SN and CR for free-style LK^{tq}: linear decorations and simulation of normalization

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Abstract

The present report is a, somewhat lengthy, addendum to Danos et al.(1997), where the elimination of cuts from derivations in sequent calculus for classical logic was studied 'from the point of view of linear logic'. To that purpose a formulation of classical logic was used, that - as in linear logic - distinguishes between *multiplicative* and *additive* versions of the binary connectives.

The main novelty here is the observation that this *type-distinction* is not essential: we can allow classical sequent derivations to use *any* combination of additive and multiplicative introduction rules for each of the connectives, and still have strong normalization and confluence of tq-reductions.

We give a detailed description of the simulation of tq-reductions by means of reductions of the interpretation of any given classical proof as a proof net of **PN2** (the system of second order proof nets for linear logic), in which moreover all connectives can be taken to be of *one* type, e.g. multiplicative.

We finally observe that dynamically the different logical cuts, as determined by the four possible combinations of introduction rules, are *independent*: it is not possible to simulate them *internally*, i.e. by only *one* specific combination, and structural rules.

1 Introduction

We'll exploit in what follows once more a specific, somewhat idiosyncratic, reading of proofs in linear logic (**LL**). That is, we consider them as (annotations of) proofs in ordinary *classical* logic. Indeed, if in a derivation

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π in linear logic of a sequent $\Gamma \vdash \Delta$ (where Γ and Δ stand for *multisets* of linear formulas) one erases all occurrences of exponentials and replaces the linear logical connectives by their classical analogues ($\multimap$ becomes $\rightarrow$, $\otimes$ and & become $\wedge$, *und so weiter*) then all occurrences of linear formulas ϕ in π become occurrences of classical formulas ϕ' in what, modulo possible repetitions of sequents, is a valid proof $\mathsf{sk}(\pi)$ of the sequent $\Gamma' \vdash \Delta'$ in a classical sequent calculus (we speak of π's *(classical) skeleton*).
Some readers will frown and object that reading linear logic as its classical trace is a bit peculiar. And they might wonder whether this is not a bit... regressive.
But lo and behold, it is not!
Not once we have put to good use the 'forgotten' linear annotations in order to properly define a notion of *normalization* of proofs in classical logic![1]
The crucial twist springs from a converse to the fact that behind every linear derivation there lurks a skeleton: every classical derivation π can, in a canonical way, be considered as the classical skeleton of some linear derivation(s) $\partial(\pi)$, its so-called *decoration(s)*[2]: π equals $\mathsf{sk}(\partial(\pi))$. Now the *dynamics* (i.e. behaviour under normalization) of the linear derivation $\partial(\pi)$ can be 'pulled back' to the original derivation π. This inspired the formulation, in a recent paper by V. Danos and two of the authors (Danos et al.(1997)), of the so-called *tq-normalization* of (classical) second order sequent calculus. It is this notion of reduction of proofs, (to be reviewed in section 5 below) which will be object of the investigations in the first, their tool in the second part of the present paper.

2 Free-style sequent calculus

The skeleton of a derivation in linear logic is a proof in a classical sequent calculus having two sorts of logical introduction rules for the binary connectives: an *additive* one, that asks for equality of contexts, and a *multiplicative* one, demanding juxtaposition of contexts. The terminology was introduced by Girard(1987). As the reader will know, in linear logic the distinction between these two *types* of binary connectives is crucial.
In a classical (or intuitionistic) setting, however, it is rather unusual. Indeed, both

$$(\mathrm{L}\wedge)\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{R}\wedge)\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

and

[1] *Normalization*, as opposed to mere eliminability of cuts (cf. Kreisel(1971)).
[2] Introduced in Joinet(1993), Schellinx(1994), Danos et al.(1995)

$$(\mathrm{L}\wedge)\,\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{R}\wedge)\,\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}$$

form a set of introduction rules for classical conjunction, that, as the structural rules of weakening and contraction can be applied freely, is *complete* for classical provability. In Danos et al.(1997) and Joinet et al.(1996) these different sets of introduction rules for a given classical connective were taken to define distinct (formal) *types* of that connective, denoted by $\wedge_a$ (a for additive), resp. $\wedge_m$ (m for multiplicative). Because in the sets above the introduction rules at both sides of the turnstile correspond to the rule(s) for the same *linear* connective, we spoke of binary connectives that are *homo-style*. But also

$$(\mathrm{L}\wedge)\,\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{R}\wedge)\,\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}$$

and

$$(\mathrm{L}\wedge)\,\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\mathrm{R}\wedge)\,\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

are complete sets of introduction rules for classical conjunction. In Joinet et al.(1996) we called them *hetero-style*, and considered corresponding *hetero-style* binary connectives, denoted - in case of the implication - by $\stackrel{am}{\to}$ (characterized by the additive unary introduction rules and the multiplicative binary introduction rule) and $\stackrel{ma}{\to}$ (given by the multiplicative unary and the additive binary rule).

In view of the equivalence with respect to classical provability of these four formulations of the introduction rules for the binary connectives, authors presenting a sequent calculus for classical logic, apply Ockham's razor and *choose* one of them.

Here however we allow ourselves to indulge in redundancy. We keep *all* the introduction rules for each of the binary connectives. But unlike in Danos et al.(1997) and Joinet et al.(1996) we will no longer make the formal (i.e. in the language) distinction between the four types of the binary connectives as described above. Our basic propositional language will as usual consist in the binary connectives $\wedge, \vee$ and $\to$, without any 'style' specification. Hence our system of departure is as presented in Appendix 1. In view of the terminology introduced in Danos et al.(1997) and Joinet et al.(1996), we will speak of *free-style* **LK**.

