Non decomposable connectives of linear logic

Roberto Maieli

Department of Mathematics and Physics – "Roma Tre" University
Largo San Leonardo Murialdo 1, 00146 Rome, Italy
maieli@mat.uniroma3.it

Abstract

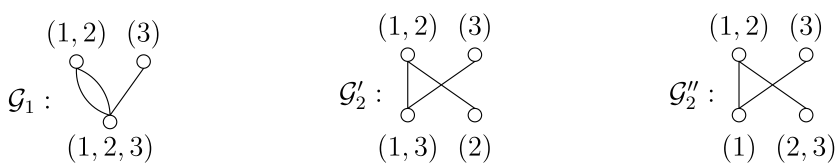
This paper studies the so-called generalized multiplicative connectives of linear logic, focusing on the question of finding the "non-decomposable" ones, i.e., those that cannot be expressed as combinations of the default binary connectives of multiplicative linear logic, $\otimes$ (*times*) and $⅋$ (*par*). In particular, we concentrate on generalized connectives of a surprisingly simple form, called "entangled connectives", and prove a characterization theorem giving a criterion for identifying the undecomposable entangled ones.

Keywords: linear logic, partitions sets, proof nets, sequent calculus.
2010 MSC: 03F03, 03F05, 03F07, 03F52, 03B47.

Contents

1. Introduction

Generalized connectives for the pure (units free) multiplicative fragment of linear logic (MLL, [6]) were introduced by Girard in his seminal paper [5] in terms of permutations but most of the results known after then are essentially due to Danos and Regnier [3] who replaced permutations by partitions of finite sets. A generalized multiplicative connective may be defined as a pair of sets of pairwise orthogonal partitions over a finite domain $\{1, 2, ...n\}$ where orthogonality is defined by a topological condition: the bipartite graph obtained by linking together classes of each partition sharing an element is acyclic and connected (ACC). E.g., partition $\{(1, 2), (3)\}$ is not orthogonal to partition $\{(1, 2, 3)\}$ since the bipartite graph $\mathcal{G}_1$ contains a cycle while the two sets of partitions, $\{\{(1, 2), (3)\}\}$ and $\{\{(1, 3), (2)\}, \{(1), (2, 3)\}\}$, are orthogonal since they are pairwise so (their respective bipartite graphs, $\mathcal{G}'_2$ and $\mathcal{G}''_2$, are ACC) as illustrated below:



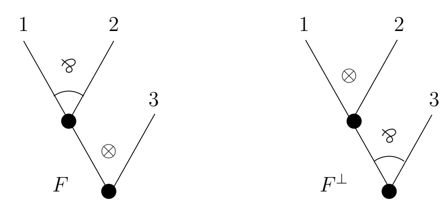
There are two ways to interpret logical connectives (formulas) of MLL by means of pairs of orthogonal sets of partitions: one based on the *sequent calculus syntax* and the other one based on the *proof nets syntax*. In the *sequential syntax*, a partition describes a sequent calculus *rule* for producing the generalized formula: the domain of the partition is the set of principal sub-formulas occurrences of the given formula and each class describes one premise of the rule; so, a multiplicative rule for an n-ary connective $F(A_1, ..., A_n)$ is completely characterized by the *organization* of its principal sub-formulas $(A_1, ..., A_n)$; indeed, since multiplicatives rules are *unconditional* about the context[1] any rule can be simply described by a *partition* over its principal sub-formulas occurrences (their indexes), omitting the contexts $(\Gamma, \Gamma_1, \Gamma_2, ...)$. E.g., the two orthogonal sets of partitions (or organizations sets), $\mathcal{O}_F = \{\{(1,2),(3)\}\}$ and $\mathcal{O}_{F^\perp} = \{\{(1,3),(2)\},\{(1),(2,3)\}\}$, are compact representations of, respectively, the sequent generalized rule below (when $F(1,2,3) = (A_1 \parr A_2) \otimes A_3$)

$$\frac{(1,2) \qquad (3)}{F(1,2,3)}\, r(F) \;\equiv\; \frac{\dfrac{\Gamma_1, A_1, A_2}{\Gamma_1, (A_1 \parr A_2)}\parr \qquad \Gamma_2, A_3}{\Gamma, F = (A_1 \parr A_2) \otimes A_3}\otimes$$

and the next two sequent n-ary rules (when $F^\perp(1,2,3) = (A_1 \otimes A_2) \parr A_3$):

$$\frac{(1,3) \qquad (2)}{F^\perp(1,2,3)}\, r_1(F^\perp) \;\equiv\; \frac{\dfrac{\Gamma_1, A_1, A_3 \qquad \Gamma_2, A_2}{\Gamma, (A_1 \otimes A_2), A_3}\otimes}{\Gamma, F^\perp = (A_1 \otimes A_2) \parr A_3}\parr \qquad \frac{(1) \qquad (2,3)}{F^\perp(1,2,3)}\, r_2(F^\perp) \;\equiv\; \frac{\dfrac{\Gamma_1, A_1 \qquad \Gamma_2, A_2, A_3}{\Gamma, (A_1 \otimes A_2), A_3}\otimes}{\Gamma, F^\perp = (A_1 \otimes A_2) \parr A_3}\parr$$

In the *graphical syntax*, a partition describes the "effects" of a *Danos-Regnier switching* over the frontier of the syntactical tree of a MLL formula. We may associate to each switching of a formula tree $F(A_1, ..., A_n)$ a partition of its top frontier $A_1, ..., A_n$ in to classes of connected components. The partitions set of the border leaves of a formula tree F, induced by all switchings, is called the *pre-type* of F (denoted $\mathcal{P}_F$). E.g., the pre-types of the two formula trees below



are resp., the partitions set $\mathcal{P}_F = \{\{(1,3),(2)\},\{(1),(2,3)\}\}$ (pre-type of $F = (A_1 \parr A_2) \otimes A_3$) and the partitions set $\mathcal{P}_{F^\perp} = \{\{(1,2),(3)\}\}$ (pre-type of $F^\perp = (A_1 \otimes A_2) \parr A_3$).

These two syntaxes describing usual MLL formulas (i.e., definable by means of the basic binary connectives $\otimes$ and $\parr$) are shown to be dual. To be precise, the organizations set of a MLL formula F (denoted $\mathcal{O}_F$) is exactly the dual of the pre-type of F that is, $\mathcal{O}_F = \mathcal{P}_F^\perp$ so, $\mathcal{O}_F$ is a *type* (i.e. it is equal

[1] Multiplicative rules do not require anything about the contexts in which appear formulas these rules apply to.

to its bi-orthogonal, $\mathcal{O}_F = \mathcal{O}_F^{\perp\perp}$), thus every sequential n-ary connective of MLL, $\mathcal{O}_F$ and $\mathcal{O}_{F^\perp}$, is reflexive that is $\mathcal{O}_F^\perp = \mathcal{O}_{F^\perp}$ and $\mathcal{O}_{F^\perp}^\perp = \mathcal{O}_F$.

We can generalize this construction in a natural way. Given two partitions sets we wonder whereas they define a generalized multiplicative connective[2] according to the two dual syntaxes as follows:

- in the sequent calculus syntax it is sufficient to describe a connective as a pair of orthogonal organizations sets $\mathcal{O}_C \perp \mathcal{O}_{C^\perp}$; each organizations set is a *type* describing *all rules* that allow to derive that connective; orthogonality is enough to ensure cut elimination;

- in the proof nets syntax, since partitions are interpreted as switchings associated to a connective and since switchings are in some sense *dense sub-sets*[3] we may define a connective as a pairs of partitions sets P and Q s.t. $P \perp Q$ and $P^\perp \perp Q^\perp$: the first condition $P \perp Q$ ensures (ACC) correctness of proof structures while the second condition $P^\perp \perp Q^\perp$ ensures the stability of correctness under cut reduction.

Then, given a pair of partitions sets, P and Q, as describing a connective (in sequential or graphical syntax), we wonder whereas this connective is definable by means of the basic binary connective of MLL ($\parr$ and $\otimes$) in which case it is called *decomposable*. Not all connectives are decomposable. To date there only exists one instance of non (binary) decomposable connective firstly discovered by Girard [5] in terms of permutations and later reformulated by Danos-Regnier [3] as a pair of orthogonal sets of partitions, G_4 and $G_4^\perp$, over the same domain $\{1, 2, 3, 4\}$:

$$G_4 = \{\{(1,2),(3,4)\},\{(2,3),(4,1)\}\} \quad \text{and} \quad G_4^\perp = \{\{(1,3),(2),(4)\},\{(2,4),(1),(3)\}\}.$$

Unfortunately, no (uniform) characterization of undecomposable connectives is known up to now. Under this respect, this work represents a first step forward: it defines a class of generalized connectives that are not decomposable, neither in sequential nor in graphical syntax, the so-called *undecomposable entangled connectives*. A pair of distinct partitions sets with same domain and same weight (cardinality) is entangled when each partition contains only binary or unary classes (Definition 21). Then a connective, P and Q,

[2]Every connective is immediately given together with its dual.

[3]"Switchings should be seen as a dense subset of para-proofs", page 41 of [7].

is entangled whenever P or Q is an entangled pair. Entangled pairs have interesting properties that are investigated throughout Section 4. The first one (Theorem 34, one of the main results of the paper), being that every entangled pair of partitions sets is a type. This result has several consequences: decomposable entangled pairs (and so, decomposable connectives) have a very simple characterization (the *normal form* of Theorem 39) which can be further used to prove that given an undecomposable entangled connective, P and Q, where P is an entangled pair, then P cannot be embedded into a decomposable type $T \supsetneq P$, s.t. T and Q become decomposable (see Corollary 43).

Although our notion of entanglement doesn't solve the admittedly difficult general problem ("try to find an uniform characterization of the full class of undecomposable connectives"), it is far from being an "ad hoc" condition. Intuitively, an entangled type is naturally obtained as soon as we "superpose" (i.e., we sum) the pre-types of two *bipoles*[4] having the same "skeleton" (i.e., the same abstract syntactical tree) up to cyclic permutation of their frontier (the border leaves; see Remarks 44). This fact is a novelty since the union of types is not in general a type while the intersection of types is always a type (Property 3 of Section 2). Indeed, entangled types are the smallest types (w.r.t. the number of partitions), if we exclude the trivial singleton types. So, entangled connectives can be considered, in some sense, "elementary connectives", since they are the "smallest" generalized multiplicative connectives (w.r.t. the number of switchings or the number of sequential rules), if we exclude, of course, the basic ones ($⅋$ and $\otimes$). However, the class of undecomposable entangled connectives is quite special and we already discovered examples of undecomposable connectives falling outside of it like e.g. G_9 below (we omit the dual $G_9^{\perp}$):

$$
\begin{aligned}
G_9 = \{ \quad & \{(1,2,3),(4,5,6),(7,8,9)\}, \\
& \{(2,3,4),(5,6,7),(8,9,1)\}, \\
& \{(3,4,5),(6,7,8),(9,1,2)\} \quad \}.
\end{aligned}
$$

We are currently working on a more general characterization[5] of the full class of primitive non decomposable connectives i.e., those ones that cannot be

[4]Naively, a *bipole* B is a special MLL formula, introduced by Andreoli in [1], with only two layers of connectives: a generalized $⅋$ of generalized $\otimes$-sub-formulas. Bipoles have the nice feature that their pre-types are already types.

[5]For this scope we consider partitions over cyclic permutations of linearly ordered sequences $1 < 2 < ... < n$; see Section 6.

defined by means of other connectives, neither binary nor entangled.

More generally, non decomposable generalized connectives witness a deep asymmetry between proof nets and sequent proofs since the former ones allow to express a kind of parallelism of proofs that the latter ones cannot do: actually, there exist proof nets in *η-expanded form*, built on non decomposable connectives, that have no correspondence with any sequential proof, if we exclude the identity axioms $\vdash G, G^{\perp}$. This significant fact, lying at the core of linear logic, was already remarked in [3]:

> "*We saw with some surprise that the realm of multiplicatives became quite complex, even handled by a careful generalization. Yet the generalization seems more natural in the non-sequential framework [...]. Maybe we witness here the limits of sequential presentations of logic*".

2. Partitions and orthogonality

Definition 1 (partitions and orthogonality). *A* **partition** *of a finite set $X = \{1, ..., n\}$ (also called* support*) is a set of nonempty subsets of X, called* classes *(also* blocks *or* parts*), such that every element $i \in X$ is in exactly one of these subsets (i.e., X is a disjoint union of the subsets).*

If p and p^* are two partitions of $X = \{1, ..., n\}$, then:

- *the* **induced graph of incidence** *of p and p^*, denoted $\mathcal{G}(p, p^*)$, is the bipartite (undirected) graph which has for vertices the classes of p and p^*, two of them being linked iff they share an element $i \in X$;*
- *p and p^* are* **orthogonal** *(denoted, $p \perp p^*$) iff the induced graph $\mathcal{G}(p, p^*)$ is acyclic and connected (i.e., it is a topological tree; shortly, ACC).*

If P and Q are two sets of partitions, we say that P and Q are orthogonal (denoted $P \perp Q$) if they are pointwise orthogonal i.e., $\forall p \in P, \forall q \in Q, p \perp q$. $P^{\perp}$ denotes the set of partitions orthogonal to all the elements of P.

As notation, we use: variables $X, Y, Z, ...$, for finite sets $\{1, ...n\}$, variables $a, b, c, ...$, for arbitrary elements of sets, variables $p, q, r, ...$, for partitions of a set, variables $x, y, z, ...$, for classes of a partition and variables $A, B, P, Q, ...$, for sets of partitions.

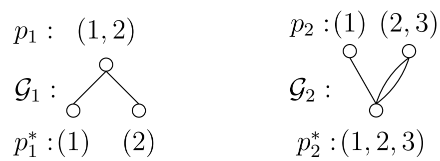


Figure 1: bipartite graphs of incidence induced by pairs of partitions

Example 2. *Given two partitions, $p_1 = \{(1,2)\}$ and $p_1^* = \{(1),(2)\})$ then, $p_1 \perp p_1^*$ since the induced graph $\mathcal{G}_1(p_1, p_1^*)$ is a tree; conversely, partitions $p_2 = \{(1),(2,3)\})$ and $p_2^* = \{(1,2,3)\}$ are not orthogonal ($p_2 \not\perp p_2^*$) since $\mathcal{G}_2(p_2, p_2^*)$ contains a cycle as illustrated in Figure 1. Moreover, observe that w.r.t the following sets of partitions given, P_1, P_2, Q_1 and Q_2, we only have $P_1 \perp Q_1$ and $P_2 \perp Q_2$ while $P_1 \not\perp Q_2$ and $P_2 \not\perp Q_1$:*

$$P_1 = \{\{(1,2),(3,4)\}\},\ Q_1 = \{\{(1,3),(2),(4)\},\{(1,4),(2),(3)\},\{(2,3),(1),(4)\},\{(2,4),(1),(3)\}\}$$
$$P_2 = \{\{(2,3),(4,1)\}\},\ Q_2 = \{\{(2,4),(3),(1)\},\{(2,1),(3),(4)\},\{(3,4),(2),(1)\},\{(3,1),(2),(4)\}\}.$$

The **degree** of a class x is the cardinality of x; the **weight** of a partition p is the cardinality of p. Given a set of partition P, we say that:
– P has weight w (so, P is **weighable**) if $\forall p \in P$, p has weight w;
– the **size** of P is the cardinality of P;
– the **dimension** of P is the cardinality of its domain X.

