logic (MNL) were, firstly, studied by Abrusci and Ruet [2] according to the idea of *trips*, originally introduced by Girard for proof nets of linear logic [4]; in particular, an MNL proof net is a graph which satisfies certain specific trip condition.

We now introduce a correctness criterion for MNL proof nets which can be considered as the non commutative counterpart to the Danos-Regnier criterion for proof nets of linear logic [3], based on the idea that a proof net is a graph that

R. Maieli: Equipe Logique de la Programmation, Institut de Mathématiques de Luminy, CNRS, 163, avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France.
e-mail: maieli@iml.univ-mrs.fr

reduces to an acyclic and connected graph (a tree) every time we mutilate (*switch*) a premise (left or right) in each disjunction link.

Our criterion allows the immediate view of a DR-switching as a seaweed. Intuitively, a non commutative proof structure $\pi$ is a proof net if any switching can be viewed as a seaweed such that once restricted to the left ($A$) and right ($B$) premises of any sequential link $\nabla$ then the seaweed contains the triple $(A, B, C)$ for any adequate conclusion $C$ of $\pi$.

Likewise the existing Abrusci-Ruet (AR) criterion of non commutative proof nets [2], the new criterion allows the mapping of any proof net onto a sequent proof via cut elimination.

This new criterion aims to develop a simple theory of modules [1] for MNL which could represent the natural non commutative counterpart of some linear correctness criteria like that one based on *switching/partitions* [3].

The rest of this paper is organized as follows: Section 2 gives the basic definitions and properties of order varieties (Section 2.1) and the sequent calculus of MNL (Section 2.2). Proof nets and cut-free sequentialization are presented, respectively, in Section 3 and Section 4. Sequentialization of non-cut free proof nets is then discussed in last Section 5 together with some future works.

## 2. Non-commutative logic

### 2.1. Order varieties

Order varieties, introduced by Ruet in [8], are structures that can be presented by partial orders in several ways. We can think an order variety as a *cycle* which becomes a *total order* once an origin or *focus* is fixed. Formally, an *order variety* on a given set $X$ is a ternary relation $\alpha$ which is:

*cyclic* : $\forall x, y, z \in X, \alpha(x, y, z) \Rightarrow \alpha(y, z, x)$ ;
*anti-reflexive* : $\forall x, y \in X, \neg\alpha(x, x, y)$ ;
*transitive* : $\forall x, y, z, t \in X, \alpha(x, y, z)$ and $\alpha(z, t, x) \Rightarrow \alpha(y, z, t)$ ;
*spreading* : $\forall x, y, z, t \in X, \alpha(x, y, z) \Rightarrow \alpha(t, y, z)$ or $\alpha(x, t, z)$ or $\alpha(x, y, t)$.

**Focusing.** Given an order variety $\alpha$ on $X$ and $x \in X$ we may define a partial order $\alpha_x$ on $X \setminus \{x\}$ by *focusing* on a point $x \in X$ such that:

$$\alpha_x(y, z) \quad \text{iff} \quad \alpha(x, y, z) \tag{1}$$

Conversely, let $\omega = (X, <)$ be a partial order, $z \in X$ and $\overset{z}{<}$ denotes the binary relation: $x \overset{z}{<} y$ iff $x < y$ and $z$ is comparable with neither $x$ nor $y$; then we may define an order variety $\overline{\omega}$ on $X$ by $\overline{\omega}(x, y, z)$ iff: $x < y < z$ or $y < z < x$ or $z < x < y$ or $x \overset{z}{<} y$ or $y \overset{x}{<} z$ or $z \overset{y}{<} x$. When $\overline{\omega} = \alpha$, we say that $\omega$ *presents* $\alpha$.

**Gluing.** If $\omega, \tau$ are two partial orders on disjoint sets, then the *gluing* $\omega * \tau$ is defined according to the following equalities:

$$\overline{\omega < \tau} = \overline{\omega \parallel \tau} = \overline{\tau < \omega} \tag{2}$$
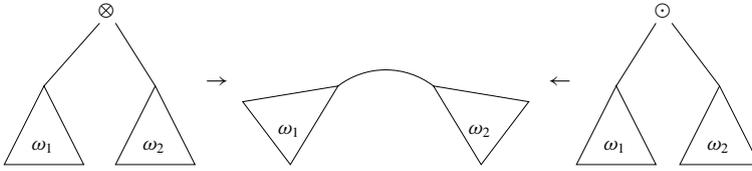
**Fig. 1.** Trees and rootless trees

The two processes of focusing on a point in an order variety and gluing orders are related by the following equalities:

$$\alpha_x * x \;=\; \alpha \quad \text{and} \quad (\omega * x)_x \;=\; \omega, \tag{3}$$

for an order variety $\alpha$ on a set $X$, $x \in X$ and a partial order $\omega$ on $X \setminus \{x\}$.

**Series-parallel order varieties.**   We are mainly interested in a particular class of order varieties: *series-parallel (s/p) order varieties*, which are precisely those order varieties which can be presented by a *series-parallel orders*. The class of *series-parallel orders* is the least class of partial finite orders containing *empty orders*, *singletons* $\{x\}$ and closed under the *series* and *parallel composition* (see [7]). Let $\omega$ and $\tau$ be partial orders on disjoint sets $X$ and respectively $Y$; their *series* and *parallel* sums $\omega < \tau$ and $\omega \parallel \tau$ are two partial orders on $X \uplus Y$ defined, respectively, by:

- $(\omega_1 < \omega_2)(x, y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$ or $(x \in X$ and $y \in Y)$ and
- $(\omega_1 \parallel \omega_2)(x, y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$.