(Thought the distinct formulations of the logical introduction rules are interderivable within classical sequent calculus, they will turn out to behave essentially different form a dynamical point of view: in section 16 we will see that the effects of e.g. a multiplicatively formulated introduction during cut elimination can, in the classical calculus, *not* be imitated by its 'additive simulation' (i.e. the interpretation of the multiplicative rule by means of the additive ones and structural rules).

3 Putting the t and the q in $\mathbf{LK}^{tq}$

Recall that there are two major sources of indeterminism in Gentzen's original 'cut-pushing' elimination procedure for classical sequent calculus: a *structural* one (should one permute upwards to the left or to the right?) and a *logical* one (related to the order of the two cuts obtained in performing a key-step). In order to be able to arrive at a *deterministic* procedure, Danos et al.(1997) introduced two more extensions, one of the language, the other of the calculus:

1. Each formula comes equipped with a mapping of the set of its subformulas into a 'colour space' $\{t, q\}$. (When necessary, we will make explicit the colour ϵ of the formula itself by means of a superscript: A^ϵ.) The rules now are supposed to preserve colours, i.e. colours should respect identity classes of formulas in a proof (cf. Danos et al.(1993)).

2. The multiplicative unary rules come in two types, prescribing whether in the 'key-step(s)' of cut-elimination in which they are involved, the cut on the left subformula comes before that on the right one, or conversely: we speak of the *orientation* of the unary multiplicative rules.

The system thus defined will be referred to as free-style $\mathbf{LK}^{tq}$.
Most of the versions of sequent calculus for classical logic known from the literature can be recovered as *fragments* of this free-style $\mathbf{LK}^{tq}$, by choosing *one* complete (for provability) set of introduction rules for each of the binary connectives and simply forgetting about colours and orientations.

We, finally, also may add the following rule to our system, the rule of *juxtaposition of proofs* (more commonly known as *mix*[3]):

$$\frac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_1 \vdash \Delta_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

It is a rule that arises quite naturally in the study of *linear logic* (see e.g. Fleury and Retoré(1994)). From the point of view of classical provability,

[3]Note that it is in fact the 0-ary instance of the rule of *multi-cut* which was called *mix* in Gentzen(1935) (and by several later authors as well).

it is highly superfluous: it can be simulated by means of weakening. However, the situation changes drastically when we consider *normalization* of proofs: it is easily seen that in general the reduction of a classical proof with juxtaposition can not be simulated by the reduction of a classical proof in which occurrences of juxtaposition have been replaced by occurrences of weakening. (While in the proof with *mix* both subderivations π_1 and π_2 of the premisses in one of its applications are available, a simulation will have to choose between one of the two: it will either contain π_1 followed by weakening with $\Gamma_2 \cup \Delta_2$, or π_2 followd by weakening with $\Gamma_1 \cup \Delta_1$. In both cases the (information contained in the) *other* subderivation will be irretrievable.) We include the rule because there are indications of its relevance in applications (cf. section 9 of Danos et al.(1997)). Moreover, in fact its presence does not change anything in the following arguments and proofs - we get it for free, it's a *bonus*.

In Danos et al.(1997), *strong normalization* and *confluence* for tq-reduction of homo-style $\mathbf{LK}^{tq}$ was proved. In Joinet et al.(1996) we extended this strong normalization result so as to include *hetero-style* derivations. Here we will prove SN and CR for tq-reductions of free-style $\mathbf{LK}^{tq}$-derivations, a result that properly contains the previous ones.

We assume that the reader has some acquaintance with linear logic, especially with *proof nets* and their reduction.

4 Some conventions and more terminology

The following conventions are used in distinguishing between the occurrences of formulas in a given logical rule, e.g. L$\rightarrow$:

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2}$$

The formula $A \rightarrow B$ is called the *main* formula of the rule with main connective $\rightarrow$; the occurrences A and B in the premisses will be referred to as the *active* formulas; all other occurrences are said to be *passive*. These are the *contextformulas* and together they constitute the *context*. In the special case of a cut, active formula occurrences are also referred to as *cutformulas*. In case of an identity axiom we say that *both* formula-occurrences are main. *We restrict the use of the term 'main' to logical rules and identity axioms.* In case of structural rules, if necessary, we speak of the *weakened*, respectively *contracted* formula.

In an occurrence in a proof π of a rule having as conclusion a sequent σ and sequent(s) σ' (and σ'') as premise(s), σ is called the *successor-sequent* of σ' (and σ'') in π, and σ' (and σ'') are *ancestor-sequents* of σ. Similarly we will speak of the successor in σ of a formula occurrence in σ', σ'' and the ancestor in σ', σ'' of a formula occurrence in σ. Using the terminology of Danos et al.(1993): formula occurrences A in σ and σ', σ'' are successor-ancestor related iff both occurrences are in the same *identity class*. E.g. in

$$\frac{A \vdash C \qquad A \vdash D}{A \vdash C \wedge D}$$

the occurrences of A in the premises are both *ancestors* of the occurrence of A in the conclusion, and the occurrence of A in the conclusion is a successor of each of the occurrences of A in the premises; the occurrence $C \wedge D$ does not have an ancestor.

A *block* (of sequents) in π is a sequence of sequents $\sigma_0, \sigma_1, \ldots, \sigma_n$ in π such that σ_{i+1} is a successor of σ_i. Note that a block of sequents unambiguously defines a block of (successive occurrences of) *rules* in π.

It is useful to think of a derivation in sequent calculus as a tree (the 'proof tree'). Each leaf (each *axiom*) uniquely defines a branch leading *downwards* to the tree's root (the derivation's *final sequent*, the conclusion). Along such a branch we can follow an occurrence A of a formula from its introduction (in an axiom, a logical rule or a weakening) to its submergence (when it is active in a logical rule) or disappearence (when it is active in a cut). See figure 1. 'Meanwhile', in between, in general many things will happen to *other* formula-occurrences (occurrences not in the identity class of A). We will call this 'meanwhile' the *main-active* or *m-a-interspace*[4] of the occurrence. If all that happens in between, are structural rules *not* involving our formula A, we say that the m-a-interspace is *lucid*. If *nothing* happens in between, we say that the m-a-interspace is *flat*: the formula is 'born', and immediately put to work.