Next facts of Property 3 concerning partitions sets can be found in the literature on multiplicatives (see e.g. [9] for detailed proofs).

Property 3 (partitions and types). *Let A, B and A_i be sets of partitions over $\{1,...,n\}$. Then the following facts hold:*

1. $A \perp B$ *iff* $A \subseteq B^\perp$ *iff* $B \subseteq A^\perp$;
2. $A \subseteq B$ *implies* $B^\perp \subseteq A^\perp$;
3. $A \subseteq A^{\perp\perp}$;
4. $A \perp B$ *iff* $A^{\perp\perp} \perp B^{\perp\perp}$;
5. $A^\perp = A^{\perp\perp\perp}$;
6. ($A \perp B$ *and* $A^\perp \perp B^\perp$) *iff* ($A^\perp = B^{\perp\perp}$) *iff* ($B^\perp = A^{\perp\perp}$);
7. *In case* $A \perp B$ *and* $A^\perp \perp B^\perp$:
   ($B^\perp = A$ *and* $B = A^\perp$) *iff* ($A^{\perp\perp} = A$ *and* $B = B^{\perp\perp}$);
8. $A = A^{\perp\perp}$ *iff* $\exists B : A = B^\perp$; *then, A is called a* **type**;
9. $(\bigcup_i A_i)^\perp = \bigcap_i A_i^\perp$;

10. $(\bigcap_i A_i)^\perp \supseteq \bigcup_i A_i^\perp$;
11. *if A is a singleton (i.e., $A = \{p\}$ where p is a partition) then A is type;*
12. *if $A \perp B$ then, all partitions in A (resp., in B) have the same weight (i.e., A and B are weighable).*

Let us prove case 6 of Property 3: assume $(A \perp B)$ and $(A^\perp \perp B^\perp)$ then, by case 1, $(A^\perp \perp B^\perp) \Rightarrow (A^\perp \subseteq B^{\perp\perp})$ and $(A \perp B) \Rightarrow (A \subseteq B^\perp)$ which implies, by case 2, $B^{\perp\perp} \subseteq A^\perp$; so $B^{\perp\perp} = A^\perp$. Vice-versa assume $B^{\perp\perp} = A^\perp$ then, by case 1, $(A^\perp \subseteq B^{\perp\perp}) \Rightarrow (A^\perp \perp B^\perp)$ and, by case 3, $(B^{\perp\perp} \subseteq A^\perp) \Rightarrow (B \subseteq A^\perp)$ from which, by case 1, $(A \perp B)$.

We draw attention to the fact that the *intersection of types is always a type* (case 9) while the *union of types is not always a type* (case 10): instances of these facts can be found in Example 5.

Case 11 follows by Lemma 4 of [9]. Finally, case 12 follows by the next well known graph theoretical property (see e.g. [8], pages 250-251).

Property 4 (Euler-Poincaré invariance). *Given a graph $\mathcal{G} = (V, E)$ then, $(|V| - |E|) = (|CC| - |Cy|)$ where $|V|$, $|E|$, $|CC|$ and $|Cy|$ denote, resp., the number of vertices, edges, connected components and primitive cycles of $\mathcal{G}$.*

By Euler-Poincaré invariance, weighable sets of partitions are indeed the only sets of partitions that are considered, as any non weighable set has empty orthogonal.

Example 5. *Assume two sets of partitions P_1 and P_2 over $\{a, b, c, d\}$ as below then, we can calculate their respective orthogonal types, by case 9 of Property 3, as follows:*

1. *if $P_1 = \{p_1 : \{(a, c), (b, d)\}, p_2 : \{(a, d), (b, c)\}\}$ then,*
   $P_1^\perp = \{p_1\}^\perp \cap \{p_2\}^\perp = \{\{(d, c), (a), (b)\}, \{(a, b), (d), (c)\}\}$;
2. *if $P_2 = \{p_1 : \{(a, d, c), (b)\}, p_2 : \{(d, b, c), (a)\}\}$ then,*
   $P_2^\perp = \{p_1\}^\perp \cap \{p_2\}^\perp = \{\{(b, a), (c), (d)\}\}$.

Concerning case 10 of Property 3, we just observe here that although every singleton $\{p_i\}$ is a type (with $i = 1, 2$, by case 11 of Property 3), only P_1 is a type while P_2 is not so, since $P_2 \subsetneq P_2^{\perp\perp} = (P_1 \cup P_2)$. We will show in Section 4.1 that if we restrict to consider pairs of singleton sets of partitions, $\{p\}$ and $\{q\}$, with same domain and same weight and s.t. their classes have degree $1 \leq d \leq 2$ then, their union, $\{p\} \cup \{q\}$, is a(n entangled*) type.*

3. Generalized connectives

Partitions sets allow to express generalized multiplicative connectives in a very abstract, uniform and compact way. A generalized (or n-ary) multiplicative connective $C(A_1, ..., A_n)$ may be defined, as in [3], by a pair of orthogonal sets of partitions, $P \perp Q$, over a same domain $\{1, ..., n\}$ that is the set of principal sub-formulas indexes of C. There are indeed two dual ways to interpret (orthogonal pairs of) sets of partitions as defining logical connectives:

- the *sequential* way, based on the sequent calculus syntax in Gentzen's style (Section 3.1) and
- the *graphical* way, based on the proof nets syntax of linear logic (Section 3.2).

3.1. Sequent calculus syntax for generalized connectives

Following [3], we want to define a generalized multiplicative connective by means of its generalized (or $n-ary$) sequential rules. We start by looking at the one-sided sequent calculus rules of the pure (units free) multiplicative fragment of linear logic (MLL [6]) given in Figure 2: the MLL rules are *unconditional* about the context (they do not require anything about the context) and *conservative* about literals[6] (the premises of each rule have exactly the same literals as in the conclusion). Hence a MLL rule is completely characterized

$$\frac{}{A, A^{\perp}}\, identity \qquad \frac{\Gamma, A \qquad \Delta, A^{\perp}}{\Gamma, \Delta}\, cut \qquad \frac{\Gamma, A_1 \qquad \Delta, A_2}{\Gamma, \Delta, A_1 \otimes A_2} \otimes (times) \qquad \frac{\Gamma, A_1, A_2}{\Gamma, A_1 \parr A_2} \parr (par)$$

Figure 2: Standard MLL Sequent Calculus

by the *organization* of its principal sub-formulas, where the "organization" describes a possible way to gather the principal sub-formulas in premise sequents of a sequent calculus subtree proving the principal formula. We want to define an abstract notion of general (or n-ary) sequent rules in such a way that these properties still hold. Thus a **general multiplicative rule** for an n-ary connective $C(A_1, ..., A_n)$ is completely characterized by the organization of its principal sub-formulas occurrences $(A_1, ..., A_n)$ as described in the l.h.s.

[6]Atoms or duals of atoms.

picture of Figure 3. Since multiplicatives general rules are unconditional about the context we can simplify the form of an n-ary rule by getting rid of contexts as in the r.h.s. picture of Figure 3. Moreover, since general rules are conservative about the literals any rule can be further simplified so as to be described by a **partition** $\{\{i^1_1, ..., i^1_{k_1}\}, ..., \{i^p_1, ..., i^p_{k_p}\}\}$ of the set of its principal sub-formulas indexes $\{1, ...n\}$.

$$\frac{\vdash \Gamma_1, A_{i^1_1}, ...A_{i^1_{k_1}} \qquad ... \qquad \vdash \Gamma_p, A_{i^p_1}, ..., A_{i^p_{k_p}}}{\vdash \Gamma_1, ..., \Gamma_p, C(A_1, ..., A_n)} R_C \qquad \frac{\vdash A_{i^1_1}, ...A_{i^1_{k_1}} \qquad ... \qquad \vdash A_{i^p_1}, ..., A_{i^p_{k_p}}}{\vdash C(A_1, ..., A_n)} R_C$$

Figure 3: generalized sequential rule

Definition 6 (generalized rules as partitions). *If R is an n-ary rule for a general connective $C(A_1, ..., A_n)$ then, it can be described by a partition p_R over $\{1, ..., n\}$ as follows: $i, j \in \{1, ..., n\}$ belong to a same class of p if formulas occurrences A_i and A_j belong to the same premise of R. Conversely, given a partition p, we can check if a rule R satisfies (i.e., corresponds to) it simply by asking if the partition induced by R is really p (modulo a possible renumbering of the principal formulas in the conclusion of R).*

E.g., the basic MLL rules *times* and *par* are described resp., by partitions $\{(1), (2)\}$ and $\{(1, 2)\}$ while the rules below, D and D^*, are described by resp., partition $\{(1, 2), (3)\}$ and partition $\{(1, 2, 3)\}$.

$$\frac{\vdash A_1, A_2 \qquad \vdash A_3}{\vdash D(A_1, A_2, A_3)} D \qquad\qquad \frac{\vdash, A_1, A_2, A_3}{\vdash D^*(A_1, A_2, A_3)} D^*$$

Anyway, single rules (single partitions) are not sufficient to define logical connectives. We need indeed a set of rules for a connective C together with a set of rules for the dual connective C^*.

Definition 7 (sequential multiplicative generalized connectives). *A* sequential multiplicative generalized connective *C consists of two sets of orthogonal partitions, $\mathcal{O}_C \perp \mathcal{O}^*_C$: a set $\mathcal{O}_C$ of partitions representing the* right rules *and a set $\mathcal{O}^*_C$ of partitions representing the* left rules *handled by the dual connective C^*. By duality, the set $\mathcal{O}^*_C$ of left rules for C are exactly the set of right rules for the dual connective C^* (i.e., $\mathcal{O}^*_C = \mathcal{O}_{C^*}$).*

Orthogonality ensures cut elimination and so the computational meaning of general connectives. E.g., $\mathcal{O}_D = \{p : \{(1,2),(3)\}\}$ and $\mathcal{O}_{D^*} = \{p^* : \{(1,2,3)\}\}$ do not define a sequential connective since $\mathcal{O}_D \not\perp \mathcal{O}_{D^*}$; we cannot indeed reduce in π below the cut between the two formulas, $D(A_1, A_2, A_3)$ and $D^*(A_1^\perp, A_2^\perp, A_3^\perp)$, introduced just above by resp., rules p and p^*, by means of "smaller" cuts between each couple of dual sub-formulas[7], A_i and $A_i^\perp$:

$$\pi : \dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma, A_1, A_2} \quad \dfrac{\vdots}{\vdash \Delta, A_3}}{\vdash \Gamma, \Delta, D(A_1, A_2, A_3)}\, p \qquad \dfrac{\dfrac{\vdots}{\vdash \Sigma, A_1^\perp, A_2^\perp, A_3^\perp}}{\vdash \Sigma, D^*(A_1^\perp, A_2^\perp, A_3^\perp)}\, p^*}{\vdash \Gamma, \Delta, \Sigma}\, cut \qquad \not\rightsquigarrow$$

If we denote by MLL^+ the standard *MLL sequent calculus* of Figure 2 extended with the sequential general multiplicative rules of Figure 3 then, MLL^+ still satisfies cut elimination[8].

3.1.1. Decomposable sequential connectives

Definition 8 (sequential decomposable connectives). *A sequential connective, $\mathcal{O}_C$ and $\mathcal{O}_{C^*}$, is (binary)* decomposable *iff there exists an MLL formula F (built by means of $\otimes$ and $⅋$) and assigned to C s.t. the two following propositions hold:*

1. *if $p \in \mathcal{O}_C$ (resp., $q \in \mathcal{O}_{C^*}$) then F (resp., $F^\perp$) is provable in MLL, from p (resp., from q), by only means of the binary rules, $⅋$ and $\otimes$;*
2. *if p (resp., q) is a partition from which we can derive F in MLL (resp., $F^\perp$) by only means of binary rules, then $p \in \mathcal{O}_C$ (resp., $q \in \mathcal{O}_{C^*}$).*

Example 9. *The generalized connective $(\mathcal{O}_Y, \mathcal{O}_{Y^*})$ below is decomposable resp., into $Y = (1⅋2) \otimes 3 \otimes 4$ and $Y^* = (1 \otimes 2)⅋3⅋4$:*

$$\mathcal{O}_Y = \{p : \{(1,2),(3),(4)\}\}$$

$$\mathcal{O}_{Y^*} = \{q_1 : \{(1,3,4),(2)\}, q_2 : \{(2,3,4),(1)\}, q_3 : \{(1,3),(2,4)\}, q_4 : \{(1,4),(2,3)\}\}$$

$$\dfrac{\vdash 1,2 \quad \vdash 3 \quad \vdash 4}{\vdash Y(1,2,3,4)}\, p \qquad \dfrac{\vdash 1,3,4 \quad \vdash 2}{\vdash Y^*(1,2,3,4)}\, q_1 \quad \dfrac{\vdash 2,3,4 \quad \vdash 1}{\vdash Y^*(1,2,3,4)}\, q_2 \quad \dfrac{\vdash 1,3 \quad \vdash 2,4}{\vdash Y^*(1,2,3,4)}\, q_3 \quad \dfrac{\vdash 1,4 \quad \vdash 2,3}{\vdash Y^*(1,2,3,4)}\, q_4$$

[7]This is known as the "*key-step*" of cut-reduction: "a key-step between two partitions succeeds iff they are orthogonal" (see Lemma 1 of [3]).

[8]See Theorem 3 of [3].