If $\alpha$ and $\beta$ are two order varieties on $X$, resp. $Y$ with $X \cap Y = \{x\}$ we then define the *series* and resp., *parallel composition* of $\alpha$ and $\beta$ as follows:

$$\alpha \odot_x \beta = \overline{\alpha_x < x < \beta_x} = (\beta_x < \alpha_x) * x$$

$$\alpha \otimes_x \beta = \overline{\alpha_x \parallel x \parallel \beta_x} = (\beta_x \parallel \alpha_x) * x \tag{4}$$

We say the order variety $\alpha$ on the non empty set $X$ is *series-parallel* if and only if there exists a s/p order $\omega$ on $X$ such that $\overline{\omega} = \alpha$.

**Seaweeds.** There is a very simple and intuitive way to represent series-parallel order varieties on a given set $X$ by means of planar graphs, called *seaweeds*, whose leaves are labeled by elements of $X$ and ternary nodes are labelled by $\otimes$ or $\odot$. Let $\alpha = \overline{\omega}$ be an order variety on a set $X$, with at least two elements, for some non unique series-parallel order $\omega$, and write $\omega$ as a non unique binary tree $t$ whose leaves are labelled by the elements of $X$, and root and nodes labelled by $\otimes$ (in the case of $\parallel$ nodes) and $\odot$ (in the case of $<$ nodes); then remove the root of $t$ like in the Figure 1. To read a seaweed $\alpha$, take three elements $a, b, c$ of the support $X$, we say the triple $(a, b, c)$ is in $\alpha$ if and only if the following two conditions are satisfied:
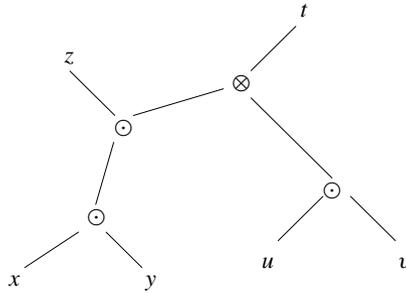
**Fig. 2.** Seaweed

1. the three paths $ab$, $bc$ and $ca$ intersect in a node labeled by $\odot$;
2. the paths $a\odot$, $b\odot$ and $c\odot$ are in this cyclic order while moving *anti-clockwise* around $\odot$.

It is easy to check in the Figure 2 that the triple $(z, y, t)$ belongs to the seaweed, whereas $(z, t, v)$ does not.

Let $\alpha$ be an order variety on X, then its *restriction* to $Y \subseteq X$, as a set of triples, is the order variety denoted $\alpha \upharpoonright_X$. In the seaweed representation of an order variety $\alpha$ the restriction to a subset $Y \subseteq |\alpha|$ is the retracted graph obtained by erasing all nodes and edges that are not on a path between two leaves of $Y$. Restriction preserves the structures of orders and order varieties, and preserves series-parallelism.

**Entropy.** We can always replace in an seaweed a node $\odot$ with a node $\otimes$ and get a smaller (w.r.t. $\subseteq$) seaweed; this operation is called *entropy*. Similarly, we can define the entropy relation $\lhd$ on the set of all partial orders for a fixed set by

$$\omega \lhd \sigma \quad \text{iff} \quad \omega \subseteq \sigma \quad \text{and} \quad \overline{\omega} \subseteq \overline{\sigma}. \tag{5}$$

The following facts on entropy are proved in [8].

*Facts 1.*
1. if $\omega$ and $\sigma$ are partial orders on X and $Y \subseteq X$ then $\omega \lhd \sigma$ implies $\omega \upharpoonright_Y \lhd \sigma \upharpoonright_Y$;
2. if $\omega \lhd \sigma$ and $\omega' \lhd \sigma'$ then $(\omega < \omega') \lhd (\sigma < \sigma')$;
3. if $\alpha$, $\beta$ are order varieties on X and $x \in X$, then $\alpha \subseteq \beta$ iff $\alpha_x \lhd \beta_x$.

We now introduce the concepts of *interior* and *wedge* of order varieties we need for defining the order varieties of proof nets.

**Interior.** Let $\alpha$ be a cyclic order on X then the *interior* $\natural\alpha$, given by

$$\natural\alpha = \bigcap_{x \in X} \alpha_x * x \tag{6}$$

is the *largest order variety* included in $\alpha$. Let $\alpha$, $\beta$ be cyclic orders on X, then the following equality holds:

$$\natural(\alpha \cap \beta) = \natural(\natural\alpha \cap \natural\beta). \tag{7}$$

**Wedge.** Let $\alpha_i$ be order varieties on $X$ for $i \in I$; we define

$$\bigwedge \alpha_i = \natural \bigcap \alpha_i \qquad (8)$$

the *wedge of order varieties* $\alpha_i$. If $\omega_i$ are partial orders on $X$ for $i \in I$ and $x \in X$, we then define the *wedge of partial oders* $\omega_i$ as follows

$$\bigwedge \omega_i = (\bigwedge \omega_i * x)_x. \qquad (9)$$

The following facts on wedge are proved in [8].

*Facts 2.*
1. $\bigwedge$ is commutative and associative;
2. if $\alpha_i$ are order varieties on $X$ for $i \in I$ then $\bigwedge \alpha_i$ is the largest order variety included in all $\alpha_i$;
3. if $\omega_i$ are partial orders on $X$ for $i \in I$ then *wedge of orders* $\bigwedge \omega_i$ is the largest partial order $\lessdot \omega_i$;
4. if $Y \subseteq |\alpha_i|$ then $(\bigwedge \alpha_i) \restriction_Y \subseteq \bigwedge \alpha_i \restriction_Y$;
5. if $Y \subseteq |\omega_i|$ then $(\bigwedge \omega_i) \restriction_Y \lessdot \bigwedge \omega_i \restriction_Y$;
6. for any $x \in |\alpha_i|$, $(\bigwedge \alpha_i)_x = \bigwedge (\alpha_i)_x$;
7. for any $x \notin |\omega_i|$, $(\bigwedge \omega_i) * x = \bigwedge (\omega_i * x)$;
8. if $|\omega_i| \cap |\sigma_j| = \emptyset$ then $(\bigwedge_{i \in I} \omega_i) * (\bigwedge_{j \in J} \sigma_j) = \bigwedge_{i \in I, j \in J} (\omega_i * \sigma_j)$.