Formally we define the m-a-interspace as follows.

Given an occurrence of a formula A in some sequent σ in a proof π, such that A has just been introduced (is *'newly born'* is either main, or has just been weakened), we inductively define a sequence of occurrences $A_0, \ldots, A_n$ of A, together with a block B_A of sequents $\sigma_0, \ldots, \sigma_n$, by

1. $\sigma_0 := \sigma, A_0 := A$;
2. if $\sigma_i \in \mathsf{B}_A$, A_i is not active, and σ_{i+1} is the successor of σ_i, then $\sigma_{i+1} \in \mathsf{B}_A$, and A_{i+1} is the successor of A_i in σ_{i+1};
3. that's all.

Hence B_A will contain σ, σ's successor, et cetera, down to either π's concluding sequent, or a sequent in which A is *active*. B_A of course is A's *main-active interspace*.

[4]Note the slight abuse of terminology: a 'main-active interspace' might actually be a '*weakening*-active interspace'.

With each *active* occurrence of a formula A, we can associate the finite set $\{B_{A_1}, \dots, B_{A_n}\}$ of m-a-interspaces, where each A_i corresponds to a leaf of A's tree of ancestors.

Whereas the m-a-interspace is obtained by looking *downward* along the prooftree, conversely, given an occurrence of a formula B in a sequent derivation π, we can look upward. There we see *the tree of B's ancestors in* π, i.e. the tree-like structure obtained by following *upwards* in π all formula occurrences in the identity class of our initial B, up to the introductions of B by axiom, logical or weakening rule (these form the ancestor-tree's leaves). See figure 1. The reader who wishes to do so, will easily provide the (long and boring) inductive definition.

5 Cut-elimination: the tq-protocol

An occurrence of a coloured formula on the left-hand side (resp. right-hand side) of the entailment-sign in a sequent is said to be *attractive* if its colour is t (resp. q): the terminology is introduced to remind us that the subproof of the sequent containing the non-attractive active cutformula 'has to move first'. We will often use the following alternative iconic notation for A^t (resp. A^q), namely $\overrightarrow{A}$ (resp. $\overleftarrow{A}$).

In each instance of a cut rule in an $\mathbf{LK}^{tq}$-proof, the cutformula will be coloured either t or q. Thus, using our iconic notation, cuts are of one of the two following forms:[5]

$$\frac{\vdash \overrightarrow{A} \qquad \overrightarrow{A} \vdash}{\vdash} \qquad \text{and} \qquad \frac{\vdash \overleftarrow{A} \qquad \overleftarrow{A} \vdash}{\vdash}$$

Let us call the subderivation containing the *attr*active occurrence of the cutformula the *attr*acting subderivation.

Definition 1 (tq-protocol) *Reduction according to tq-protocol proceeds via two possible types of steps, 'structural' ones, S1 and S2, and 'logical' ones, L ('key-steps'):*

— An L-step applies when both cutformulas are main in a logical rule. L-steps have to be specified for each one of the connectives and, in case of the binary ones, for each of the possible combinations of a left and a right introduction rule. (We obtain as descendants one or two cuts on the immediate subformula(s) of the cutformula. In case of two descendants, the order in which these cuts are applied is determined by the orientation of the unary rule.)

[5] In depicting proof figures, we will often indicate the context only when this is relevant to the argument.