Remarks 10 (**The packaging problem**). *"Partitions of a decomposable connective C do not describe all proofs involving it, but only those ones that are independent from the context" [3]. E.g., assume we want to prove a sequent S with only two conclusions, $D = (A_1 \parr B_2) \otimes C_3$ and $D^\perp = (A_1^\perp \otimes B_2^\perp) \parr C_3^\perp$ then, either we introduce D (resp., $D^\perp$) only by its sequential rules $\mathcal{O}_D = \{(1,2),(3)\}$ (resp., $\mathcal{O}_{D^\perp} = \{\{(1),(2,3)\},\{(1,3),(2)\}\}$) or we build S stepwise by exploiting the decomposition of D (resp., $D^\perp$). Clearly the latter method is more powerful since there are proofs, like the one below, that cannot be derived by the former one. Last rule for D (reps., for $D^\perp$) cannot be binary (i.e., with two premises) because any attempt to derive S would build a premise with the other conclusion $D^\perp$ (resp., D) together with only some (not all) of the principal formulas, A, B and C (resp., $A^\perp, B^\perp$, $C^\perp$): such sequents are never derivable from atomic logical axioms.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A, A^\perp \qquad \vdash B, B^\perp}{\vdash A, B, (A^\perp \otimes B^\perp)}\otimes}{\vdash (A \parr B), (A^\perp \otimes B^\perp)}\parr \qquad \vdash C, C^\perp}{\vdash (A \parr B) \otimes C, (A^\perp \otimes B^\perp), C^\perp}\otimes}{\vdash (A \parr B) \otimes C, (A^\perp \otimes B^\perp) \parr C^\perp}\parr \quad \equiv \quad \dfrac{\dfrac{\dfrac{\vdash A, A^\perp \qquad \vdash B, B^\perp}{\vdash A, B, (A^\perp \otimes B^\perp)}\otimes \qquad \vdash C, C^\perp}{\vdash (A \parr B) \otimes C, (A^\perp \otimes B^\perp), C^\perp} D}{\vdash (A \parr B) \otimes C, (A^\perp \otimes B^\perp) \parr C^\perp}\parr$$

In other words, all the binary rules yielding a decomposable connective X in the sequent calculus can be packed *in an single rule iff these rules can be permuted in order to appear consecutively (as a "package").*

3.2. Proof nets syntax for generalized connectives

Definition 11 (**graphical generalized connectives**). *A* generalized multiplicative connective in graphical syntax *consists of two sets of partitions, P and Q, over the same domain $\{1, ..., n\}$, s.t. $P \perp Q$ and $P^\perp \perp Q^\perp$; we therefore denote a connective by a pair (P, Q).*

Dually to the sequential case, $P_\parr = \{\{(a),(b)\}\}$ and $Q_\otimes = \{\{(a,b)\}\}$ denote resp., the graphical basic connectives "$\parr$" and "$\otimes$". In the following, unless differently specified, the expression "generalized connectives" means "generalized connectives in the graphical setting" according to Definition 11.

3.2.1. Decomposable graphical connectives

Definition 12 (**Danos-Regnier switchings, pre-types and types**). *Let F be a MLL formula built by the binary multiplicative connectives $\parr$ and $\otimes$ from its top border X (i.e. the literals of F); given the parse (or syntactical) tree of F, a* Danos-Regnier switching *[3] of F is the graph obtained after the* *mutilation of one of the two premises for each* $\parr$ *node of* F*; after the mutilation, the elements of the border belonging to a same connected component, constitute a class of the induced partition. Then, the* pre-type *of* F*, denoted by* $\mathcal{P}_F$*, is given by the set of partitions over* X *induced by all DR-switchings of the parse tree of* F*. We denote* $\mathcal{T}_F$ *the* type $\mathcal{P}_F^{\perp\perp}$ *of* F*.*

Definition 13 (decomposable connectives). *A connective* (P, Q) *is* MLL binary decomposable *(or* definable*) iff there exists a formula* F*, only built with binary connectives of MLL, s.t.* P *is the pre-type of the parse tree of* F *and* Q *is the pre-type of the parse tree of the dual formula* $F^{\perp}$*.*

In the following we say that *a set of partitions* P *is decomposable* if there exists a formula F, s.t. P is the pre-type of F (i.e., $P = \mathcal{P}_F$). Next Fact 14 on decomposable sets of partitions corresponds to Theorem 7 of [9].

Fact 14 (orthogonal of a decomposable set of partitions). *If* F *is a MLL formula then,* $\mathcal{P}_F^{\perp} = \mathcal{T}_{F^{\perp}}$*.*

Remarks 15 (expressiveness of the graphical syntax). *Connectives defined according to Definition 11 are very expressive. Given a set of partitions* P *there exists a connective* (P, Q) *for every* Q *s.t.* $Q^{\perp\perp} = P^{\perp}$*, by case 6 of Property 3. E.g., given (the type)* $P = \{p_1 = \{(1,2),(3,4),(5)\},\ p_2 = \{(1,3),(2,4),(5)\}\}$*, there exist at least two graphical connectives, as follows:*

1. $(P, P^{\perp})$ *where, by case 9 of Property 3,* $P^{\perp} = \{p_1\}^{\perp} \cap \{p_2\}^{\perp}$ *is the set of the flagged partitions below:*

$\{p_1\}^{\perp}$:					$\{p_2\}^{\perp}$:			
(145)	(3)	(2)	✓		(145)	(2)	(3)	✓
(135)	(4)	(2)			(125)	(4)	(3)	
(235)	(4)	(1)	✓		(325)	(4)	(1)	✓
(245)	(3)	(1)			(345)	(2)	(1)	
(45)	(13)	(2)			(45)	(12)	(3)	
(25)	(13)	(4)			(35)	(12)	(4)	
(25)	(14)	(3)	✓		(35)	(14)	(2)	✓
(35)	(14)	(2)	✓		(25)	(14)	(3)	✓
(15)	(23)	(4)	✓		(15)	(32)	(4)	✓
(15)	(24)	(3)			(15)	(34)	(2)	
(45)	(23)	(1)	✓		(45)	(32)	(1)	✓
(35)	(24)	(1)			(25)	(34)	(1)	

2. (P, R) *with* $R = \{\{(1,4,5),(2),(3)\},\ \{(3,2,5),(4),(1)\}\} \subsetneq P^{\perp}$ *(s.t.* $R^{\perp\perp} = P^{\perp}$*).*

3.2.2. Proof structures with generalized links

A *generalized multiplicative proof structure* is a graph which has for edges formulas of MLL^+ and for vertices *links* as depicted on the l.h.s. of Figure 4 and defined as follows:

- an *axiom link*, is a vertex with only a pair of dual atomic conclusions (incident edges), $\{\{(a, a^\perp)\}\}$;
- a *cut link*, is a vertex with only a pair of dual premises (incident edges), $\{\{(A, A^\perp)\}\}$;
- a *generalized link* P, denoted λ_P, is a vertex with premises $A_1, ..., A_{n\geq 2}$ and exactly one conclusion $F(A_1, ..., A_n)$ (also denoted by the special symbol $\sharp$) where (P, Q) is a generalized connective with domain $1, .., n$ and a bijection $\varphi : i \mapsto A_i$; λ_P is immediately given together with its dual link λ_Q. We assume there exactly exists a generalized link for each connective.

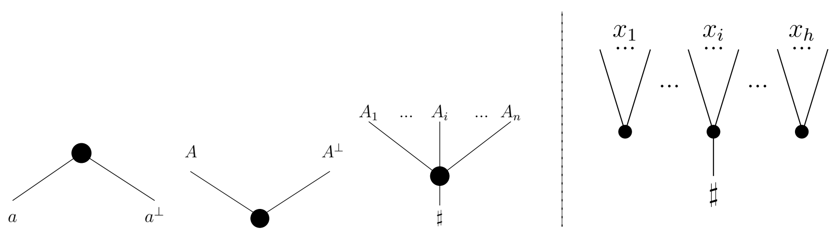


Figure 4: axiom, cut and generalized links on the l.h.s. and switching on the r.h.s.

In a proof structure, each formula is conclusion of exactly one link and premise of at most one link; in the special case of the *conclusions* of a proof structure, i.e., formulas that are not premises of any link, we add a handling vertex. Not all proof structures are *correct* that is, correspond (sequentialize) to derivable sequent proofs. In order to characterize those sequentializable among all proof structures we need an intrinsic (i.e. non inductive) *correctness criterion.* First, we extend the usual notion of DR-switching to generalized links. For the switch of a generalized link λ_P (where P is a generalized connective), chose a partition $p = \{x_1, ..., x_h\} \in P$, then chose a class $x_i \in p$, that is elect the upper edges belonging to this class and adjacent to the corresponding vertex of λ_P, disconnect all other upper edges, then for each

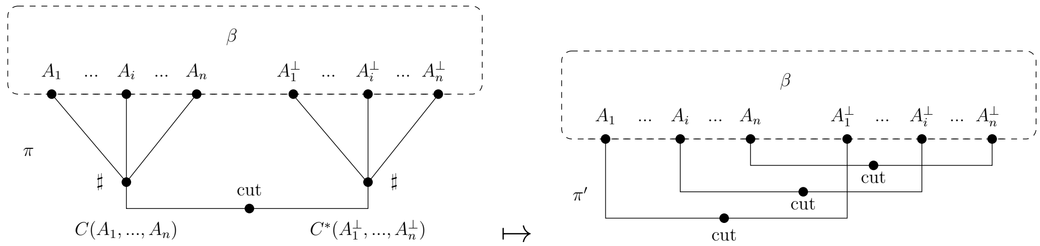
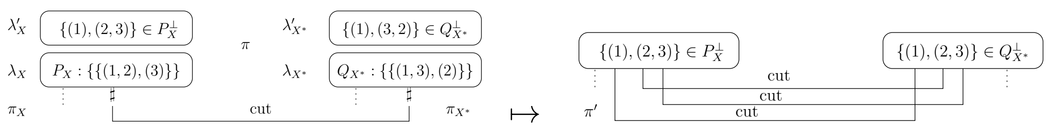


Figure 5: reduction step of a generalized cut link

remaining class of $p \setminus \{x_i\}$ connect together the pending edges (disconnected before) to a new vertex as in the rightmost side picture of Figure 4. Then, the correctness criterion for generalized proof structures remains the same as the Danos-Regnier's one for MLL proof nets: *a generalized proof structure* π *is* correct *(it is a* generalized proof net*) iff for each choice of the generalized switches, the associated graph is acyclic and connected.* So, a generalized proof net is an *MLL proof net* when each link is an axiom, a cut or a basic binary ⅋ and ⊗ link. We may also compactly describe a basic link by the set of its switches interpreted as partitions over the whole border, i.e., the top border A, B together with the bottom border $\sharp = (A \otimes B)$ or $\sharp = (A ⅋ B)$: $\lambda_⅋ = \{\{(A, \sharp)), (B)\}, \{(A), (B, \sharp)\}\}$ and $\lambda_\otimes = \{\{(D, \sharp, C)\}\}$.

Remarks 16 (on cut reduction). *The "computational meaning" of generalized connectives is assured by the fact that reducing in a proof net a cut between (dual) generalized connectives,* C *and* C^**, preserves the correctness criterion [3]. This fact follows by Definition 11 of a graphical connective where unlike the sequential syntax, requiring only orthogonality between sets of partitions,* $P \perp Q$*, is not enough for getting the stability of correctness under cut reduction; orthogonality "must pass to the contexts" that is,* $P^\perp \perp Q^\perp$*; thus, the first condition* $P \perp Q$ *ensures (ACC) correctness of proof structures while the second condition* $P^\perp \perp Q^\perp$ *ensures the stability of correctness under cut reduction. A pictorial view is given in Figure 5. E.g., the sequential generalized connective* $P_X = \{\{(1,2),(3)\}\}$ *and* $Q_{X^*} = \{\{(1,3),(2)\}\}$ *is not a connective in proof net syntax since, by Definition 11,* $P_X^\perp = \{\{(1,3),(2)\},\{(2,3),(1)\}\}$ *is not orthogonal to* $Q_{X^*}^\perp = \{\{(1,2),(3)\},\{(2,3),(1)\}\}$*. This intuitively means that if we admit the generalized links* λ_X *and* λ_{X^*} *(corresponding, resp., to connectives* P_X *and* Q_{X^*}*) then, the proof net* π *of Figure 6 (obtained by cutting* π_X*, with conclusion* $\sharp_X$*, against* π_{X^*}*, with conclusion* $\sharp_{X^*}$*) does not reduce to*



(correct proof structure) (non-correct proof structure)

Figure 6: a cut reduction between "pseudo orthogonal" links

a correct proof net: actually after the reduction of the cut link (X, X^*)*, the reductum* π' *will contain both a cycle and a pair of disconnected components. For simplicity reasons, we assumed in Figure 6 that* π_X *(resp.,* π_{X^*}*) is the proof net built by gluing two orthogonal generalized links (modules)* λ'_X *and* λ_X *(resp.* λ'_{X^*} *and* λ_{X^*}*) through their common border* $\{1, 2, 3\}$.

3.3. Sequentialization of decomposable connectives

The most natural way to compare the two syntaxes, the sequential and the graphical ones, is through their *sequentialzation*, i.e. a way to set a precise correspondence between sequential proofs and proof nets. Here we focus only on the comparison w.r.t. decomposable connectives (we refer to Section 5 the comparison w.r.t the undecomposable connectives studied in Section 3.4).

There is a strong link between sequential and parallel decomposable connectives as exemplified by the basic cases $\mathcal{O}_\otimes = \{\{(1,2)\}\}^\perp = \mathcal{P}_\otimes^\perp$ and $\mathcal{O}_⅋ = \{\{(1),(2)\}\}^\perp = \mathcal{P}_⅋^\perp$. Actually, the two syntaxes are orthogonal views of a same decomposable connective, as stated in Proposition 17 proved in [3].

Proposition 17 (**Danos-Regnier sequentialization**)**.** *Assume* F *is a decomposable connective s.t.* $\mathcal{P}_F$ *is its pre-type (in proof net syntax) and* $\mathcal{O}_F$ *is its set of right rules (the organizations set in the sequential syntax) then:*

- $\mathcal{O}_F \subseteq \mathcal{P}_F^\perp$ *(*de-sequentialization part*) and*
- $\mathcal{P}_F^\perp \subseteq \mathcal{O}_F$ *(*sequentialization part*).*

E.g., assume $\mathcal{P}_F = \{\{(A, C), (B)\}, \{(B, C), (A)\}\}$ and $\mathcal{O}_F = \{\{(A, B), (C)\}\}$ with $F = (A ⅋ B) \otimes C$, then $\mathcal{O}_F = \mathcal{P}_F^\perp$ as illustrated in Figure 7.

Next corollary follows by Proposition 17 and case 8 of Property 3.