We now show that the wedge of orders commutes with s/p compositions.

**Lemma 1.** *Let* $|\omega_i| = X$ *be and* $|\sigma_j| = Y$, *for all* $i \in I$, $j \in J$, *such that* $X \cap Y = \emptyset$ *then:*

*1.* $\bigwedge (\omega_i \parallel \sigma_j) = \bigwedge \omega_i \parallel \bigwedge \sigma_j$;
*2.* $\bigwedge (\omega_i < \sigma_j) = \bigwedge \omega_i < \bigwedge \sigma_j$.

*Proof.* To prove proposition 1 observe that $\bigwedge (\omega_i \parallel \sigma_j)$ must be of the form $\tau \parallel \tau'$, with $|\tau| = |\omega_i|$ and $|\tau'| = |\sigma_j|$, since, by fact 2 (3), $\bigwedge (\omega_i \parallel \sigma_j) \lessdot \omega_i \parallel \sigma_j$ and so, by proposition (5), $\bigwedge (\omega_i \parallel \sigma_j) \subseteq (\omega_i \parallel \sigma_j)$; then, by restriction $\tau \lessdot \bigwedge \omega_i$ and $\tau' \lessdot \bigwedge \sigma_j$ and so $(\tau \parallel \tau') \lessdot (\bigwedge \omega_i \parallel \bigwedge \sigma_j)$, since $\lessdot$ is compatible with parallel composition. Besides, $(\bigwedge \omega_i \parallel \bigwedge \sigma_j) \lessdot (\omega_i \parallel \sigma_j)$ for all $i \in I$, $j \in J$ then $(\bigwedge \omega_i \parallel \bigwedge \sigma_j) \lessdot \bigwedge (\omega_i \parallel \sigma_j)$, since $\bigwedge (\omega_i \parallel \sigma_j)$ is the largest solution $\lessdot (\omega_i \parallel \sigma_j)$, and so we conclude $\bigwedge (\omega_i \parallel \sigma_j) = \bigwedge \omega_i \parallel \bigwedge \sigma_j$.
The proof of proposition 2 is analogous.

In the sequent calculus, particularly in the $\invamp$ rule, we will make use of a particular case of wedge.

**Identification.** Let $\alpha$ be an order variety on a set $X \uplus \{x\} \uplus \{y\}$ and $z \notin X \uplus \{x\} \uplus \{y\}$; we define the *identification* $\alpha[z/x, y]$ of $x$ and $y$ into $z$ in $\alpha$ by:

$$\alpha[z/x, y] = \natural (\alpha \restriction_{X \uplus x} [z/x] \cap \alpha \restriction_{X \uplus y} [z/y]) \qquad (10)$$

or equivalently, by proposition 8,

$$\alpha[z/x, y] = \alpha \restriction_{X \cup \{x\}} [z/x] \wedge \alpha \restriction_{X \cup \{y\}} [z/y]. \qquad (11)$$

The following proposition, showed in [2] (Lemma 3.22), relates interior, wedge and identification:

$$(\natural\alpha)[z/x, y] = \alpha[z/x, y] \tag{12}$$

when $\alpha$ is a cyclic order on $X$ with $x, y \in X$, $x \neq y$ and $z \notin X$.

Finally, in order to get the sequentialization we need to show that wedge commutes with identification.

**Lemma 2.** *Let $\alpha_i$ be an order variety on $X$ for any $i \in I$ with $x, y \in X$, $x \neq y$ and $z \notin X$, then the wedge of order varieties commutes with the identification* $(\bigwedge_i \alpha_i)[z/x, y] = \bigwedge_i(\alpha_i[z/x, y])$.

*Proof.*

$$
\begin{aligned}
(\textstyle\bigwedge_i \alpha_i)[z/x, y] &= (\natural \textstyle\bigcap_i \alpha_i)[z/x, y] && \text{by 8,}\\
&= (\textstyle\bigcap \alpha_i)[z/x, y] && \text{by 12,}\\
&= \natural((\textstyle\bigcap_i \alpha_i)\restriction_{X \uplus x}[z/x] \cap (\textstyle\bigcap_i \alpha_i)\restriction_{X \uplus y}[z/y]) && \text{by 10,}\\
&= \natural((\textstyle\bigcap_i \alpha_i\restriction_{X \uplus x}[z/x]) \cap (\textstyle\bigcap_i \alpha_i\restriction_{X \uplus y}[z/y])) && \restriction \text{ comm. with } \cap,\\
&= \natural((\natural \textstyle\bigcap_i \alpha_i\restriction_{X \uplus x}[z/x]) \cap (\natural \textstyle\bigcap_i \alpha_i\restriction_{X \uplus y}[z/y])) && \text{by fact 7,}\\
&= \natural((\textstyle\bigwedge_i \alpha_i\restriction_{X \uplus x}[z/x]) \cap (\textstyle\bigwedge_i \alpha_i\restriction_{X \uplus y}[z/y])) && \text{by 8,}\\
&= \textstyle\bigwedge_i \alpha_i\restriction_{X \uplus x}[z/x] \wedge \textstyle\bigwedge_i \alpha_i\restriction_{X \uplus y}[z/y] && \text{by 8,}\\
&= \textstyle\bigwedge_i(\alpha_i[z/x, y]) && \text{by assoc. and 11.}
\end{aligned}
$$

### 2.2. Sequent calculus

*Formulas* of MNL are built from atoms $p, q, ..., p^\perp, q^\perp, ...$ and the following multiplicative connectives:

– the non commutative conjunction $\odot$ (*next*);
– the non commutative disjunction $\nabla$ (*sequential*);
– the commutative conjunction $\otimes$ (*times*);
– the commutative disjunction $\parr$ (*par*).