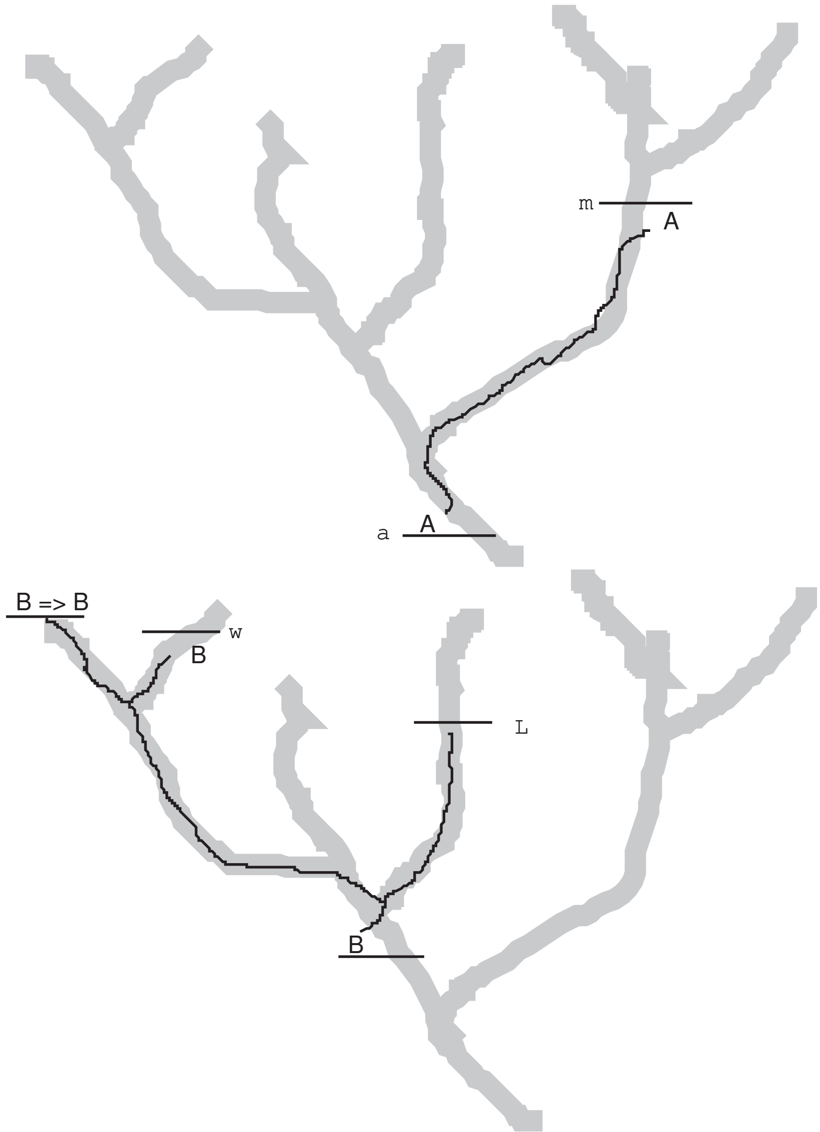

Figure 1: A proof tree π in which we traced (above) the m-a-interspace of a formula occurrence A and (below) the tree of ancestors of a formula occurrence B.

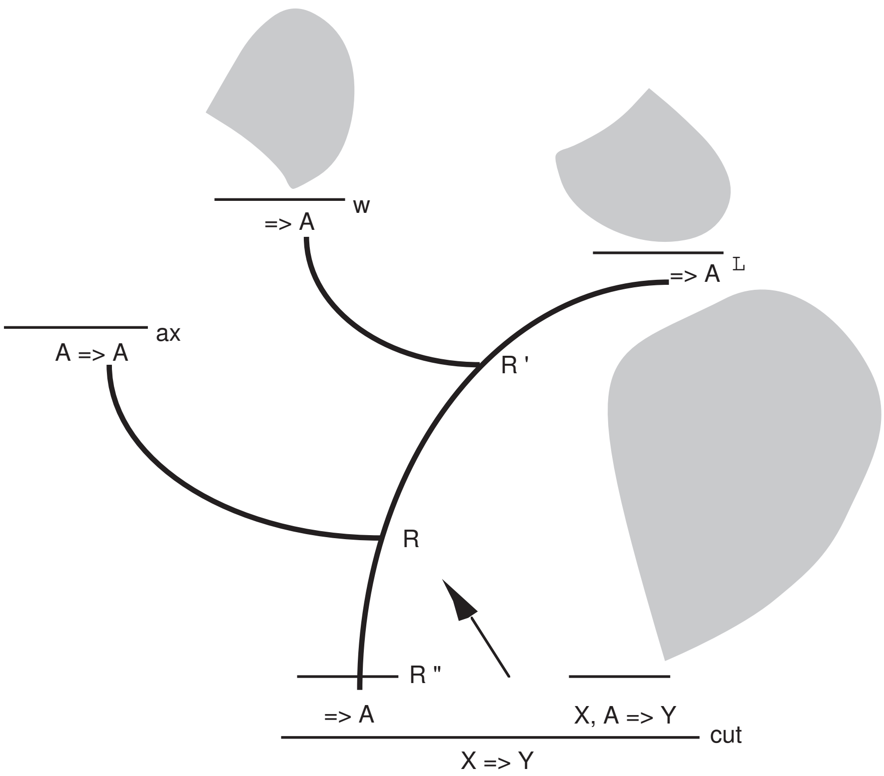

Figure 2: A structural cut, either of type S1 or of type S2, depending on whether the occurrence of the cut-formula in the left premiss of the cut is attractive or not. If it is, the reduction is of type S1. If it isn't, then the occurrence of the cut-formula in the right premiss of the cut is main in a logical rule, and the reduction is of type S2.

— *In case no L-step is applicable, necessarily an S-step applies, which consists in 'transporting' one of the cut's subderivations up the tree of the cutformula's ancestors in the other one, duplicating it and contracting the context whenever passing an instance of contraction (or via the context of a binary additive rule); this process ends when reaching instances of introduction in an axiom, in which case the resulting 'axiom-cuts' are reduced* immediately, *when reaching an introduction by weakening, which are replaced by weakenings of the context formulas, or when reaching instances of introduction of the main connective of the cutformula.*

Of course, now one needs to know which *of the two subderivations has to move. This is decided by asking whether or not the attractive cutformula is main* in a logical rule. *If the answer is* "yes!", *we transport the attracting subderivation (S2); if it is* "no!", *we transport the other one (S1).*
And that's it.

Note that neither the choice of orientations nor that of colours has to do with imposing a reduction *strategy*. We do not select redexes, but rather the way we reduce them.
Figures 2 and 3 show, schematically, the 'movement' involved in performing a structural reduction step.

The crucial difference between *tq*-reduction and the standard definitions of cut-elimination steps in sequent calculus is in the definition of the *structural steps*, where the *complete* tree of ancestors of one of the occurrences of the cutformula A is involved: we raise (a copy of) the transported subderivation right up to the leaves, disregarding all 'events' within the m-a-interspaces that make up the tree, to each of the spots where an ancestor of the cutformula was introduced. Then we push the contextformulas of the final sequent of the transported subderivation 'back down' along the tree. At branchings originally due to explicit contractions on A, the contractions now are inherited by these contextformulas. (Cf. the 'complete exponential reduction' of proof nets, as described e.g. in Regnier(1992).)
This procedure of course forms a radical departure from the usual 'one-up' reduction steps that are considered in most proofs of eliminability of cut.