Corollary 18 (**organizations sets are types**)**.** *The organizations set* $\mathcal{O}_F$ *of a decomposable formula* F *is a type.*

$$\dfrac{\dfrac{\vdash A, A^{\perp} \qquad \vdash B, B^{\perp}}{\vdash A, B, (A^{\perp} \otimes B^{\perp})} \otimes \qquad \vdash C, C^{\perp}}{\vdash (A ⅋ B) \otimes C, (A^{\perp} \otimes B^{\perp}), C^{\perp}} F \qquad \Leftrightarrow$$

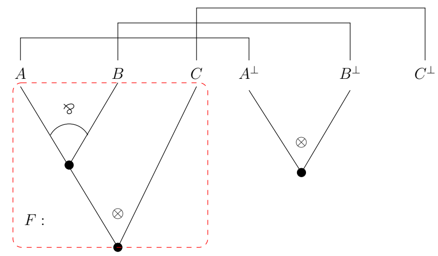


Figure 7: an instance of sequentialization

Corollary 19 (reflexive connectives). *A connective (P, Q) is* reflexive *when $P^{\perp} = Q$ and $Q^{\perp} = P$. All decomposable connectives are reflexive in the sequential syntax.*

Proof. Let (P, Q) be a decomposable connective in sequential syntax, then:
1) by Definitions 7 and 8, $P = \mathcal{O}_F$ and $Q = \mathcal{O}_{F^{\perp}}$ for some $F, F^{\perp}$ of MLL;
2) by Property 3, $(\mathcal{P}_F \perp \mathcal{P}_{F^{\perp}}) \Leftrightarrow (\mathcal{P}_F^{\perp\perp} \perp \mathcal{P}_{F^{\perp}}^{\perp\perp})$;
3) by Proposition 17, $(\mathcal{O}_F = \mathcal{P}_F^{\perp}) \wedge (\mathcal{O}_{F^{\perp}} = P_{F^{\perp}}^{\perp})$;
4) by 1, $\mathcal{O}_F \perp \mathcal{O}_{F^{\perp}}$ then, by 2 and 3, $\mathcal{O}_F^{\perp} \perp \mathcal{O}_{F^{\perp}}^{\perp}$;
Finally, by 4 and Property 3 (case 7), we get $(\mathcal{O}_F = \mathcal{O}_{F^{\perp}}^{\perp}) \wedge (\mathcal{O}_{F^{\perp}} = \mathcal{O}_F^{\perp})$ since by 3, $(\mathcal{O}_F = \mathcal{O}_F^{\perp\perp}) \wedge (\mathcal{O}_{F^{\perp}} = \mathcal{O}_{F^{\perp}}^{\perp\perp})$. □

By reflexivity, it is enough to indicate only one of the two organizations sets of a sequential connective, $\mathcal{O}_C$ or $\mathcal{O}_{C^*}$, while for a graphical connective we have to indicate a pair of pre-types.

Example 20. *We may now interpret the two pairs of orthogonal sets of partitions, (P_1, Q_1) and (P_2, Q_2) of Example 2, as two decomposable connectives, according to the two dual points of view seen above: the sequential one as in Figure 8 and the graphical one as in Figure 9 where the decomposable graphical connectives are displayed as binary trees enclosed in dotted boxes:*

$$P_1 = \{\{(1,2),(3,4)\}\} \quad \textit{and} \quad Q_1 = \{\{(1,3),(2),(4)\},\{(1,4),(2),(3)\},\{(2,3),(1),(4)\},\{(2,4),(1),(3)\}\}$$
$$P_2 = \{\{(2,3),(4,1)\}\} \quad \textit{and} \quad Q_2 = \{\{(2,4),(3),(1)\},\{(2,1),(3),(4)\},\{(3,4),(2),(1)\},\{(3,1),(2),(4)\}\}.$$

3.4. Undecomposable connectives

Not all generalized connectives are binary decomposable; e.g. the sequential connective $\mathcal{O}_X = \{\{(1,2),(3)\}\}$ and $\mathcal{O}_{X^*} = \{\{(1,3),(2)\}\}$ seen in

$$\frac{(1,2)\qquad(3,4)}{(1\parr 2)\otimes(3\parr 4)}\,p_1 \qquad \frac{(1,3)\quad(2)\quad(4)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^1 \quad \frac{(1,4)\quad(2)\quad(3)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^2$$

$$\frac{(2,3)\quad(1)\quad(4)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^3 \quad \frac{(2,4)\quad(1)\quad(3)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^4$$

$$\frac{(2,3)\qquad(4,1)}{(2\parr 3)\otimes(4\parr 1)}\,p_2 \qquad \frac{(2,4)\quad(3)\quad(1)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^1 \quad \frac{(2,1)\quad(3)\quad(4)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^2$$

$$\frac{(3,4)\quad(2)\quad(1)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^3 \quad \frac{(3,1)\quad(2)\quad(4)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^4$$

Figure 8: generalized decomposable connectives in sequent calculus syntax

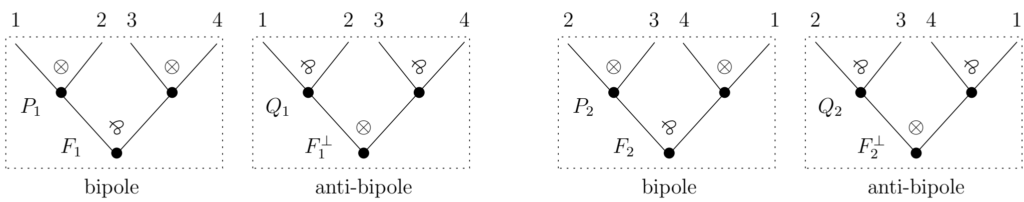


Figure 9: generalized decomposable connectives in proof net syntax

Remark 16 is undecomposable w.r.t. the basic connectives $\parr$ and $\otimes$ (it is neither a connective in the graphical syntax, indeed). Anyway, if we complete $\mathcal{O}_{X^*}$, by adding the extra partition $\{(2,3),(1)\}$, we get the decomposable connective $\mathcal{O}_D$, $\mathcal{O}_{D^*}$ seen in Remark 10. Similarly, $\mathcal{O}_Z = \{\{(1,2),(3),(4)\}\}$ and $\mathcal{O}_{Z^*} = \{\{(1,3,4),(2)\},\{(2,3,4),\ (1)\}\}$, is not a decomposable connectives but, as soon as we complete $\mathcal{O}_{Z^*}$ by adding a couple of extra partitions, $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$, we get the decomposable connective $\mathcal{O}_Y$ and $\mathcal{O}_{Y^*}$ of Example 9. A more serious example of undecomposable connective is the famous *Girard's connective* $(G_4, G_4^\perp)$ below:

$$G_4 = \{\{(1,2),(3,4)\},\{(2,3),(4,1)\}\} \qquad G_4^\perp = \{\{(1,3),(2),(4)\},\{(2,4),(1),(3)\}\}$$

along with some following instances[9] (we omit the orthogonal):

[9]We use index "$i \geq 4$" to indicate an undecomposable connective G_i, where "i" indicates the size (cardinality) of the connective support; since there may be multiple undecomposable

$$G'_6 = \{ \ \{(1,2),(3,4),(5,6)\}, \ \{(2,3),(4,5),(6,1)\} \ \} \qquad G''_6 = \{ \ \{(1,2,3),(4,5,6)\}, \ \{(2,3,4),(5,6,1)\}, \ \{(3,4,5),(6,1,2)\} \ \}$$

$$G_9 = \{ \ \{(1,2,3),(4,5,6),(7,8,9)\}, \ \{(2,3,4),(5,6,7),(8,9,1)\}, \ \{(3,4,5),(6,7,8),(9,1,2)\} \ \}.$$

As we will see in Section 4.3, the pregnancy of Girard's connectives is given by the fact that, unlike the two previous ones, (X, X^*) and (Z, Z^*), there is no chance to complete them in such a way they become decomposable connectives neither in the sequential nor in the graphical setting. That is consequence of the fact that connectives like $(G_4, G_4^\perp)$ are described by pairs of types. Moreover, sequents like $\vdash G_4, G_4^\perp$ are not provable *tout court* in MLL^+ (if we exclude the trivial non atomic logical axioms) because of the "packaging problem" seen in Remark 10. Finally, since G_4 (resp., $G_4^\perp$) is not decomposable and since there is no chance to embed it into a decomposable connective[10], sequent $\vdash G_4, G_4^\perp$ is not even stepwise derivable.

In the following we study a "minimal" class of undecomposable connectives.

4. Entangled connectives

In this section we introduce a special class of partitions sets, the class of *entangled pairs of partitions sets* (Definition 21) which is then exploited to define the class of *entangled connectives* (Definition 35). Entangled pairs have interesting properties that are investigated throughout this Section 4: the first one (Theorem 34), being that every entangled pair of partitions sets is a type. This result has several consequences: decomposable entangled pairs (and so, decomposable connectives) have a very simple characterization (the *normal form* of Theorem 39) which can be further used to prove that given an undecomposable entangled connective, P and Q, where P is an entangled pair, P cannot be embedded into a decomposable type $T \supsetneq P$, s.t. T and Q become decomposable (Corollary 43).

4.1. Entangled types

Definition 21 (entangled pair of partitions). *Let P be a pair of nonempty distinct partitions, p_1 and p_2, over the same (finite) domain $X = \{1, ..., n\}$. P is an* entangled pair *if it satisfies the following two conditions:*

1. *p_1 and p_2 have same weight (that is, P is weighable);*
2. *each class belonging to p_1 or p_2 has degree 1 or 2.*

---

connectives with same support we use super-indexes to distinguish them as e.g., $G'_6, G''_6, \ldots$.

[10] This fact will formally be stated later in Corollary 43 of Section 4.4.

Definition 22 (**restriction of entangled pairs**). *Given a weighable set of partitions P on a finite domain X and given an element $a \in X$, the* **restriction of** P **by point** a *(also* a**-restriction of** P*), denoted $P^{(\downarrow a)}$, is the set of partitions obtained after erasing a from each partition of P. A restriction of P by a is* **conservative** *when P and $P^{(\downarrow a)}$ have the same weight.*

Given a restriction of an entangled pair P by a point a s.t. the two partitions are not identified by this restriction, we say that the restriction of P by a is **admissible** *when $P^{(\downarrow a)}$ is in turn an entangled pair; hence, an entangled pair P is* **restrictable** *if there exists an admissible restriction of P by a point. We say that an entangled pair P is* **restrictable only in one point** *(also, P* **admits only one restriction by a single point***) iff $\exists a$ s.t. $P^{(\downarrow a)}$ is an entangled pair and $\neg \exists b \neq a$ s.t. $P^{(\downarrow b)}$ is an entangled pair.*

Restrictions can be extended to subsets of points of the support X of a weighable partitions set P, i.e., $P^{(\downarrow Y)}$ where $Y \subseteq X$.

Example 23. *Both P_1 and $P_1^{\perp}$ of Example 5 are entangled pairs while neither P_2 nor $P_2^{\perp}$ is so. Sets P_3, P_4 and P'_4 below are not restrictable entangled pairs; P_5 and P_6 below are both conservative restrictable entangled pairs: in particular, P_5 admits two restrictions, one by point c and another one by point d, while P_6 admits only one restriction, that one by single point a:*

$$P_3 = \{\ \{(a,b),(c)\},\ \{(a,c),(b)\}\ \} \qquad P_5 = \left\{ \begin{array}{l} \{\{(a,c),(b),(d,e),(f)\}, \\ \{(b,c),(a),(d,f),(e)\}\} \end{array} \right\}$$

$$P_4 = \{\ \{(a,b),(c),(d)\},\ \{(c,d),(a),(b)\}\ \} \qquad P_6 = \left\{ \begin{array}{l} \{(a,b),(c),(d,e),(f),(g)\}, \\ \{(a,c),(b),(f,g),(d),(e)\} \end{array} \right\}$$

$$P'_4 = \left\{ \begin{array}{l} \{(a,b),(c,d),(e),(f),(g),(h)\}, \\ \{(e,f),(g,h),(a),(b),(c),(d)\} \end{array} \right\}.$$

Lemma 24 (**forms of non restrictable entangled pairs**). *If P is a non restrictable entangled pair then, P is either in form (1) or in form (2), with $i = 1, ..., n \geq 4$, over a disjoint sum support $X = \{a_1, ..., a_{2n}\} \uplus \{b_1, ..., b_{2n}\}$.*

$$\left\{ \begin{array}{c} p_1 : \{(a,b),(c)\}, \\ p_2 : \{(a,c),(b)\} \end{array} \right\} \tag{1}$$

$$\left\{ \begin{array}{l} p_1 : \{(a_1,a_2),...,\ (a_{2i-1},a_{2i}),...,\ (a_{2n-1},a_{2n}),\ (b_1),\ (b_2),...,\ (b_{2i-1}),\ (b_{2i}),...,\ (b_{2n-1}),\ (b_{2n})\} \\ p_2 : \{(b_1,b_2),...,\ (b_{2i-1},b_{2i}),...,\ (b_{2n-1},b_{2n}),\ (a_1),\ (a_2),...,\ (a_{2i-1}),\ (a_{2i}),...,\ (a_{2n-1}),\ (a_{2n})\} \end{array} \right\} \tag{2}$$

Proof. Since $P = \{p_1, p_2\}$ is an entangled pair, every element of the support, $a \in X$, occurs in each $p_{i=1,2}$ either as singleton class (a) or in a pair class

(a, b) for some $b \in X$ with $a \neq b$. Moreover, since P is not restrictable, point a can occur:

1. neither as singleton class (a) in both p_1 and in p_2,
2. nor in a pair class in both p_1 and p_2 that is, it cannot be the case that $(a, b) \in p_1$ and $(a, c) \in p_2$ for some $a, b, c \in X$, except when P is in form (1).

That is to say, if P is a non restrictable pair then either P is in form (1) or $\forall a \in X$ only one of the following two cases holds:

(i) in case the singleton class (a) occurs in p_1 (resp., occurs in p_2) then a occurs in a pair class (a, b) of p_2 (resp., in a pair class (a, b) of p_1);

(ii) in case a occurs in a pair class (a, b) of p_1 (resp., of p_2) then a occurs as singleton class (a) in p_2 (resp., in p_1);

otherwise the erasing of point a in the non restrictable pair P would produce two restricted sets, $p_1^{(\downarrow a)}$ and $p_2^{(\downarrow a)}$, that result to be either equal ($p_1^{(\downarrow a)} = p_2^{(\downarrow a)}$), in case P is in form (1), or with distinct weights, $w_1 \neq w_2$, contradicting Definition 21. This means that P appears as a generalization of P_4 or P_4' of Example 23, that is, the support X of P has dimension $4n$ and it can be thought as the disjoint sum, $\{a_1, ..., a_{2n}\} \uplus \{b_1, ..., b_{2n}\}$, of two segments of points, $S_1 = \{a_1, ..., a_{2n}\}$ and $S_2 = \{b_1, ..., b_{2n}\}$ s.t.:

- S_1 (resp., S_2) contains exactly all points occurring both inside pair classes in p_1 and inside singleton classes in p_2;

- S_2 (resp., S_1) contains exactly all points occurring both inside singleton classes in p_1 and inside pair classes in p_2.