*Negation* is defined by De Morgan rules:

$$
\begin{aligned}
(p)^\perp &= p^\perp & (p^\perp)^\perp &= p\\
(A \otimes B)^\perp &= B^\perp \parr A^\perp & (A \parr B)^\perp &= B^\perp \otimes A^\perp\\
(A \odot B)^\perp &= B^\perp \nabla A^\perp & (A \nabla B)^\perp &= B^\perp \odot A^\perp
\end{aligned}
$$

Negation is then an involution, i.e., for any formula $F$, $F^{\perp\perp} = F$.

About the notation, some times we use the symbol $\blacktriangledown$ to denote a $\parr$ or $\nabla$ connective, and the symbol $\bullet$ to denote a $\otimes$ or $\odot$ connective.

An MNL *sequent* is of the form $\vdash \alpha$, where $\alpha$ is a series-parallel order variety on a set of MNL formulas occurrences. The rules of the sequent calculus of MNL are given in the Table 1.
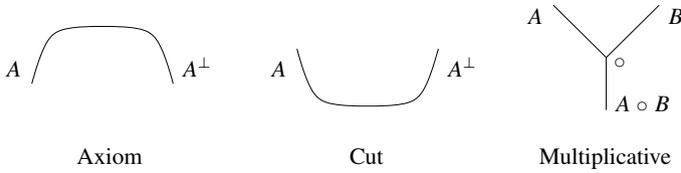
**Identity and cut**

$$\frac{}{\vdash A^{\perp} * A} \text{ Id} \qquad \frac{\vdash \omega * A \qquad \vdash \omega' * A^{\perp}}{\vdash \omega * \omega'} \text{ Cut}$$

**Commutative rules:**

$$\frac{\vdash \omega * A \qquad \vdash \omega' * B}{\vdash (\omega \parallel \omega') * A \otimes B} \otimes \qquad \frac{\vdash \boldsymbol{\alpha}[A, B]}{\vdash \boldsymbol{\alpha}[A \,\mathfrak{R}\, B/A, B]} \,\mathfrak{R}$$

**Non-commutative rules:**

$$\frac{\vdash \omega * A \qquad \vdash \omega' * B}{\vdash (\omega' < \omega) * A \odot B} \odot \qquad \frac{\vdash \omega * (A < B)}{\vdash \omega * A \nabla B} \nabla$$

**Table 1.** MNL sequent calculus



**Fig. 3.** Links of MNL

## 3. Proof nets

**Definition 1 (Proof structures).** *An MNL* proof structure *is a graph $\pi$ whose nodes are called* links *and whose* edges *are labelled by formulas of MNL. Links are given in Figure 3:*

– *an* axiom link *has no premise and two conclusions labelled, resp., by dual formulas;*
– *a* cut link *has two premises labelled, resp., by dual formulas and no conclusion;*
– *a* multiplicative link $\circ$ *(of type $\otimes$, $\odot$, $\mathfrak{R}$ or $\nabla$) has a left premise A, a right premise B and a conclusion $A \circ B$.*

*Each edge is the conclusion of a unique link and the premise of at most one link. Edges which are not premises of a link are the* conclusions *of the proof structure.*

**Definition 2 (Switchings).** *Given a non-commutative proof structure $\pi$ with conclusions $\Gamma$, then:*

1. *a DR switching $s(\pi)$ is the graph we obtain from $\pi$ by mutilating, arbitrarily, one premise, left or right, for each ▼ link of $\pi$;*
2. *fixed a DR switching $s(\pi)$ and a $\frac{X \quad Y}{X \nabla Y} l$ link in $\pi$, we denote $\langle s(\pi), l \rangle$ the (possibly not connected) retracted graph obtained by:*
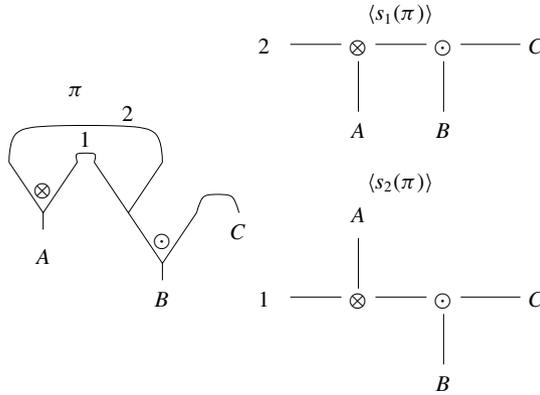   – *mutilating in $s(\pi)$ both premises $X$, $Y$ of $l_i$, and*

**Fig. 4.** Switchings and seaweeds

– *erasing any link and edge which is not connected to X or Y.*
*We then call $\langle s(\pi), l \rangle$ the* restriction *of $s(\pi)$ to $l$, and we will denote $|\langle s(\pi), l \rangle|$ the set of the handling edges of $\langle s(\pi), l \rangle$ that are conclusions of $\pi$ or premises of $l$.*

*Fact 1.* Clearly, any connected switching $s(\pi)$ for a given proof structure $\pi$ is a tree, with nodes the links of MNL and leaves the handling edges; moreover, $s(\pi)$ can be immediately viewed as a seaweed $\langle s(\pi) \rangle$ on the handling edges when we restrict to consider only the tensor links; we then call $\langle s(\pi) \rangle$ the *seaweed induced* by $s(\pi)$. Similarly, any connected graph $\langle s(\pi), l \rangle$ can be viewed as a seaweed on the set of handling edges; it is then called the *restriction* of the seaweed $\langle s(\pi) \rangle$ to $l$.

As an example of the above fact, consider the two seaweeds drawn on the right side of Figure 4: they are induced by the only two possible switchings $s_1(\pi), s_2(\pi)$ for the structure $\pi$ (on the left side). Observe the two induced order varieties are different since they have different supports (clearly they are equal when we restrict only to the conclusions $A, B, C$ of $\pi$).

In the following we use the notation $G/\{l_1, ...l_n\}$ to denote the retracted graph obtained by erasing links $l_1, \ldots, l_n$ from the graph $G$. Naively, the union $G_1 \uplus G_2$ denotes the graph obtained by merging the two graphs $G_1, G_2$ two through their common border formulas (when this makes sense).