6 Weak equivalence

A well-known, sometimes execrated, property of sequent calculus, is the high number of possible *variations* on a given derivation that is obtainable by simply permuting the order of applications of rules. An interesting and rewarding line of research is that into the question which sort of permutations induce an equivalence relation on the set of proofs of a given sequent that is *compatible* with our reduction, meaning that if π' is obtained by a number of certain permutations of applications of rules in π, than the *normal form* of π'

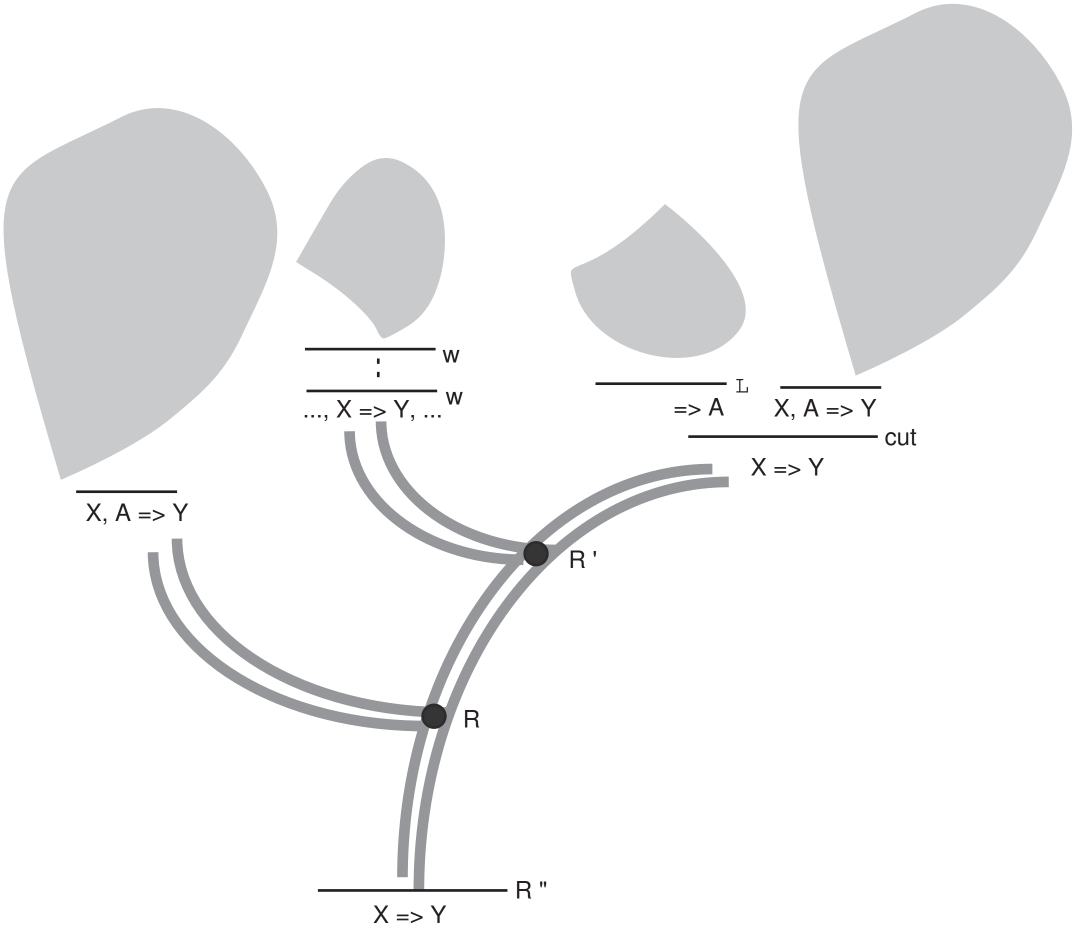

Figure 3: The result of performing the one-step structural *tq*-reduction induced by the structural cut in figure 2

can be obtained by a number of similar permutations from the *normal form* of π. Danos et al.(1997) consider two types of rule permutation-induced equivalence relations on $\mathbf{LK}^{tq}$-proofs: *weak* equivalence (induced by permutations of occurrences of structural rules) and *strong* equivalence (induced by weak equivalence and permutations of occurrences of reversible rules). Of these it is the first one, the *weak* equivalence, that will be of interest to us also in the present paper.
Two proofs are said to be *weakly equivalent* if they differ only up to contracted weakenings, re-arrangement of multiple contractions on the same formula (associativity, commutativity) and permutations of structural rules as, e.g. for a contraction and a binary multiplicative rule:

$$\dfrac{\dfrac{C, C \vdash A}{C \vdash A} \qquad B \vdash}{A \to B, C \vdash} \quad \overset{w}{\approx} \quad \dfrac{\dfrac{C, C \vdash A \qquad B \vdash}{A \to B, C, C \vdash}}{A \to B, C \vdash}$$

or a contraction and a binary additive rule:

$$\dfrac{\dfrac{C, C \vdash A}{C \vdash A} \qquad \dfrac{B, C, C \vdash}{B, C \vdash}}{A \to B, C \vdash} \quad \overset{w}{\approx} \quad \dfrac{\dfrac{C, C \vdash A \qquad B, C, C \vdash}{A \to B, C, C \vdash}}{A \to B, C \vdash}$$

The following was proved by Danos et al.(1997) for homo-style $\mathbf{LK}^{tq}$, but it is easily seen to hold also in the free-style case:

Proposition 2 (weak equivalence is tq-compatible) *If two $\mathbf{LK}^{tq}$-proofs are weakly equivalent, then so are their normal forms.* $\boxtimes$

Intuitively, proposition 2 tells us, among other things, that we may consider occurrences of structural rules in $\mathbf{LK}^{tq}$-derivations as being *not localized.* As a consequence, we may very well consider sequents $\Gamma \vdash \Delta$ in which Γ, Δ are not multi-sets but rather *sets* of formulas. Indeed, any $\mathbf{LK}^{tq}$-derivation π has 'set-like' weakly equivalent forms, namely those in which
(i) after each application of a logical or cut rule *all* possible contractions are performed (in some order);
(ii) occurrences of weakening are as high up in the proof as possible, i.e. all weakenings are done immediately following the occurrences of axioms in the proof tree.
The set-like equivalents of π provide a *canonical* localization of occurences of structural rules. They correspond to derivations in versions of sequent calculus that lack explicitly formulated structural rules. Instead sequents $\Gamma \vdash \Delta$ consist in sets of formulas (which takes care of *contraction*), and axioms are of the form $\Pi \vdash \Sigma$, with $\Pi \cap \Sigma \neq \emptyset$ (thus accounting for *weakening*).
Proposition 2 ensures that such calculi are innocent variations on the multiset calculus with the simple axioms and *explicit* structural rules.