Thus, P has form (2), that is, for $i = 1, ..., n$, point a_{2i-1} (resp., b_{2i-1}) occurs together with point a_{2i} (resp., b_{2i}) in a class of p_1 (resp., of p_2) if and only if a_{2i-1} and a_{2i} (resp., b_{2i-1} and b_{2i}) occur as singleton classes, (a_{2i-1}) and (a_{2i}) (resp., (b_{2i-1}) and (b_{2i})), in p_2 (resp., in p_1). □

Lemma 25 (orthogonality of non restrictable entangled pairs). *If P is a non restrictable entangled pair then, its orthogonal $P^\perp$ is not empty.*

Proof. By Lemma 24, any non restrictable entangled pair is either in form (1) or in form (2). If P is in form (1) then, $P^\perp = \{\{(a), (b, c)\}\}$ by calculation. Otherwise, when P is in form (2), we first notice that P has weight $3n$. We then build a partition $q = \{y_1, ..., y_m\}$ over $X = \{a_1, ..., a_{2n}\} \uplus \{b_1, ..., b_{2n}\}$, with weight $m = n + 1$, as displayed both on the top and on the bottom side of Figure 10 and we finally show that q belongs to $P^{\perp}$. Partition q is built as follows:

- first elect a class, let us say e.g. y_1, containing all the elements of the domain X s.t. they do not form pairwise a class either in p_1 or in p_2, that is $y_1 = \{a_{2i-1}, b_{2i-1}\}_{i=1,...,n}$; so, y_1 has degree $2n - 2$;
- then, chose each remaining class y_j, with $2 \leq j \leq m$, in such a way that it consists of only two elements: one belonging to a class of p_i with two elements and the other one belonging to a class of p_i with a single element, as in the top (resp., in the bottom) side picture of Figure 10; that is, $y_j = (a_{2i}, b_{2i})$ with $i = 1, ..., n$; so, there are n such classes.

By calculation, partition q is orthogonal to each $p_{i=1,2}$ since by the Euler-Poincaré law, $|V| - |E| = 1$ where V and E are resp., vertexes and edges of each induced graph $\mathcal{G}_i(q, p_i)$ with $|V| = 3n + n + 1$ and $|E| = 4n$ as depicted in Figure 10; therefore $P^{\perp} \neq \emptyset$. $\square$

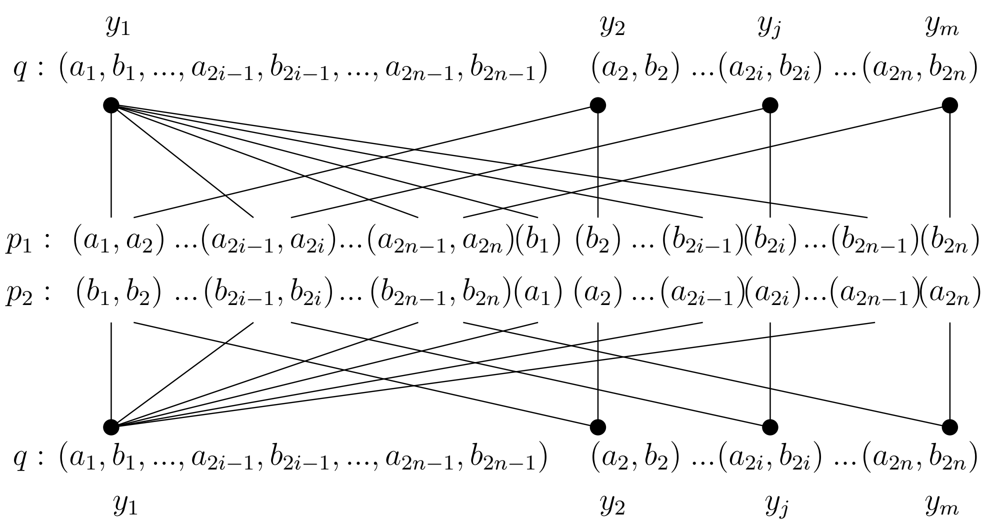


Figure 10: orthogonal of non restrictable entangled pair

Lemma 26 (restriction and orthogonality). *Assume p and q are two partitions over the same support X with an element $a \in X$ occurring in a* *singleton class of* p *(i.e.,* $p = \{x_1 = (a), x_2, ..., x_n\}$*) and in a class of* q *with degree* ≥ 2 *(i.e.,* $q = \{y_1 = (a, b, ...), y_2, ..., y_m\}$*) then,* $p \perp q$ *iff* $p^{(\downarrow a)} \perp q^{(\downarrow a)}$*.*

Proof. Consequence of the Euler-Poincaré Property 4: $\mathcal{G}(p^{(\downarrow a)}, q^{(\downarrow a)})$ is ACC iff $\mathcal{G}(p, q)$ is ACC. □

Proposition 27 (restrictable entangled pairs). *Let* $P = \{p_1, p_2\}$ *be an entangled pair that is restrictable by point* a*; assume the orthogonal of the* a*-restriction of* P*, i.e.* $(P^{(\downarrow a)})^\perp = \{q_1, ..., q_m\}$*, is nonempty then, the set of partitions* $P^\sharp$ *built according to the two cases below is orthogonal to* P*.*

1. *If* a *is singleton class in both* p_1 *and* p_2*, i.e.,* P *is in form (3)*

$$P = \left\{ \begin{array}{l} \{x_1 = (a), x_2, ..., x_n\}, \\ \{y_1 = (a), y_2, ..., y_n\} \end{array} \right\} \tag{3}$$

   then, $P^\sharp$ *is built as follows: for each partition* q_i*,* $1 \leq i \leq m$*, insert point* a *in a class* x_i^j *of* q_i*; repeat this insertion of point* a *for each class* x_i^j *in* q_i *(each insertion produces a "pumped" partition also denoted* $q_i^a \in P^\sharp$*).*
2. *Otherwise, if* a *belongs to a class with two elements, both in* p_1 *and* p_2*, i.e.* P *is in form (4) below*

$$P = \left\{ \begin{array}{l} p_1 : \{x_1 = (a, b), x_2, ..., x_n\}, \\ p_2 : \{y_1 = (a, c), y_2, ..., y_n\} \end{array} \right\}. \tag{4}$$

   then, $P^\sharp$ *is built as follows: insert (or "pump") the singleton class* (a) *in each partition* $q_i \in P^{(\downarrow a)\perp}$*; so* $P^\sharp$ *is as in form (5) below (each insertion produces a "pumped" partition also denoted* $q_i^a \in P^\sharp$*):*

$$P^\sharp = \{q_1 \cup \{(a)\}, ..., q_i \cup \{(a)\}, ...q_m \cup \{(a)\}\} \tag{5}$$

Proof. Consequence of Lemma 26. □

Lemma 28 (entangled pair of dimension 3). *If the dimension of an entangled pair* P *is* 3 *(i.e.* $|X| = 3$*) then* P *is in form (1) and its weight is* 2*.*

Proof. It follows by calculation: actually, since P is an entangled pair, it has weight $w \geq 2$ and in case the weight of P is $w = 2$ then, at least one class of P has degree 2; thus, if $|X| = 3$ then P is in form (1). □

Lemma 29 (orthogonality of entangled pairs). *The orthogonal* $P^\perp$ *of an entangled pair* P *is not empty.*

Proof. By induction on the dimension $|X|$ of the support of P.

The **base of induction** is when $|X| = 3$ then P is in form (1), by Lemma 28, so its weight w is 2. Thus, P is non restrictable and so, by Lemma 25, its orthogonal is not empty.

For the **induction step** $|X| > 3$ we split our reasoning in two main cases:

1. if P **is restrictable by point** a, it is in form (3) or (4), then the orthogonal of the restriction $P^{(\downarrow a)}$, let us say $(P^{(\downarrow a)})^{\perp} = \{q_1, ..., q_m\}$, is not empty by hypothesis of induction, so by Proposition 27, $P^{\sharp} \subseteq P^{\perp}$;
2. otherwise, if P **is not restrictable**, then we conclude by Lemma 25.

□

Theorem 30 (non restrictable entangled types). *If P is a non restrictable entangled pair (i.e., $\neg\exists a \in X$ s.t. $P^{(\downarrow a)}$ is an entangled pair) then P is a type.*

Proof. If $P = \{p_1, p_2\}$ is a non restrictable entangled pair then, by Lemma 24 either P has form (1) or P has form (2). In case P has form (1) then it is a type, simply by calculation. Otherwise, if P has form (2) then, we reason by absurdum, assuming that P is not a type, that is assuming there exists a partition $p_3 \neq p_{i=1,2}$ belonging to $P^{\perp\perp}$. We reason on the classes of p_3. Observe that p_3 cannot contain only singleton classes, otherwise p_3 will have weight $4n$ greater than weight $3n$ of p_1 and p_2, contradicting case 12 of Property 3 (by Lemma 29, $P^{\perp}$ is nonempty); thus p_3 contains at least a class with at least two points. Moreover, since p_3 differs from p_1 and p_2, it satisfies at least one of the following three cases:

1. p_3 contains a class x with two points, a' and a'' of S_1 (resp., b' and b'' of S_2) belonging to two distinct classes of p_1 (resp., p_2);
2. p_3 contains a class x with two points, a and b, s.t. $a \in S_1$ and $b \in S_2$;
3. p_3 contains two classes x_1 and x_2 s.t. x_1 belongs to p_1 and x_2 belongs to p_2 (resp., x_1 belongs to p_2 and x_2 belongs to p_1).

We show that in each of these three cases we get a contradiction. Indeed, since P has form (2) then, for each p_3 satisfying (at least) one of cases 1–3, we can always build a partition as $q = \{y_1, ..., y_m\}$ of Figure 10 (modulo possibly re-numbering) with weight $m = n+1$ and s.t. $q \in P^{\perp}$ (by Lemma 25, $P^{\perp} \neq \emptyset$) and $p_3 \not\perp q$ (the induced graph of incidence $\mathcal{G}(p_3, q)$ will not be ACC).

1. If p_3 satisfies case 1 then we can build a partition q as in Figure 10 with a class as e.g. $y_1 = (a_1, b_1, ..., a_{2i-1}, b_{2i-1}..., a_{2n-1}, b_{2n-1})$ containing the two points, $a', a'' \in \{a_1, ..., a_{2i-1}, ..., a_{2n-1}\}$ (resp., $b', b'' \in \{b_1, ..., b_{2i-1}, ..., b_{2n-1}\}$) with $i = 1, ..., n$, occurring resp., in two distinct classes of p_1 (resp., p_2). Thus $q \in P^\perp$ (by Lemma 25) and $p_3 \not\perp q$ since the induced graph of incidence $\mathcal{G}(p_3, q)$ contains a cycle.
2. If p_3 satisfies case 2 then we can build a partition q as the one in Figure 10 with some class $y_j = (a_{2i}, b_{2i})$ containing exactly the two points, a and b, with $a = a_{2i}$ and $b = b_{2i}$ for some $1 \leq i \leq n$. Thus $q \in P^\perp$ and $p_3 \not\perp q$ since the graph $\mathcal{G}(p_3, q)$ contains a cycle.
3. Finally, assume p_3 satisfies case 3 that is, p_3 contains two classes as e.g. (a_{2i-1}, a_{2i}) and (b_{2i-1}, b_{2i}) for some $i = 1, ..., n$; then we can build q as in Figure 10 with the two classes, $y_1 = (a_1, b_1, ..., a_{2i-1}, b_{2i-1}..., a_{2n-1}, b_{2n-1})$ and $y_j = (a_{2i}, b_{2i})$, for some $j = 2, ..., m$. Thus $q \in P^\perp$ and $p_3 \not\perp q$ since the induced graph of incidence $\mathcal{G}(p_3, q)$ contains a cycle. Observe that in case p_3 contains two classes as e.g. $(a_{2i-1}, a_{2i}) \in p_1$ and $(a_k) \in p_2$ (resp., $(b_{2i-1}, b_{2i}) \in p_2$ and $(b_k) \in p_1$), for some $i = 1, ..., n$ and $k = 1, ..., 2n$ then, by reasoning on the weight of p_3, partition p_3 satisfies one of the three cases already discussed.

□

Lemma 31 (restrictable entangled pairs). *Let $P = \{p_1, p_2\}$ be a restrictable entangled pair and let $p_3 \neq p_{i=1,2}$ be a partition in the type $P^{\perp\perp}$. If some a-restriction of P is a type (i.e., $P^{\downarrow a} = \{p_1^{(\downarrow a)}, p_2^{(\downarrow a)}\} = (P^{(\downarrow a)})^{\perp\perp}$ for some $a \in X$) then, the restriction $p_3^{(\downarrow a)}$ belongs to the restriction $P^{\downarrow a}$.*

Proof. Assume P admits a **non conservative restriction** by a point $a \in X$ that occurs as singleton class (a) both in p_1 and in p_2 so, P is in form (3), $P = \{p_1 = \{(a), x_2, ..., x_n\}, p_2 = \{(a), y_2, ..., y_n\}\}$, and let $p_3 \neq p_{i=1,2}$ be a partition belonging to the type $P^{\perp\perp}$. Then, point a also occurs in p_3 as singleton class (a), i.e. $p_3 = \{(a), z_2, ..., z_n\}$, otherwise we can find a partition $q_i^a \in P^\sharp \subseteq P^\perp$ (by Proposition 27 and Lemma 29) s.t. the induced graph $G(p_3, q_i^a)$ contains a cycle, where q_i^a denotes a partition of $\{q_1, ..., q_m\} = (P^{(\downarrow a)})^\perp$ in which point a has been "pumped" as in case (1) of Proposition 27. This implies $p_3^{(\downarrow a)} \perp q_i$, $\forall q_i \in (P^{(\downarrow a)})^\perp$, i.e., $\{p_3^{(\downarrow a)}\} \perp (P^{(\downarrow a)})^\perp$, since, by Lemma 26, $\mathcal{G}(p_3^{(\downarrow a)}, q_i)$ is ACC iff $\mathcal{G}(p_3, q_i^a)$ is ACC. Now, by assumption $P^{(\downarrow a)} = (P^{(\downarrow a)})^{\perp\perp}$ so, $\{p_3^{(\downarrow a)}\} \subseteq P^{(\downarrow a)}$, by case 1 of Property 3.