**Definition 3 (Proof nets).** *An MNL proof structure $\pi$ is a* proof net *(or it is* correct*) if and only if for any switching $s(\pi)$ and for any sequential link $\frac{A \quad B}{A \triangledown B} l$ of $\pi$:*

1. *$s(\pi)$ is acyclic and connected;*
2. *the triple $(A, B, C)$ belongs to the restricted seaweed $\langle s(\pi), l \rangle$ for any conclusion $C$ of $\pi$ in $|\langle s(\pi), l \rangle|$.*
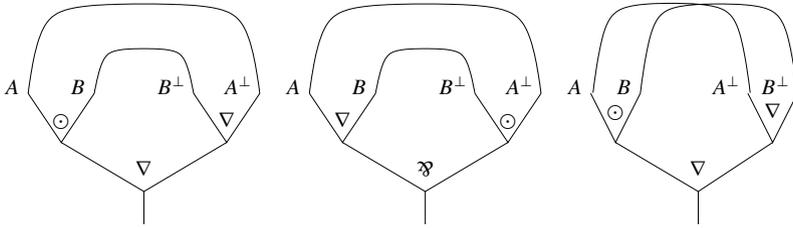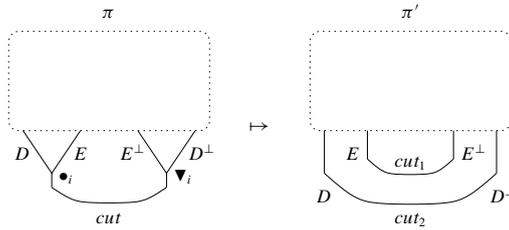
**Fig. 5.**



**Fig. 6.** A step of cut reduction

*Example 1.* The two MNL proof structures drawn on the left side of Figure 5 are correct, whereas, the third is not so, since for any switching $s(\pi)$, $\neg s(\pi)(A^\perp, B^\perp, A \odot B)$.

In the following we assume that all proof nets contain only atomic axiom links.

**Theorem 1 (Cut elimination).** *The proof structure $\pi'$ obtained from an MNL proof net $\pi$ after one step of cut elimination is still correct.*

*Proof.* Assume $\pi$ is an MNL proof net containing cut links. The only crucial case is when we want reduce a cut involving a compound cut-formula, like in the Figure 6. We denote $G_2 = \{\frac{D\ E}{D\bullet_i E}, \frac{E^\perp\ D^\perp}{E^\perp \blacktriangledown_i D^\perp}, cut\}$, $G_1 = \pi/G_2$ and $G_3 = \{cut_1, cut_2\}$, so, $\pi = G_1 \uplus G_2$ and $\pi' = G_1 \uplus G_3$. We proceed by absurdum. We assume there exists a switching $s'(\pi')$ and a sequential link $\frac{X\ Y}{X\triangledown Y}l$ such that $\langle s'(\pi'), l\rangle(X, C, Y)$ for some conclusion $C \in \pi'$ in $|\langle s'(\pi'), l\rangle|$, and assume $\bullet_j$ is the tensor link where the three paths $XY$, $XC$ and $YC$ intersect for the fixed switching. We need to consider three main cases, depending if the path $EE^\perp$ or $DD^\perp$ is included in $X\bullet_j$, $Y\bullet_j$ or $C\bullet_j$ for $s'(\pi')$.

1. If $EE^\perp \subseteq X\bullet_j$ and $DD^\perp \subseteq Y\bullet_j$, then, following Figure 7, we restore $\pi$ and build the switching $s(\pi) = s'(G_1) \uplus s_2(G_2)$ for an arbitrary switching $s_2(\pi_2)$. Now :
   – if the cut formula is of type $D \otimes_i E$, then we can easily observe that in the restricted seaweed $\langle s(\pi), l\rangle$ the three paths $XY$, $XC$, $YC$ intersect in $\otimes_i$ contradicting the assumption $\pi$ is correct;
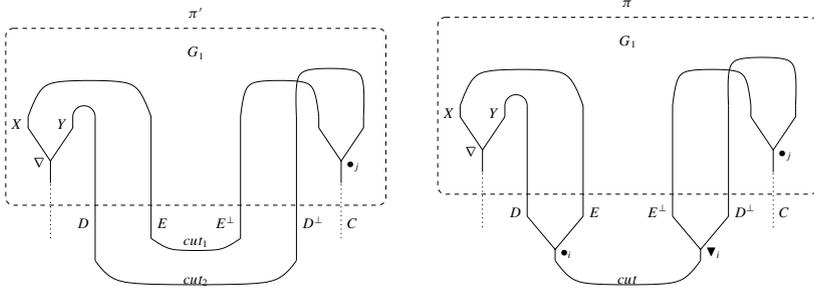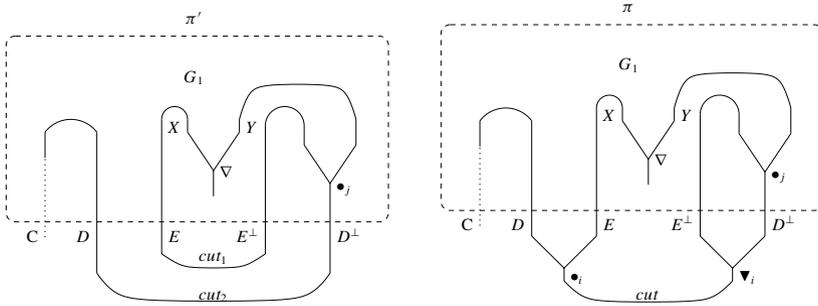
Fig. 7.



Fig. 8.