Another version of the calculus without explicit structural rules is obtained by adding to the 'set-like' treatment of contraction the convention that the introduction of formula occurrences by *weakening* will be done *as low as possible*: just before the weakened occurrence is active in a logical or cut rule, or, if it is not, at the very end of the proof.
If we stick to these conventions of localization, we say that in our derivation the structural rules are treated *implicitly*, and we can consider (a multi set version of) $\mathbf{LK}^{tq}$ with *implicit* structural rules. Again tq-compatibility of weak equivalence ensures that this is an innocent variation of the calculus with *explicit* structural rules.

7 Stability

Let π' denote a one-step tq-reduct of π. Note that, trivially, *any* occurrence of a logical rule r in π' in which some occurrence of a formula, say A, is active, stems from a *unique* occurrence r_* of (the same) logical rule in π in which an occurrence A_* of (the same up to substitution) formula is active (we call r_*, resp. A_*, the *lift* of the occurrence r, resp. A). Conversely, we define the set of *residues* of a given occurrence s of a logical rule in π as the set of all occurrences r of logical rules in π' such that $r_* = s$.
Similarly, any occurrence of a cut c in π' with cutformula A stems from a unique occurrence c_* in π with cutformula A_*, except for those occurrences of cut that are created in an L-step (which are said to have *no* lift in π, and the reduced logical cut does *not* have a residue in π', cf. Danos et al.(1997)).
As a consequence, given an m-a-interspace $B_A = \sigma_0, \ldots, \sigma_n$ in π, such that σ_n's successor a is a logical or a cut rule, we can define the set of residues of B_A in π' as the set of all m-a-interspaces in π' associated with the corresponding active formulas in all residues of a in π'.
The following lemma, a weak version of a corresponding lemma in Danos et al.(1997), says that when B_A is *flat*, under certain conditions the same will hold for all of B_A's residues.

Lemma 3 (stability) *Let π be an $\mathbf{LK}^{tq}$-proof, π' a one-step tq-reduct of π, and B_A a flat m-a-interspace of some formula occurrence A. Suppose A is attractive or m is a logical rule or a weakening. Then all residues of B_A in π' are flat.*

proof: Consider the schematic form of such an m-a-interspace in π:

$$\dfrac{\dfrac{\Delta}{\Delta', A}\,\mathrm{m} \qquad \Sigma}{\Delta', \Sigma'}\,\mathrm{a}$$

where 'm' denotes the rule that introduces A, 'a' the rule where A is active.

Reducing a cut in π obviously can only affect B_A in case $\Delta' \cup \{A\}$ contains (an ancestor of) the cutformula. Suppose A itself is the cutformula (hence 'a' is a cut). If the reduction step is an L-step, then B_A has no residue in π', and our claim vacuously holds.
If π_1 ends in a weakening on A or is an axiom $A \vdash A$ and the cutformula is attractive, again there is no residue; similarly there is no residue in case π_1 ends in a weakening and the cut is of type S2. In all other cases π_1 will be the transported subderivation, independent of A's colour. For the leaves of A's tree of ancestors in π_2 corresponding to introductions in an axiom or a weakening, there are no residues of B_A; all residues will correspond to the introductions of ancestors of the cutformula in π_2 in a logical rule, and all of these are flat.
In case it is Δ' that contains an ancestor of the cutformula: either a subproof of π containing B_A will be transported, resulting in $m \geq 0$ flat residues; or the cut will 'just pass through' B_A, not affecting its flatness - unless π_1 is an identity-axiom, but then, by hypothesis, the occurrence of A that concerns us is attractive, therefore the cutformula is non-attractive. Hence the substitution will necessarily replace the axiom by a derivation having a last rule in which A is main. ⊠

Let us say that an m-a-interspace B_A is of type m (resp. w, ax) if m is a logical rule (resp. a weakening, an identity axiom). Then inspection of the proof shows that in fact:
- if B_A is flat of type w, all its residues are flat of type w;
- if B_A is flat of type m (corresponding to rule r), all its residues are flat of type m (corresponding to the same rule r);
- if A is attractive and B_A is flat of type ax, its residues are flat of type ax or of type m.
Ordering the three types of cut in the obvious way, by $S1 > S2 > L$, a simple corollary of the stability-lemma is that tq-reduction never increases a cut's type:

Corollary 4 *If c is a cut in π, and π tq-reduces to π', then (i) if c is a logical cut, all of c's residues in π' are logical cuts; (ii) if c is of type S2, none of c's residues in π' is of type S1.* ⊠

Stability is a crucial property of tq-reduction, with interesting applications (cf. Danos et al.(1997), Joinet et al.(1996)). In the present paper, however, we will invoke it (or rather, the corollary) only once, in the proof of the structural simulation proposition 6.