Otherwise, assume P admits only **conservative restrictions** by a point, i.e., assume P is a-restrictable as in form (4), $P = \{p_1 = \{(a,b), x_2, ..., x_n\}, p_2 = \{(a,c), y_2, ..., y_n\}\}$, for some a in X. Then, point a cannot occur in p_3 as singleton class (a), otherwise, as before, we can find a partition $q_i^a \in P^\sharp \subseteq P^\perp$ (by Proposition 27 and Lemma 29) s.t. graph $G(p_3, q_i^a)$ has a disconnected component, where q_i^a denotes a partition of $\{q_1, ..., q_m\} = (P^{(\downarrow a)})^\perp$ in which the singleton class (a) has been "pumped" as in case (2) of Proposition 27. So p_3 is in form $\{(a,d), z_2, ..., z_n\}$ and this means $p_3^{(\downarrow a)} \perp q_i$, $\forall q_i \in (P^{(\downarrow a)})^\perp$, i.e., $\{p_3^{(\downarrow a)}\} \perp (P^{(\downarrow a)})^\perp$, since, by Lemma 26, $\mathcal{G}(p_3^{(\downarrow a)}, q_i)$ is ACC iff $\mathcal{G}(p_3, q_i^a)$ is ACC. Now, since by assumption $P^{(\downarrow a)}$ is a type i.e., $P^{(\downarrow a)} = (P^{(\downarrow a)})^{\perp\perp}$, we conclude that $\{p_3^{(\downarrow a)}\} \subseteq P^{(\downarrow a)}$, by case 1 of Property 3. □

Proposition 32 (conservative single point restrictable pairs). *Assume $P = \{p_1, p_2\}$ is a* conservative single point restrictable entangled pair *that is, P is an entangled pair s.t. (see Definition 22):*

1. *it admits only one restriction by a single point a and*
2. *the restriction $P^{(\downarrow a)}$ is conservative.*

Then,

1. *the support X of P is given by the disjoint sum of the three segments*

$$S_0 : \{a, b, c\} \uplus S_1 : \{a_1, ..., a_{2n}\} \uplus S_2 : \{b_1, ..., b_{2n}\}$$

where S_1 and S_2 are defined as in Lemma 25; moreover

2. *P is obtained by merging the entangled pair $P_1 = \{p_1^1, p_2^1\}$ of form (1) together with the entangled pair $P_2 = \{p_1^2, p_2^2\}$ of form (2) that is, $P = \{p_1^1 \uplus p_1^2, p_2^1 \uplus p_2^2\}$ is in form (6) below:*

$$\left\{ \begin{array}{l} p_1 : \{(a,b),(c),(a_1,a_2),...,(a_{2i-1},a_{2i}),...,(a_{2n-1},a_{2n}),(b_1),(b_2),...,(b_{2i-1}),(b_{2i}),...,(b_{2n-1}),(b_{2n})\} \\ p_2 : \{(a,c),(b),(b_1,b_2),...,(b_{2i-1},b_{2i}),...,(b_{2n-1},b_{2n}),(a_1),(a_2),...,(a_{2i-1}),(a_{2i}),...,(a_{2n-1}),(a_{2n})\} \end{array} \right\} \quad (6)$$

where the restriction $P^{(\downarrow S_0)}$ is exactly that one of form (2).

Proof. Since the a-restriction is conservative, a occurs both in one class of p_1 together with a point b and in one class of p_2 together with a point c of the support that is, $(a,b) \in p_1$ and $(a,c) \in p_2$ for some $b, c \in X$; moreover, since P is only restrictable by the single point a, we have that:

1. $b \neq c$ and

2. b and c cannot occur together with other points of the support in some class of p_2 resp., p_1 otherwise P would be restrictable also in these two points, contradicting the assumption P is only restrictable by a.

This means that if $(a, b) \in p_1$ and $(a, c) \in p_2$ then $(b) \in p_2$ and $(c) \in p_1$. Now, since P is restrictable, p_1 and p_2 cannot consist of only these two pairs of classes that is, P cannot be reduced in form (1). Thus P is in form (7) below

$$P = \left\{ \begin{array}{l} p_1 = \{(a, b), (c)\} \uplus Q_1 \\ p_2 = \{(a, c), (b)\} \uplus Q_2 \end{array} \right\}. \tag{7}$$

where Q_1 and Q_2 are two distinct sets of partitions over the support $X \setminus S_0$ with $S_0 = \{a, b, c\}$. Moreover, since P is restrictable only by point a, Q_1 and Q_2 will be non restrictable pairs in form (2) over $S_1 \uplus S_2$; therefore, P is an entangled pair in form (6) with support $S_0 : \{a, b, c\} \uplus S_1 : \{a_1, ..., a_{2n}\} \uplus S_2 : \{b_1, ..., b_{2n}\}$. Observe that Q_1 and Q_2 cannot be a non restrictable pair in form (1) otherwise P of form (7) will be restrictable in exactly two points contradicting the assumption that P is only restrictable in a point. □

Lemma 33 (conservative single point restrictable entangled pairs). *Let $P = \{p_1, p_2\}$ be an entangled pair that admits only one conservative restriction by point a so, P is in form (4) with $b \neq c$ and $\{x_2, ..., x_n\}, \{y_2, ..., y_n\}$ is a non restrictable pair. If $p_3 \neq p_{i=1,2}$ is a partition in $P^{\perp\perp}$ then, p_3 contains either class $x_1 = (a, b)$ or class $y_1 = (a, c)$ of P in form (4).*

Proof. Assume P admits only one conservative restriction by point a then, by Proposition 32, P is in form (6). Now, consider the two partitions below, q_1 and q_2, belonging resp., to $P_1^{\perp}$ and $P_2^{\perp}$:

$$q_1 : \{z_1 = (a), z_2 = (b, c)\} \in P_1^{\perp}$$

$$q_2 : \{y_1 = (a_1, b_1 ..., a_{2i-1}, b_{2i-1} ..., a_{2n-1}, b_{2n-1}), y_2 = (a_2, b_2), ..., y_j = (a_{2i}, b_{2i}), ..., y_m = (a_{2n}, b_{2n})\} \in P_2^{\perp}$$

where $P_1^{\perp}$ and $P_2^{\perp}$ are the orthogonal of resp., P_1 in form (1) and P_2 in form (2), by Proposition 32, as displayed on top and bottom side of Figure 11.

Fix a bijection $\varphi : \{a, b, c, a_1, b_1, ..., a_{2i}, b_{2i}, ..., a_{2n}, b_{2n}\} \to \{1, ..., 4n+3\} \subset \mathcal{N}$ then, build a partition q of the form $q_1 q_2(i, j)$, for some $1 \leq i, j \leq (4n + 3)$, where the notation "$q_1 q_2(i, j)$" means that we concatenate q_1 and q_2 by mixing classes $z_h \in q_1$ and $y_k \in q_2$ containing, resp., points i and j. E.g., partition $q = \{z_1 \uplus y_1, z_2, y_2, ..., y_m\}$, displayed in the bottom side of Figure 12,

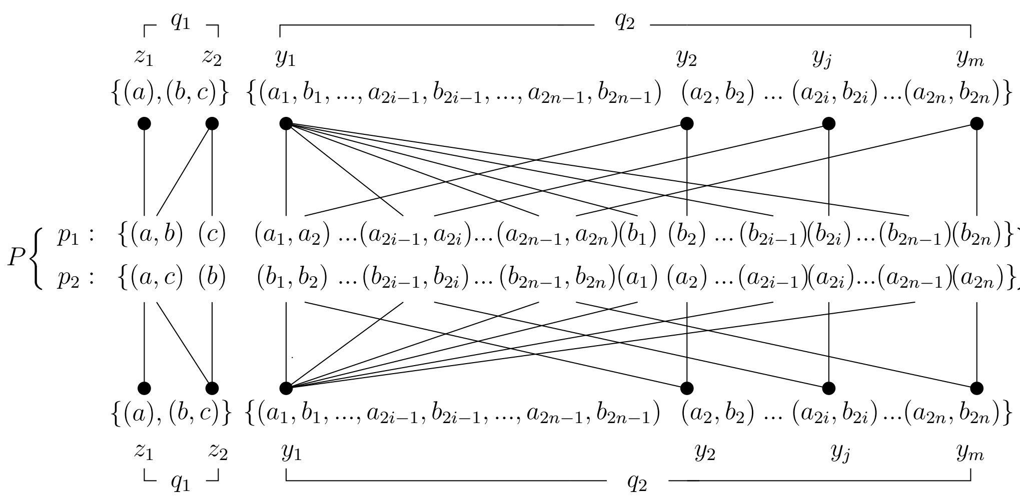


Figure 11: a conservative single point restrictable entangled pair

is obtained by concatenating q_1 and q_2 and mixing (by union) classes z_1 and y_1. By Property 4, $q \in P^{\perp}$, since each induced graph $\mathcal{G}(p_{i=1,2}, q)$ is ACC.

Assume $p_3 \neq p_{i=1,2}$ belongs to $P^{\perp\perp}$. By Proposition 27 (case 2), a cannot occur in p_3 as singleton class (a) so, assume by absurdum that a occurs in a class of $p_3 \neq p_{i=1,2}$ that is distinct from both classes (a,b) and (a,c) of form (4). Let us say a belongs to a class containing at least two point such as $(a,d,...)$ with $d \neq b$ and $d \neq c$. Then it is easy to build a partition like q or q', displayed resp., in the bottom and top sides of Figure 12, s.t. $q, q' \in P^{\perp}$ but the induced graph $\mathcal{G}(p_3, q)$ resp., $\mathcal{G}(p_3, q')$, contains a cycle (i.e., $q \not\perp p_3$ and $q' \not\perp p_3$). The remaining case, when $(a,b,c) \in p_3$, is also excluded because of partitions such as q or q' contain class $z_2 = (b,c)$ so, $p_3 \not\perp q$ and $p_3 \not\perp q'$. □

Theorem 34 **(entangled types).** *Every entangled pair of partitions sets $P = \{p_1, p_2\}$, with domain $X = \{1, ..., n\}$, is a type ($P = P^{\perp\perp}$) thus, entangled pairs are called* entangled types*.*

Proof. By **induction on the dimension** of P, i.e. on the cardinality of the domain $X = \{1, ..., n\}$ of P.

The **base of induction** is when $n = 3$ since there is no entangled pair of dimension ≤ 2; then, by Lemma 28, P is in form (1), so it is not restrictable; therefore P is a type, by Theorem 30.

For the **induction step**, assume P has dimension $n > 3$. If P is not

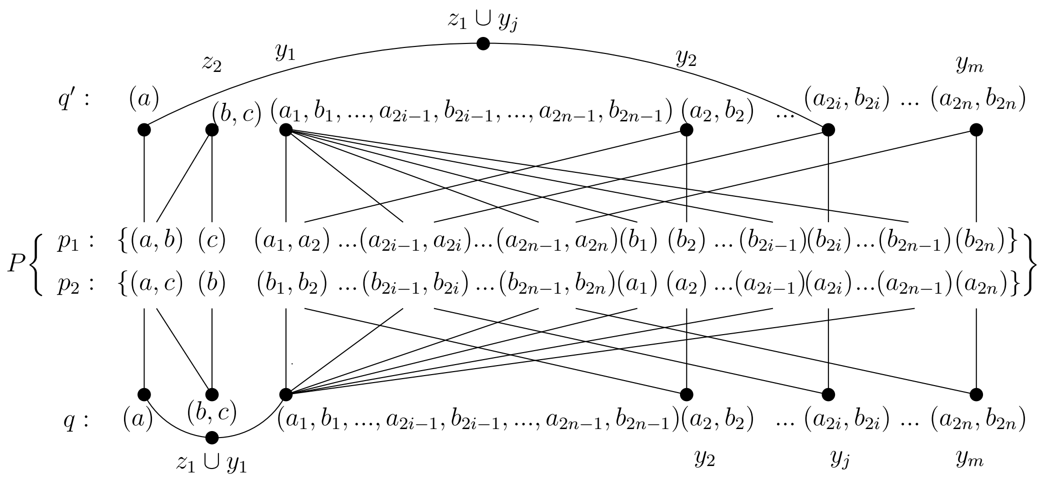


Figure 12: orthogonal of a conservative only single point restrictable entangled pair

restrictable then, by Theorem 30, P is a type. Otherwise, when P is restrictable, we split our reasoning in two main cases, depending on whether P admits or not non conservative restrictions.

Case 1. Assume P **admits a non conservative restriction** by a point, then P is in form (3) for some $a \in X$. Assume, by absurdum, there exists $p_3 \in P^{\perp\perp}$ s.t. $p_3 \neq p_{i=1,2}$. By Proposition 27 (case 1), a also occurs in p_3 as singleton class, i.e. $p_3 = \{(a), z_2, ..., z_n\}$ otherwise, we get a cycle in the induced graph $\mathcal{G}(q_i^a, p_3)$ for some pumped $q_i^a \in P^\perp$ (observe that in particular, by Proposition 27 (case 1), $\forall b \in X \backslash \{a\}$, $\exists q_i^a \in P^\sharp \subseteq P^\perp$ s.t. q_i^a contains a class with both a and b). By hypothesis of induction, $P^{(\downarrow a)}$ is a type so, by Lemma 31, either $p_3 = p_1$ or $p_3 = p_2$, contradicting the assumption.

Case 2. Assume P **admits only conservative restrictions** by a point so, P is in form (4) for some $a \in X$. Assume by absurdum there exists $p_3 \in P^{\perp\perp}$ s.t. $p_3 \neq p_{i=1,2}$. By Proposition 27 (case 2), a cannot occur as singleton class (a) in p_3 otherwise, we get a disconnection in the induced graph $\mathcal{G}(q_i^a, p_3)$ for some pumped $q_i^a \in P^\perp$ (notice there exists at least a pumped partition q_i^q containing a singleton class "(a)"); therefore the restriction of p_3 by a is conservative i.e., $p_3^{(\downarrow a)}$ and p_3 have the same weight. Now, by hypothesis of induction, the restriction $P^{(\downarrow a)}$ is a type thus, by Lemma 31, either $p_3^{(\downarrow a)} = p_1^{(\downarrow a)}$ or $p_3^{(\downarrow a)} = p_2^{(\downarrow a)}$; this implies the following constraint (8):

"a cannot belong to a class of p_3 *whose degree is greater than* 3*".* (8)

Then, we split our reasoning in two main sub-cases depending on whether b is or not equal c in P of form (4).