- if the the cut formula is of type $D \odot_i E$ then we can consider the restricted seaweed $\langle s(\pi), \blacktriangledown_i \rangle$ and observe that it contains the triple $(E^\perp, C, D^\perp)$, contradicting the assumption $\pi$ is correct;

2. If $EE^\perp \subseteq X \bullet_j$ and $DD^\perp \subseteq C \bullet_j$ (see Figure 8) we then proceed as before in order to get a contradiction. The case when $DD^\perp \subseteq X_1 \bullet_j$ and $EE^\perp \subseteq C \bullet_j$ is symmetric.

3. In all the other cases we build a switching for $\pi$, $s(\pi) = s'(G_1) \uplus s_2(G_2)$ with an arbitrary switching $s_2(\pi_2)$. Trivially, the restriction $\langle s(\pi), l \rangle$ still contains the triple $(X, C, Y)$, contradicting the assumption $\pi$ is correct.

**Definition 4 (Order variety of a proof net).** *If $\pi$ is an MNL proof net with conclusions $\Gamma$ and $\langle s_i(\pi) \rangle \restriction_\Gamma$ is the series-parallel order variety induced by any switching $s_i$ on $\Gamma$, then the order variety of $\pi$ is given by*

$$\bigwedge (\langle s_i(\pi) \rangle \restriction_\Gamma)$$

## 4. Sequentialization

**Definition 5 (Splitting condition).** *A non commutative proof net $\pi$ is in* splitting condition *when it has at least a cut or a tensor conclusion and no $\blacktriangledown$ conclusions.*

*A link with premises A and B is* splitting *if by erasing it we get two sub proof nets, with resp., A and B among their conclusions.*

Given an MNL proof structure $\pi$, we denote $\pi^*$ its *commutative translation* obtained by replacing any $\odot$ with $\otimes$ and any $\nabla$ by $\mathbin{⅋}$ (both in the structure and in the formulas). Recall that any commutative proof net in splitting condition contains always a splitting link (see [4]).

**Lemma 3 (Splitting).** *If $\pi$ is an MNL proof net in splitting condition, then it is splitting at $\bullet$ iff $\pi^*$ is so at $\bullet^*$.*

*Proof.* Assume $\pi^*$ is splitting at a $\bullet^*$ link $\frac{A \quad B}{A \bullet^* B}$ (we omit the obvious $*$ renaming over the formulas) into two proof nets $\pi_1^*$ and $\pi_2^*$ with conclusions $\Gamma_1$, $A$, resp., $\Gamma_2$, $B$ and assume, by absurdum, that $\pi_1$ (or $\pi_2$) is not correct. That implies there exists in $\pi_1$ a $l$ link $\frac{D \quad E}{D \nabla E}$ and a switching $s_1(\pi_1)$ such that $\langle s_1(\pi_1), l \rangle \restriction_{\Gamma_1', D, E}$ $(D, C, E)$ for some conclusion $C$ of $\pi$ in $|\langle s_1(\pi_1), l \rangle|$. Now observe that the conclusion $C$ must be $A$, otherwise there would already exist a seaweed, induced by a switching $s'$ for $\pi$, containing the triple $(D, C, E)$, contradicting the hypothesis that $\pi$ is correct. That implies we can build a switching $s(\pi) = s_1(\pi_1) \cup s_2(\pi_2) \cup \{\frac{A \quad B}{A \bullet B}\}$ such that $\langle s(\pi), l \rangle \restriction_{\Gamma, A \bullet B}$ $(D, A \bullet B, E)$ contradicting the assumption $\pi$ is correct.

*Fact 2 (Merging).* *If $\pi_A$ and $\pi_B$ are two MNL proof nets then the proof structure obtained by merging them through the tensor link $\frac{A \quad B}{A \bullet B}$ is still correct.*

The next sequentialization theorem shows that all cut free proof nets are sequentializable. This is enough to prove, via cut elimination (Theorem 1), that general proof nets are sequentializable.

**Theorem 2 (Sequentialization).** *Any MNL cut free proof net is* sequentializable *w.r.t the sequent calculus of Table 1, i.e., a cut free proof net $R$ with order variety $\alpha_R$ on its conclusions $\Gamma$ can be mapped onto a sequent proof $\pi$ of $\Gamma$ with the same order variety.*

*Proof.* We assume a proof net $R$ with order variety $\alpha_R$ on the conclusions $\Gamma, C$ and we show, by the induction on the number of links of $R$, how to map $R$ onto a sequent proof $\pi$ with order variety $\alpha = \alpha_R$.

1. The case when $R$ is an axiom link is trivial.
2. If $R$ contains a conclusion $C = A \mathbin{⅋} B$

$$R: \quad \frac{\begin{array}{c} R' \\ A \quad B \end{array}}{A \mathbin{⅋} B} \ l$$

   we then erase the link $l$ with its conclusion edge $A \mathbin{⅋} B$ and get the proof structure $R'$ whose order variety is $\alpha_R' = \bigwedge \alpha_i$ with $\alpha_i = \langle s_i(R') \rangle$ on $\Gamma, A, B$ for any switching $s_i(R')$. Then, by definition 4 and identification, $\alpha_R$ is

$$\begin{aligned}
&= \bigwedge(\alpha_i \restriction_{\Gamma, A} [C/A]) \wedge \bigwedge(\alpha_i \restriction_{\Gamma, B} [C/B]) && \text{by definition 4,} \\
&= \bigwedge(\alpha_i \restriction_{\Gamma, A} [C/A] \wedge \alpha_i \restriction_{\Gamma, B} [C/B]) && \text{by associativity,} \\
&= \bigwedge(\alpha_i [C/A, B]) && \text{by identification,}
\end{aligned}$$

with $\alpha_i \upharpoonright_{\Gamma, A} [C/A]$ (resp., $\alpha_i \upharpoonright_{\Gamma, B} [C/B]$) the seaweed induced by the switching $s_i(R')$ where the left (resp., the right) premise of $l$ has been mutilated. By hypothesis of induction applied to $R'$ we build the sequent proof $\pi'$ with order variety $\alpha = \alpha_{R'}$ and then, by an instance of $\mathfrak{P}$-rule applied to $A, B$, the proof $\pi$ with order variety $\alpha[C/A, B] = (\bigwedge \alpha_i)[C/A, B]$ equal, by Lemma 2, to $\alpha_R$.