8 The method

We will show that tq-reductions starting from a given derivation in free-style $\mathbf{LK}^{tq}$ always are *finite* (the S(trong)N(ormalization)-property; one

also says: *"the reduction is* noetherian*"*) and always lead to the same cut-free derivation of the final sequent, i.e. there is a *unique normal form* (the C(hurch)R(osser)-property; one also says: *"the reduction is* confluent*"*).
As already mentioned in the very first paragraph, our main tool consists in the construction of structure preserving mappings from $\mathbf{LK}^{tq}$-derivations to **LL**. The first one of such mappings that we define, in section 13, maps a classical derivation into the *additive-free* fragment of the system of second order proof nets for linear logic ($\mathbf{PN}_2$). We will refer to this fragment as **naLL** (non-additive linear logic). In **naLL** all binary linear connectives used are of the multiplicative type.[6]
We show that this mapping is *homoreductive* - it is a homomorphism with respect to the image's reduction as a proof net. The idea is to then 'pull back' the properties enjoyed by the normalization of non-additive proof nets to the set of classical proofs by means of a *simulation* of the *tq*-reductions by reductions of proof nets.
The proofs of strong normalization and confluence for the non-additive fragment of $\mathbf{PN}_2$ are reasonably well documented: the quantifier-free fragment was introduced in Girard(1987); one should supplement the treatment there by that of quantifiers as in Girard(1991), and complete it with the 'striction lemma' from Danos(1990). A combinatorial proof of strong normalization for the quantifier free fragment can be found in Joinet(1993). Observe that in none of the arguments that make up the proof of the SN- and CR-property of reductions of **naLL**-nets, connectivity of the 'correctness graphs' associated to proof-structures is invoked. It therefore is immediate that strong normalization and Church-Rosser continue to hold for the calculus with *mix*-rule: the same proof works.)
This first mapping in fact suffices to establish SN and CR for free-style $\mathbf{LK}^{tq}$.

Still, we show in section 14 that one alternatively can define a *second* mapping, which maps a classical derivation into the *multiplicative-free* fragment of the system of second order for linear logic, where all linear connectives used are of the additive type.
Of course one then wonders whether it is possible to repeat the argument, and deduce SN and CR also via this second embedding.
This question is addressed in section 15.

Despite the fact that the results of this paper seems to suggest that the distinction in classical and intuitionistic proof theory between binary connectives of additive, multiplicative or mixed styles henceforth can be of only

[6]Note that an interpretation of additively formulated logical introduction rules in **naLL** can not be obtained by a decoration in a strict sense. Indeed, as we will see, *both* our mappings are 'decorations' only if the (multiplicative or additive) style of the original rule corresponds to that of the linear image; otherwise what we have is a 'decoration' of the (additive or multiplicative) *simulation* of the original (multiplicative or additive) rule. In what follows we will use the word 'decoration' in this extended sense.

anecdotical interest, we demonstrate in the final section 16, that nevertheless there is a certain pertinence in the 'style'-distinction as maintained in our earlier work: the specific ('multi')dynamics of the four different logical cuts that go with each binary connective can not be simulated by only *one* of them: dynamically the four (complete for provability) sets of introduction rules are *disjoint*.

9 Proof net reduction: a technical lemma

We extend the reduction of **naLL**-nets by adding the following *weakening/contraction ([cow])*-reduction step to the usual cut-elimination steps:

$$\dfrac{\overline{?A} \qquad \begin{array}{c}\vdots\\ ?A\end{array}}{\begin{array}{c}?A\\ \vdots\end{array}} \quad \stackrel{[cow]}{\rightarrow} \quad \begin{array}{c}\vdots\\ ?A\\ \vdots\end{array}$$

Lemma 5 *The thus extended reduction is strongly normalizing and confluent.*

proof: Clearly, any sequence of successive [*cow*] steps has to be finite (the size of the net decreases). As also the usual cut-elimination for non-additive nets is noetherian, an infinite reduction, when it exists, has to be 'eternally alternating'. However, one easily checks that instead of performing a [*cow*]-step followed by a cut-elimination step, one can always obtain the same result by *first* applying a (non void!) sequence of relevant cut-elimination steps to the corresponding cut, and *then* a sequence of [*cow*] steps. In the presence of an infinite reduction, this procrastinating attitude would enable us to generate cut-reduction sequences of unbounded length. This contradicts noetherianity.
To prove confluence, it suffices to verify *local* confluence. This is left to the reader. ⊠

N.B.: *In what follows, by reduction* $\twoheadrightarrow$ *of* **naLL**-*nets we will always mean the extended reduction as defined in this section.*

10 The structural simulation

We inductively associate to a free-style $\mathbf{LK}^{tq}$-proof π of $\Gamma^t, \Lambda^q \Rightarrow \Delta^t, \Sigma^q$ a **naLL** proof net $D^-(\pi)$ with conclusions $(!?D(\Gamma))^\perp, (!D(\Lambda))^\perp, ?D(\Delta), ?!D(\Sigma)$, where $D(.)$ stands for some, as yet unspecified, *modal translation* of $\mathbf{LK}^{tq}$-formulas ϕ into non-additive linear formulas $D(\phi)$, later to be defined by induction on the complexity of such formulas. (For notational convenience

in fact we will usually represent a proof net by an element of the equivalence class of sequent proofs corresponding to it, i.e by a linear *sequent proof*.) The *modalities* (i.e. sequences of occurrences of !, ?) $!?, !, ?$ and ?! prefixing the D-translated formulas are called the *context*- or *surface*-modalities.
The general pattern of what we call an *inductive decoration* is as follows: the axioms $A^t \vdash A^t$ and $A^q \vdash A^q$ become respectively

$$\frac{?D(A) \vdash ?D(A)}{!?D(A) \vdash ?D(A)} \qquad \frac{!D(A) \vdash !D(A)}{!D(A) \vdash ?!D(A)}$$

Now suppose we have inductively decorated the proof(s) π_i of $!?D(\Gamma_i), !D(\Lambda_i) \Rightarrow ?D(\Delta_i), ?!D(\Sigma_i)$, corresponding to the subderivation(s) of the premise(s) in the final application of a rule in our derivation π. (Let us write μD(premises), where μ stands for the context-modalities.)
(-i-) If this last rule is a *logical rule*, we proceed as in

$$\begin{array}{c} D^-(\pi_i) \\ \vdots \\ \mu D(\text{premises}) \\ \hline \text{pre-modalization of active formula(s)} \\ \hline \text{non-additive linear version of the rule} \\ \hline \text{post-modalization of the main formula} \\ \hline \mu D(\text{conclusion}) \end{array}$$