Case 2.1. If $b = c$ then, form (4) of P becomes as below:

$$P = \left\{ \begin{array}{l} p_1 : \{x_1 = (a, b), x_2, ..., x_n\}, \\ p_2 : \{y_1 = (a, b), y_2, ..., y_n\} \end{array} \right\}.$$

By Proposition 27 (case 2) a cannot occur as singleton class (a) in p_3. In case a does not occur together with b in a class of p_3 then, by Lemma 31 (by hypothesis of induction, $P^{(\downarrow a)}$ is a type), b will occur alone as singleton class (b) in p_3. Now, by Proposition 27 (case 2) there exists at least a partition q_i in $P^{\perp}$ containing the singleton class (b), so the induced graph $\mathcal{G}(p_3, q_i)$ contains a disconnected component, contradicting the assumption $p_3 \in P^{\perp\perp}$. Otherwise, in case a occurs together with b in a class of p_3 then, by Lemma 31 (by hypothesis of induction, $P^{(\downarrow a)}$ is a type), we get either $p_3 = p_1$ or $p_3 = p_2$, contradicting the assumption.

Case 2.2. In case $b \neq c$, w.r.t. form (4) of P, we distinguish two main sub-cases, depending on whether a belongs to a class of p_3 containing two or three elements of the domain, by constraint (8).

Case 2.2.1. Assume a **occurs in a class of** p_3 **containing exactly two elements** (including a) so, let us say that p_3 has the form below

$$p_3 = \{(a, d), z_2, ..., z_n\};$$

then, $d \neq b$ and $d \neq c$ otherwise we get either $p_3 = p_1$ or $p_3 = p_2$, by Lemma 31 (via the hypothesis of induction, $P^{(\downarrow a)}$ is a type), contradicting the assumption. This implies that P is restrictable by at least one more point than a otherwise, by Lemma 33, we get either $d = b$ or $d = c$. So, by Lemma 31, it does not matter whether d is or not an admissible restriction for P, we have either $d = b$ or $d = c$, contradicting the assumption $p_3 \neq p_{i=1,2}$.

Case 2.2.2. Assume that a **occurs in a class of** p_3 **containing exactly three elements** so, let us say p_3 is in form (9) below

$$p_3 = \{(a, d, e), z_2, ..., z_n\}. \tag{9}$$

By Lemma 33, P will be restrictable by at least one more point than a; therefore, w.r.t. all possible admissible restrictions of P, there are only three cases involving points d and e.

1. **Neither** d **nor** e **is an admissible restriction** of P. Assume P is restrictable by a point $f \neq a$. By hypothesis of induction the restriction $P^{(\downarrow f)}$ is a type, so by Lemma 31, either either $p_3^{(\downarrow f)} = p_1^{(\downarrow f)}$ or $p_3^{(\downarrow f)} = p_2^{(\downarrow f)}$, which is inconsistent, since $p_3^{(\downarrow f)}$ (notice p_3 has form (9)) contains a class with three elements, (a, d, e).
2. **Both** d **and** e **are admissible restrictions** of P. This means that, by Lemma 31 (via hypothesis of induction, $P^{(\downarrow a)}$ is a type and p_3 has form (9)) $d = b$ and $e = c$ (resp., $d = c$ and $e = b$); then, by playing with admissible restrictions, we get that p_1 (resp., p_2) will be an inconsistent partition since it will contain by assumption class (a, b) (resp., class (a, c)) together with class (b, c), this latter occurring as consequence, by Lemma 31, of the restriction of p_3 by point a, i.e.:

$$\text{either } p_1 = \{(a, b), (b, c), x_3 ..., x_n\} \in P$$
$$\text{or } p_2 = \{(a, c), (b, c), y_3 ..., y_n\} \in P.$$

3. **Only one between** d **and** e **is an admissible restriction** of P. Assume d is an admissible restriction of P (in addition to that one by a). Since by Lemma 31, via hypothesis of induction, the restriction of p_3 in form (9) by d is either equal to $p_1^{(\downarrow d)}$ or equal to $p_2^{(\downarrow d)}$), we conclude that either $e = b$ or $e = c$. Assume $e = c$ then, p_3 has form (10) below

$$p_3 = \{(a, d, c), z_2, ..., z_n\}. \tag{10}$$

Since class (a, c) belongs to partition p_2 and since P is not restrictable by c (by hypothesis), c will occur as singleton class (c) in p_1; so, form (4) of P will become as the one below:

$$P = \left\{ \begin{array}{l} p_1 : \{(a, b), (c), x_3, ..., x_n\}, \\ p_2 : \{(a, c), y_2,\ y_3, ..., y_n\} \end{array} \right\}.$$

By Lemma 31, via hypothesis of induction, the restriction of p_3 (in form (10)) by a will be either equal to $p_1^{(\downarrow a)}$ or equal to $p_2^{(\downarrow a)}$; this means, that class (d, c) of $p_3^{(\downarrow a)}$ will also occur either in p_1 or in p_2: in both cases we get an inconsistent partition since c will simultaneously occur in two classes belonging either to p_1 or to p_2, that is:

$$\text{either } p_1 = \{(a, b), (c), (d, c), x_4 ..., x_n\} \in P$$
$$\text{or } p_2 = \{(a, c), (d, c), y_3 ..., y_n\} \in P.$$

□

4.2. Entangled connectives

If P is an entangled pair then, by Lemma 25, its orthogonal $P^\perp$ is nonempty so, it makes sense to define the class of *entangled connectives* as follows.

Definition 35 (entangled connectives). *A generalized connective* (P, Q)*, in graphical or sequential syntax, is called* entangled *whenever* P *or* Q *is an entangled pair of partitions sets.*

Entangled connective enjoy the following Property 36, consequence of case 6 of Property 3 and Theorem 34.

Property 36 (entangled connectives). *If* P *is an entangled pair then,* (P, Q) *is an entangled generalized connective iff* $P = Q^\perp$*.*

Remarks 37. *If* $P = \{p_1, p_2\}$ *is an entangled pair over* $X = \{1, ..., n\}$ *then, each singleton* $\{p_i\}$ *can be interpreted as the pre-type of a special decomposable formula with border* $\{1, ..., n\}$*, called* **bipole** *[1]. A bipole* B *is a MLL formula with only two layers of connectives: a generalized* $\parr$ *of* $(F_{i-1}) \otimes F_i$ *sub-formulas (with* $1 < i \leq n$*) that is,* $B = \hat{\parr}(\hat{\otimes}_1, ..., \hat{\otimes}_m)$ *as in the l.h.s. picture of Figure 13. Dually,* $B^\perp = \hat{\otimes}(\hat{\parr}_1, ..., \hat{\parr}_m)$ *is called* anti-bipole*, that is a MLL formula with only two layers of connectives: a generalized* $\otimes$ *of* $(F_{i-1}) \parr F_i$ *sub-formulas (with* $1 < i \leq n$*) as in the r.h.s. picture of Figure 13. Every bipole satisfies the property that its pre-type is a type (see Lemma 4 of [9]). To be precise, observe that in case an entangled pair* P *is interpreted as the (pre-)type of a bipole* B *then some* $\otimes$*-subtrees of* B *may be unary.*

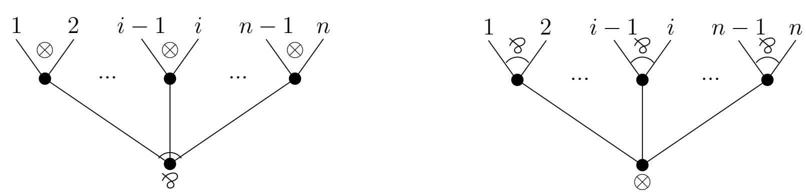


Figure 13: a *bipole* skeleton on l.h.s. and an *anti-bipole* skeleton on r.h.s.

*The "*smallest*" (w.r.t. the dimension of the domain) entangled decomposable connective in graphical syntax is* $\mathcal{P}_F = \{\{(1,3),(2)\},\{(2,3),(1)\}\}$ *and* $\mathcal{P}_{F^\perp} = \{\{(1,2),(3)\}\}$*, where* F *is the anti-bipole* $(1 \parr 2) \otimes 3$ *and* $F^\perp$ *is the bipole* $(1 \otimes 2) \parr 3$.

In the following we characterize, by means of Theorem 39 and Theorem 42, the *class of non binary decomposable entangled connectives*. We then show that these connectives can neither be approximated to decomposable connectives (Corollary 43) nor sequentialized in the MLL^+ sequent calculus (Section 5).

4.3. Decomposable normal form *of entangled connectives*

Definition 38 (decomposable normal form). *Let $P = \{p_1, p_2\}$ be an entangled pair of partitions sets then: P is decomposable iff it is the pre-type of the parse tree of a MLL formula; moreover, P is in* (decomposable) normal form *iff P is the pre-type of a formula tree F with literals indexes as in the l.h.s picture of Figure 14 that is, F is built by making a $\parr$ between:*

- *the smallest decomposable entangled connective* $\{\{(1,3),(2)\},\{(2,3),(1)\}\}$, *enclosed by the dashed line of Figure 14, and*
- *the possibly empty bipole with possibly unary classes,* $\{\{(4,5),...,(n-1,n)\}\}$, *enclosed by the dotted line of Figure 14;*

hence P in the following normal form *(11).*

$$P : \left\{ \begin{array}{l} p_1 = \{x_1 : (1,3), x_2 : (2), x_3 : (4,5), ..., x_{m\geq 2} : (n-1,n)\}, \\ p_2 = \{y_1 : (2,3), y_2 : (1), y_3 : (4,5), ..., y_{m\geq 2} : (n-1,n)\} \end{array} \right\}. \quad (11)$$

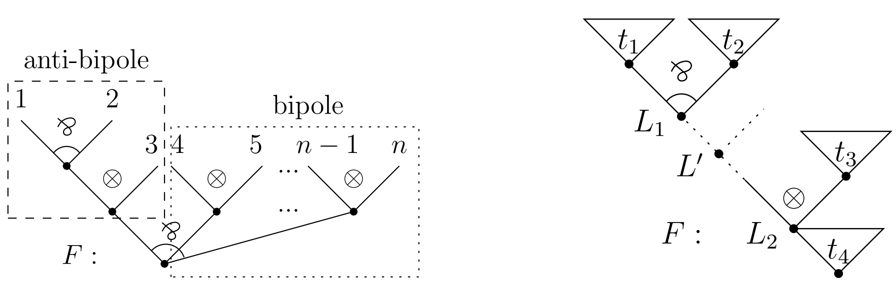


Figure 14: normal form of decomposable entangled connectives

E.g., $P = \{\{(a,b),(c,e),(d)\},\ \{(a,b),(d,e),(c)\}\}$ is an entangled pair in decomposable normal form since $P = \mathcal{P}_F$ with $F = (a \otimes b) \parr ((c \parr d) \otimes e)$ while the entangled pair $Q = \{\{(a,c),(b),(d,e),(f)\},\ \{(b,c),(a),(d,f),(e)\}\}$ is not in normal form (it is not even decomposable). Observe that normal form is defined up to commutativity of MLL.

Theorem 39 (decomposable normal form). *An entangled pair of partitions sets* $P = \{p_1, p_2\}$ *is decomposable iff* P *is in* normal form *(11).*

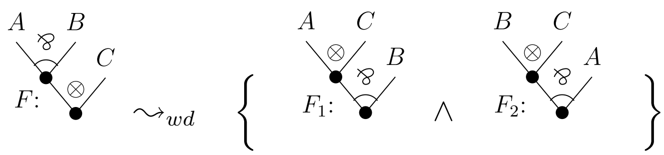
Proof. Assume P is a decomposable entangled pair of partitions, hence by Definition 13, there exists a MLL formula F s.t. P is the pre-type $\mathcal{P}_F$ of the syntactical tree of F. Since P is entangled (its size is 2 and its weight is ≥ 2), the syntactical tree of F contains at least a $⅋$-node otherwise, by reasoning on the switchings of F, the pre-type of F will consist of only one partition with a single class, contradicting the assumption P is entangled. For similar reasons F cannot consist of only $⅋$-nodes otherwise the (pre-)type of F will contain exactly one partition, contradicting the assumption P is entangled. So, F contains at least a $\otimes$-node and a $⅋$-node. Moreover, there will be a $⅋$-node L_1 above a $\otimes$-node L_2 in the tree of F otherwise F will be a bipole and therefore its (pre-)type would consist of a single partition, contradicting the assumption P is entangled pair. Thus, assume F is as in the r.h.s. picture of Figure 14, with a $⅋$-node L_1 above a $\otimes$-node L_2 and with t_1, t_2, t_3 and t_4 sub-trees of the F formula tree. Now, observe that no other link can stay between L_1 and L_2, otherwise such a link, let us say, $L' =⅋$ (resp., $L' = \otimes$) would increase the size of P (resp., the maximal degree $1 \leq \delta \leq 2$ allowed for each class of p_i) contradicting the assumption P is an entangled pair. Then observe that every tree t_i, for $i = 1, 2, 3$, will consist of a single unary node (i.e. a node with a single incident premise), otherwise:

1. in case t_i is a tree whose root is a $\otimes$-node (with two premises), we will get a class with degree > 2, contradicting the assumption P is an entangled pair;
2. in case t_i is a tree whose root is a $⅋$-node (with two premises), by reasoning on the switchings of F (since there would be two $⅋$-nodes dominating a $\otimes$-node), we get the size of P will be greater than 2, contradicting the assumption P is entangled pair.

So, t_1, t_2 and 3 are trees consisting of single points; moreover t_4 (if any) is a bipole as that one enclosed by dotted line in the l.h.s. of Fig. 14 otherwise, by computing of all possible switchings of F, we get that the pre-type of F, $\mathcal{P}_F = P$, is not an entangled pair, contradicting the assumption. Thus, F appears as in the l.h.s. picture of Figure 14, with its pre-type consisting of partitions p_1 and p_2 as in normal form (11).