3. If $R$ contains a conclusion $C = A \nabla B$, then we proceed as above. Observe that, when we remove a terminal $\nabla$ link, the remaining proof structure is still correct.

4. If $R$ contains only tensor conclusions and no $\mathfrak{P}$ or $\nabla$ conclusions, then by property 3, there exits a splitting link $l$. Let us assume, for instance, $l$ is labeled by the conclusion $C = A \otimes B$

$$R: \quad \cfrac{\overset{R'}{A} \qquad \overset{R''}{B}}{A \otimes B} \; l$$

Now, split $R$ at $l$ into two proof nets $R'$ and $R''$: $R'$ has order variety $\alpha_{R'} = \bigwedge (\omega_i * A)$ on $\Gamma', A$, with $\omega_i * A = \langle s_i'(R') \rangle$ for any switching $s_i'(R')$, $R''$ has order variety $\alpha_{R''} = \bigwedge (\sigma_j * B)$ on $\Gamma'', B$, with $\sigma_j * B = \langle s_j''(R'') \rangle$ for any switching $s_j''(R'')$ and $\Gamma = \Gamma' \uplus \Gamma''$. By induction hypothesis applied to $R'$ and $R''$ we build two corresponding sequent proofs $\pi'$ and $\pi''$ with, respectively, order variety $\alpha' = \alpha_{R'}$ and $\alpha'' = \alpha_{R''}$. Then, by means of a $\otimes$ rule we get a sequent proof $\pi$ of the order variety $\alpha = (\alpha'_A \parallel \alpha''_B) * A \otimes B$, which is equal to $(\bigwedge \omega_i \parallel \bigwedge \sigma_i) * A \otimes B$, since, by Fact 2-(iv), $\alpha'_A = \bigwedge \omega_i$ and $\alpha''_B = \bigwedge \sigma_i$. Finally we need to show $\alpha = \alpha_R$, indeed

$$
\begin{aligned}
\alpha_R &= \overline{(\alpha_{R'})_A \parallel A \otimes B \parallel (\alpha_{R''})_B} && \text{by Proposition 4,} \\
&= \overline{(\bigwedge \omega_i) \parallel A \otimes B \parallel (\bigwedge \sigma_i)} && \text{by Fact 2-(iv),} \\
&= (\bigwedge \omega_i \parallel \bigwedge \sigma_i) * A \otimes B && \text{by Proposition 2,} \\
&= \bigwedge (\omega_i \parallel \sigma_i) * A \otimes B && \text{by Lemma 1,} \\
&= \bigwedge ((\omega_i \parallel \sigma_i) * A \otimes B) && \text{by Fact 2-(iv).}
\end{aligned}
$$

5. The case when the splitting link is labelled by a $A \odot B$ is analogous to the previous one.

**Theorem 3 (Adequacy).** *Definition 3 is* adequate *w.r.t. the sequent calculus of Table 1, i.e., we can always associate to each sequent proof $\pi$ with order variety $\alpha$ on the conclusions $\Gamma$ a proof net $R$ with the same order variety $\alpha_R = \alpha$.*

*Proof.* We proceed by induction on the length of a given sequent proof $\pi$ with order variety $\alpha$ on the conclusions $\Gamma$. The base of induction is given by $\pi$ equal to $\cfrac{}{\vdash A^\perp * A}$, which, trivially, corresponds to the axiom link. The other cases depend on the last rule of $\pi$.

1. If last rule of $\pi$ is a $\mathfrak{P}$ rule

$$\pi: \quad \cfrac{\overset{\pi'}{\vdash \alpha[A, B]}}{\vdash \alpha[A \mathfrak{P} B/A, B]}$$

then, by induction hypothesis on $\pi'$, there exists a proof net $R'$ with order variety $\alpha_{R'} = \alpha[A, B]$ and $\alpha_{R'}$ equal by definition to $\bigwedge \alpha_i$, where $\alpha_i = \langle s_i(R')\rangle$ for any switching $s_i$ of $R'$. We can so add a link $\frac{A\;\;B}{A\mathbin{\invamp}B}$ to $R'$ and obtain the proof net $R$ with order variety

$$\alpha_R = \bigwedge (\alpha_i)\upharpoonright_{\Gamma,A} [A\mathbin{\invamp}B/A] \wedge \bigwedge (\alpha_i\upharpoonright_{\Gamma,B})[A\mathbin{\invamp}B/B]$$

that is, by associativity of wedge and identification, equal to

$$\bigwedge (\alpha_i[A\mathbin{\invamp}B/A, B])$$

Finally, by Lemma 2 we get the equality

$$\bigwedge (\alpha_i[A\mathbin{\invamp}B/A, B]) \; = \; \alpha[A\mathbin{\invamp}B/A, B].$$

2. If last rule of $\pi$ is a $\nabla$ rule

$$\pi: \frac{\overset{\pi'}{\vdash \omega * (A < B)}}{\vdash \omega * A\nabla B}$$

then by induction hypothesis on $\pi'$ there exists a proof net $R'$ with order variety $\bigwedge (\omega_i * (A < B)) = \omega * (A < B)$ with $\omega_i * (A < B) = \langle s_i(R')\rangle$ for any switching $s_i$ of $R'$. We can so add a new link $\frac{A\;\;B}{A\nabla B}$ to $R'$ and get the proof net $R$ whose order variety

$$\bigwedge (\omega_i * (A < B))\upharpoonright_{\Gamma,A} [A\nabla B/A] \wedge \bigwedge (\omega_i * (A < B))\upharpoonright_{\Gamma,B} [A\nabla B/B]$$

is equal to

$$\begin{aligned}
\bigwedge (\omega_i * A\nabla B) &= (\bigwedge (\omega_i * (A < B)))[A\nabla B/A, B] && \text{by Lemma 2,}\\
&= \omega * (A < B)[A\nabla B/A, B] && \text{by assumption,}\\
&= \omega * A\nabla B && \text{by identification.}
\end{aligned}$$