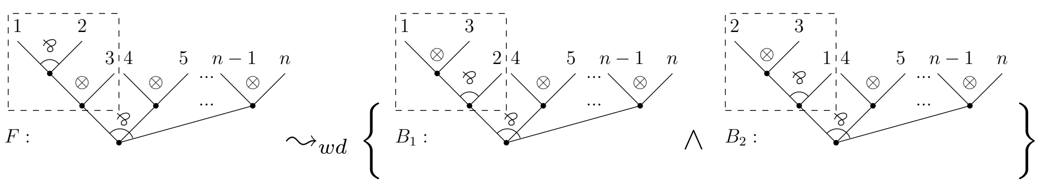
For the other way round, it is easy to calculate that an entangled pair in normal form (11) is decomposable. □

Remarks 40 (normal form and semi-distributive law). *Intuitively we can say that an entangled pair P in decomposable normal form can be thought as the set of the pre-types (types indeed) of the two bipoles (reductum), B_1 and B_2, that are obtained after one step of* weak *(or* linear *or* semi-*)* distributive law *[2] applied to the (redex) formula F (where $P = \mathcal{P}_F$) whose abstract skeleton tree is the one depicted on the l.h.s. of Figure 14. Actually, following [9], we may define a* rewrite relation *"$\rightsquigarrow_{wd}$" on MLL formulas trees generated by associativity and commutativity of $\otimes$ and $\parr$ plus the* **weak** *or* **semi-distributive law** *synthetically defined as follows:*



$$(A \parr B) \otimes C \vdash (A \otimes C) \parr B \quad \wedge \quad (A \parr B) \otimes C \vdash (B \otimes C) \parr A.$$

Then it is easy to calculate the equality $P = \mathcal{P}_F = \mathcal{P}_{B_1} \cup \mathcal{P}_{B_2}$ once F is as in the l.h.s. of Figure 14 and $F \rightsquigarrow_{wd} B_i$, for $i = 1, 2$, as below:



Alternatively to Theorem 34, in the particular case "an entangled pair P is decomposable", we can show that "P is a type*", via Theorem 41 stated below and proved in [9]. Actually, if P is a decomposable entangled pair then, by Theorem 39, P is the pre-type of the formula tree F in the l.h.s. of Figure 14 and so, by Theorem 41, P is a type. In other words, if P is a decomposable set of partitions then, by Theorem 41, both types $\mathcal{P}_F^{\perp\perp}$ and $\mathcal{P}_F^{\perp}$ (since, by Fact 14, $\mathcal{P}_F^{\perp} = \mathcal{T}_{F^\perp}$) can be computed by the exhaustive $\rightsquigarrow_{wd}^*$-rewriting of F and $F^\perp$. So, in some sense, Theorem 34 can be thought as a generalization of Theorem 41.*

Theorem 41 (Maieli-Puite: type generation of a MLL formula tree). *If F is a MLL formula then, its type $\mathcal{T}_F$ can be generated by the exhaustive $\rightsquigarrow_{wd}$-rewriting of F, as follows:*

$$\mathcal{T}_F = \bigcup_i \mathcal{P}_{F_i} \ s.t., F \rightsquigarrow_{wd}^* F_i$$

where each F_i is a bipole whose type $\mathcal{P}_{F_i}$ is obtained by transitive closure of $\rightsquigarrow^*_{wd}$ relation from F.

Proof. Given in [9]. $\square$

4.4. Undecomposable entangled connectives

By Theorem 39, an entangled connective, P and Q, is binary decomposable iff P or Q is in normal form. We now show that if P (resp., Q) is a non decomposable entangled type then neither Q will be decomposable (by Theorem 42), so *a fortiori* P and Q is an undecomposable connective.

Theorem 42 (undecomposable entangled connectives). *Let E and Q be a generalized connective. If E is an entangled pair that is not binary decomposable then, Q is neither decomposable.*

Proof. Let (E, Q) be a generalized connective with E an undecomposable entangled pair. Suppose, for the sake of absurdity, that Q is decomposable, i.e. $Q = \mathcal{P}_A$ for some formula A. By Property 36 and by Fact 14, we have

$$E =_{\text{(Property 36)}} Q^{\perp} =_{\text{(hyp. ab.)}} \mathcal{P}_A^{\perp} =_{\text{(Fact 14)}} \mathcal{T}_{A^{\perp}}$$

that is, E is the bi-orthogonal of $\mathcal{P}_{A^{\perp}}$. Therefore, $\mathcal{P}_{A^{\perp}}$ is included in E, by case 3 of Property 3. Now, since E has only two elements, if $\mathcal{P}_{A^{\perp}}$ were strictly included in E, it would be either the empty set or a singleton set so, its bi-orthogonal would not be E; therefore we have $E = \mathcal{P}_{A^{\perp}}$, a contradiction. $\square$

Next corollary is consequence of Property 36 and Theorem 42.

Corollary 43 (completion). *Let E and Q be an entangled connective with E being a non (binary) decomposable entangled pair; then, E cannot be completed in such a way to become decomposable (i.e., $\neg\exists D \supsetneq E$ s.t. D is binary decomposable with D and Q being a connective).*

Remarks 44 (entangled connectives and bipoles). *Although our notion of "entanglement" doesn't solve the (admittedly, difficult) general problem ("try to find a characterization of the full class of undecomposable connectives"), it is far from being and "ad hoc" condition. It is rather a natural condition that can be observed as soon as we "superpose" (sum) pairs of bipoles with the same "skeleton" (i.e.,bipoles with the same abstract syntactical tree), like e.g. the two ones, F_1 and F_2, enclosed by dotted lines in Figure 9 of Section 3.3. More* *precisely, an entangled connective, $P = \{p_1, p_2\}$ and $P^\perp$, can be interpreted as the union (resp., the intersection) of the types of two bipoles (resp., the types of two anti-bipoles) which are equivalent up to cyclic permutations of the literals indexes of the top border that is, bipoles (resp., anti-bipoles) s.t.:*

1. *they have the same syntactical tree skeleton;*
2. *they have the same border up to cyclic permutation of literals indexes.*

Formally, given two bipoles as F_1 and F_2 of Figure 9, with $\mathcal{T}_{F_1} = \{p_1\}$ and $\mathcal{T}_{F_2} = \{p_2\}$, then $P = \mathcal{T}_{F_1} \cup \mathcal{T}_{F_2}$ and $P^\perp = \mathcal{T}_{F_1^\perp} \cap \mathcal{T}_{F_2^\perp}$ is an entangled connective that is undecomposable, by Definition 38 and Theorem 39. E.g., consider the famous Girard's undecomposable connective, G_4 and $G_4^\perp$, which is an entangled connective (by Definition 35): G_4 (resp., $G_4^\perp$) results by the union (resp., by the intersection) of the types of bipoles $F_1 = (1 \otimes 2) \parr (3 \otimes 4)$ and $F_2 = (2 \otimes 3) \parr (4 \otimes 1)$[11] (resp., of the types of anti-bipoles $F_1^\perp$ and $F_2^\perp$) of Figure 9 i.e., $G_4 = \mathcal{T}_{F_1} \cup \mathcal{T}_{F_2}$, (resp., $G_4^\perp = \mathcal{T}_{F_1^\perp} \cap \mathcal{T}_{F_2^\perp}$). This fact (Theorem 34) is a novelty since the union of types is not in general a type while the intersection of types is always a type (Property 3). Indeed, entangled types are the smallest types (w.r.t. the number of partitions), if we exclude the trivial singleton types (every set consisting of a single partition is a type by case 11 of Property 3). So, entangled connectives can be considered in some sense the "smallest" generalized multiplicative connectives (w.r.t. the number of switchings or the number of rules), if we exclude the basic ones, $\parr$ and $\otimes$, and bipoles of course.

5. Sequentialization of undecomposable connectives

The natural correspondence (sequentialization, Proposition 17) between sequential decomposable connectives and graphical decomposable connectives is broken by non decomposable connectives! There exist proof nets, containing non decomposable links, without counterpart in the sequential calculus MLL^+, if we exclude the trivial correspondence with (non atomic) logical axioms. Let G and $G^\perp$ be an entangled non decomposable connective. By Theorem 42, neither G nor $G^\perp$ is binary decomposable. There is indeed no η-expanded proof of $\vdash G, G^\perp$ since each rule for G, resp., $G^\perp$ has at least two premises (see Section 3.4). Actually, sequents of non decomposable entangled

[11] Notice that the respective literals indexes of the top borders of F_1 and F_2 are cyclic permutations of the linear sequence $1 < 2 < 3 < 4$.

formulas $\vdash G, G^{\perp}$, are not provable from atomic logical axioms in the extended MLL sequent calculus (that's because of the "packaging problem" seen in Remark 10). This situation is quite different with proof nets. Consider e.g. the proof structure π of Figure 15 with only two conclusions, G and $G^{\perp}$, built as follows:

1. take the two non decomposable links λ_G and $\lambda_{G^{\perp}}$;
2. label the elements of the top border $X = \{1, ..., n\}$ of λ_G (resp., of $\lambda_{G^{\perp}}$) by n occurrences of literals, $a_1, ..., a_n$ (resp., $a_1^{\perp}, ..., a_n^{\perp}$);
3. put an axiom link for each matching pair of dual literals, a_i and $a_i^{\perp}$.

Clearly π is an η-expanded proof net: each global switching induces an acyclic and connected correction graph; nevertheless, π cannot be "sequentialized" in MLL^{+}, if we exclude the trivial (non atomic) axiom $\overline{\vdash G, G^{\perp}}$. Actually, any attempt of starting by sequentializing G (resp., $G^{\perp}$) will induce a single connected proof structure $\pi_{G^{\perp}}$ (resp., π_G), with $G^{\perp}$ (resp., G) among its conclusions, which is not correct: there exist two switchings for $\pi_{G^{\perp}}$ (resp., π_G) with at least two disconnected components (consequence of the entanglement conditions of Definition 21). Moreover, since neither G nor $G^{\perp}$ is binary definable (by Corollary 43, neither G nor $G^{\perp}$ can be be completed to a decomposable connective), π is not even stepwise sequentializable. This fact

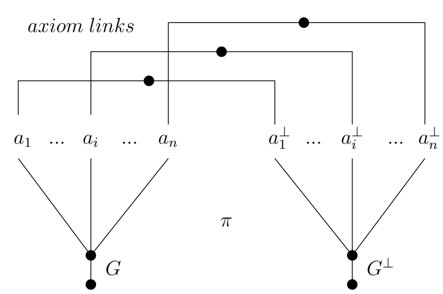


Figure 15: non sequentializable proof net with non decomposable conclusions $G, G^{\perp}$

witnesses an asymmetry between proof nets and sequent proofs since the former ones allow us to express a kind of parallelism of proofs that the latter ones cannot do.

6. Conclusions and future works

We gave the first characterization (via Theorems 34, 39 and 42) of a class of multiplicative undecomposable connectives: namely, the entangled connectives

(Definition 35) that are not in (decomposable) normal form (Definition 38). Anyway, there exist non decomposable connectives besides the entangled ones. We are currently working on a more general characterization of the full class of primitive non decomposable connectives (i.e., those ones non definable by means of other connectives). Naively our idea is that a non decomposable connective is rather defined by a set of partitions over the cyclic permutations of the linearly ordered support $1 < ... < n$; that's the case with e.g. connective G_9 below (we omit its orthogonal $G_9^{\perp}$ for the sake of simplicity):

$$\begin{aligned} G_9 = \{ \quad & \{(1,2,3),(4,5,6),(7,8,9)\}, \\ & \{(2,3,4),(5,6,7),(8,9,1)\}, \\ & \{(3,4,5),(6,7,8),(9,1,2)\} \quad \}. \end{aligned}$$

Concerning larger (than MLL) fragments of linear logic, like e.g. MELL or MALL, we don't know at this moment about non definable connectives for such fragments. Certainly, if we restrict to consider the pure additive fragment (ALL) then, there does not exist any generalized connective that cannot be decomposed by the basic additive ones, & and $\oplus$. There is no "packaging problem" in ALL since contexts in which we derive generalized formulas are simply duplicated. Roughly speaking, non decomposability concerns rather the multiplicative partition of contexts than the additive superposition (better, *slicing*) of contexts.

The fact that one can compute by means of cut elimination, using such non decomposable connectives, is certainly a good starting point: nevertheless, the study of their connection with *concurrency* (typically, the Pi-Calculus [10]) rather than the *Curry-Howard correspondence* should be further developed[12].

Finally, it would be also useful to have a *coherence semantics* for such non decomposable connectives. We could e.g. use the so-called *experiment method* of Girard [5] for the analysis of proof structures built on such non decomposable connectives. Examples of what can be done with these kind of semantic techniques can be found in [11] and more recently in [4].

[12] Naively, since, by the Curry-Howard isomorphism, "proofs are supposed to correspond to programs", *what should be the program* $[\pi]$ *corresponding to the non decomposable proof net* π *of* G, G^* *(we met in Section 5)?* if such a program does exist, is there any correspondence between the non-sequentializability of π and the possibly *concurrent nature* of such a *program* π^*? *Have we a chance to write "a sequential version" of such a "concurrent program"* $[\pi]$*?*

Acknowledgements. Thanks to Jean-Yves Girard, Michele Abrusci, Thomas Ehrhard, Paolo Pistone and the anonymous referee for their valuable feedback.

References

[1] Andreoli, J.-M.: Focussing and Proof Construction. *APAL* 107(1), pp. 131-163, 2001.

[2] Cockett, J.R.B. and Seely, R.A.G.: Weakly Distributive Categories. Journal of Pure and Applied Algebra, vol. 114, pp. 133-173, 1997.

[3] Danos, V. and Regnier, L.: The structure of multiplicatives. AML, 28, pp.181-203, 1989.

[4] Ehrhard, T.: A new correctness criterion for MLL proof nets. Proc. of the Twenty-Third Conf. CSL-LICS'14. Vienna, July 14-18, 2014.

[5] Girard, J.-Y.: Multiplicatives. Rendiconti del Seminario Matematico dell'Univ. di Torino, Special Issue on Logic and Computer Science, pp. 11-33, 1987.

[6] Girard, J.-Y.: Linear Logic, Theor. Comp. Sci., London Math. vol. 50(1), pp. 1-102, 1987.

[7] Girard, j.-Y.: On the meaning of logical rules I: syntax vs. semantics. Computational Logic, pp. 215-272, SV, Heidelberg, 1999.

[8] Girard, J.-Y.: Le point aveugle. In: Hermann (ed.), Vers la Perfection, Paris, vol. I, 2006

[9] Maieli, R. and Puite, Q.: Modularity of proof nets: generating the type of a module. Archive for Math. Logic, vol. 44(2), pp. 167-193, 2005.

[10] Milner, R.: Communicating and Mobile Systems: the Pi-Calculus, Cambridge University Press, 1999.

[11] Retoré, C.: A Semantic Characterisation of the Correctness of a Proof Net. Math. Struct. in Comp. Sci., vol. 7(5), pp. 445-452, 1997.