3. If last rule of $\pi$ is a $\otimes$ rule

$$\pi: \frac{\overset{\pi'}{\vdash \omega * A} \qquad \overset{\pi''}{\vdash \sigma * B}}{\vdash (\omega \parallel \sigma) * A \otimes B} \otimes$$

then, by induction hypothesis applied to $\pi'$ and $\pi''$, we get two proof nets $R'$ and $R''$ with order varieties, respectively, $\bigwedge (\omega_i * A)$ and $\bigwedge (\sigma_j * B)$. We can then merge $R'$ and $R''$ by means of a link $\frac{A\;\;B}{A\otimes B}$ and get, by Proposition 2, the proof net $R$. Finally, we need show that the $\alpha_R = \alpha$. We know $\omega * A = \bigwedge (\omega_i * A)$ and $\sigma * B = \bigwedge (\sigma_j * B)$, by hypothesis of induction, and so, by Facts 2-(iv), $\omega = \bigwedge \omega_i$ and $\sigma = \bigwedge \sigma_j$; then by parallel composition $\omega \parallel \sigma = \bigwedge \omega_i \parallel \bigwedge \sigma_j$ and so, by Lemma 1, $\omega \parallel \sigma = \bigwedge (\omega_i \parallel \sigma_j)$. Finally, by parallel composition and gluing $(\omega \parallel \sigma) * A \otimes B = \bigwedge (\omega_i \parallel \sigma_j) * A \otimes B$.

4. The case when last rule of $\pi$ is a $\odot$ rule is analogous to the previous one.
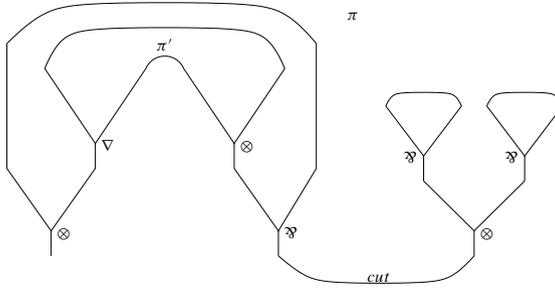
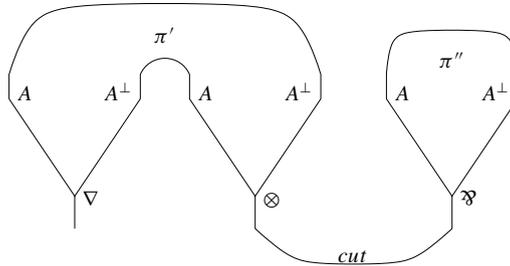**Fig. 9.** Non-directly sequentializable proof net



**Fig. 10.** Correct and sequentializable proof structure

## 5.  Open problems and forthcoming works

Section 4 left open the question of the direct sequentialization of non cut free proof nets. Actually, the direct sequentialization of non cut-free proof net fails as shown in Figure 9 : the proof structure $\pi$, containing only one cut link and no terminal splitting tensor, is correct according to Definition 3 but not directly sequentializable, since after removing the cut link we get, immediately, the non correct proof structure $\pi'$. Now, there is no way to fix this problem by interpreting the (binary) cut link as a (ternary) tensor link ($\otimes$ or $\odot$) with a "dummy conclusion", like in the linear case. Indeed, this solution is not adequate, since some correct proof nets may become immediatly non correct, even though they are sequentializable: that is the case, for instance, of the proof net in Figure 10 which sequentializes as follows:

$$\frac{\dfrac{\overline{\vdash A * A^{\perp}}\qquad\overline{\vdash A * A^{\perp}}}{\vdash (A \parallel A^{\perp}) * A^{\perp} \otimes A}\qquad\dfrac{\overline{\vdash A * A^{\perp}}}{\vdash A \bindnasrepma A^{\perp}}}{\dfrac{\vdash A * A^{\perp}}{\vdash A^{\perp} \nabla A}}$$

However, all the above drawbacks[1] are also common to the Abrusci-Ruet (AR) theory of proof nets, even though our correctness criterion is not equivalent to the
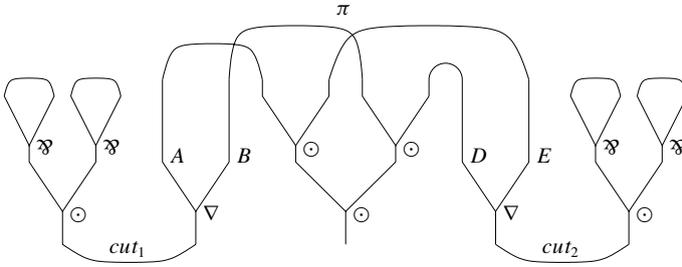
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[1]  See also the Appendix A of [2].

**Fig. 11.** Non-directly sequentializable proof net

AR-criterion, as shown in Figure 11[2] : in fact, if we assume $A = B = C = D = (p^{\perp} \otimes p)$ then $\pi$ is correct according to Definition 3 but not so according to AR-criterion; moreover, observe that $\pi$ is not directly sequentializable.

By the way, [2] proposes in the Appendix a solution to the direct sequentialization of non cut free proof net, which at the end seems not very satisfatory since it makes the criterion very difficult to follow. At this moment we are exploring some ideas in order to get a satisfactory treatment of the direct sequentialization of non cut free proof nets.

As future work we aim to exploit the new criterion in order to contribute to a simple theory of non-commutative *modules*[3] [5, 1]. Naively, the question is to try to define a notion of *module type* and *orthogonal composition* between types of modules obtained after splitting a correct proof structure.

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[2] This example is due to Paul-André Mellies.

[3] "Imagine to cut a proof net and obtain two (very bad) components linked together by their common border" J.-Y. Girard.

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