In the, possibly empty, *pre-modalization* phase we, when necessary, apply rules of dereliction and promotion in order to add exponentials to the formula occurrences $\mu D(A)$, where A is a formula occurrence active in the logical rule being decorated. We then perform the non-additive linear version of the rule: a block consisting in a linear logical rule, sometimes pre- and/or postceded by structural rules - this part of the decoration will be specified below for each of the logical rules. The decoration ends with the *post-modalization* phase, in which we apply rules of dereliction and promotion in order to obtain $\mu D(X)$, where X stands for the formula occurrence main in the logical rule.
(-ii-) If the last rule is a *structural rule*, we proceed as in (i), but now there are *neither* pre- and *nor* post-modalization steps. Due to the exponential 'surface-structure' of μD(premises), exponential structural rules are freely applicable. Hence we have a decoration pattern as in:

$$\begin{array}{c} D^-(\pi) \\ \vdots \\ \mu D(\text{premises}) \\ \hline \text{exponential structural rule} \\ \hline \mu D(\text{conclusion}) \end{array}$$

E.g. if we end by

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \Gamma^t, A^q, A^q, \Delta^q \vdash \Pi^q, \Sigma^t\end{array}}{\Gamma^t, A^q, \Delta^q \vdash \Pi^q, \Sigma^t}$$

our decoration becomes

$$\dfrac{\begin{array}{c}D^-(\pi)\\ \vdots\\ !?D(\Gamma), !D(A), !D(A), !D(\Delta) \vdash ?!D(\Pi), ?D(\Sigma)\end{array}}{!?D(\Gamma), !D(A), !D(\Delta) \vdash ?!D(\Pi), ?D(\Sigma)}$$

(-iii-) Finally, in case the last rule is a *cut rule*, we distinguish the three possible types of cut.

1. If the cut is logical, both occurrences of the cutformula A have just been main in a logical rule. E.g.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Delta^q \vdash A}\text{intro} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Gamma^t, A \vdash \Sigma^t}\text{intro}}{\Gamma^t, \Delta^q \vdash \Sigma^t}$$

   We then *do not* perform their respective post-modalizations, and cut directly on $D(A)$:

$$\dfrac{\dfrac{\begin{array}{c}D^-(\pi_1)\\ \vdots\end{array}}{!D(\Delta) \vdash D(A)}\text{intro} \qquad \dfrac{\begin{array}{c}D^-(\pi_2)\\ \vdots\end{array}}{!?D(\Gamma), D(A) \vdash ?D(\Sigma)}\text{intro}}{!?D(\Gamma), !D(\Delta) \vdash ?D(\Sigma)}$$

   Therefore, in the decoration of an L-cut, the cutformula is *without* surface-modality.

2. If the cut is of type S2, e.g.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Delta^q \vdash A^q}\text{intro} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma^t, A^q \vdash \Sigma^t\end{array}}{\Gamma^t, \Delta^q \vdash \Sigma^t}$$

then the attractive occurrence of the cutformula A has just been main in a logical rule. We do *not* perform the second post-modalization (dereliction) in the corresponding premise, and cut on $!D(A)$ ($?D(A)$ in case A is coloured t):

$$\dfrac{\dfrac{\dfrac{D^-(\pi_1)}{\vdots}}{!D(\Delta) \vdash D(A)}\ \text{intro} \quad\quad}{!D(\Delta) \vdash !D(A)} \quad \dfrac{D^-(\pi_2)}{\vdots} \atop {!?D(\Gamma), !D(A) \vdash ?D(\Sigma)}$$

$$\dfrac{!D(\Delta) \vdash !D(A) \qquad !?D(\Gamma), !D(A) \vdash ?D(\Sigma)}{!?D(\Gamma), !D(\Delta) \vdash ?D(\Sigma)}$$

We see that in the decoration of an S2-cut the cutformula has a surface-modality *of length 1.*

3. If the cut is of type S1, e.g.

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma^q \vdash A^t\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Delta^t, A^t \vdash \Sigma^q\end{matrix}}{\Gamma^q, \Delta^t \vdash \Sigma^t}$$

we add a post-modalization (promotion) to the decoration of the premise that contains the non-attractive occurrence of the cutformula, hence cut on $!?D(A)$ (resp. $?!D(A)$ in case the cutformula is coloured q).

$$\dfrac{\dfrac{\begin{matrix}D^-(\pi_1)\\ \vdots\\ !D(\Gamma) \vdash ?D(A)\end{matrix}}{!D(\Gamma) \vdash !?D(A)} \qquad \begin{matrix}\pi_2\\ \vdots\\ !?D(\Delta), !?D(A) \vdash ?!D(\Sigma)\end{matrix}}{!D(\Gamma), !?D(\Delta) \vdash ?!D(\Sigma)}$$

We thus find that in the decoration of an S1-cut the cutformula has a surface-modality *of length 2.*

(Observe how the length of surface-modalities reflects the natural order of the types of cuts.)

At this point the reader might get somewhat uneasy, and impatiently object that we still did not define the formula translation D. Let her be patient! For ultimately we will. (But, by the way, we can (and we shall), without loss of generality, limit ourselves to the case of implication, universal quantification and negation, as the D-translation of the other connectives and quantifier follows by a simple 'duality correspondence'.)

However, even *without* specifying the translation on the level of formulas, we already can prove that our mapping enables a simulation of the *structural* parts of a tq-reduction.

Proposition 6 *If $\pi \rightarrow_{tq} \pi'$ by a structural reduction step, then there is a non-void proof net reduction $D^-(\pi) \twoheadrightarrow D^-(\pi')$.*

proof: Let us consider a structural cut c in π.

- If c is a cut on A of type S1, then in the decoration of c in $D^-(\pi)$ the cutformulas have a surface-modality of length 2 and the decoration of the premise containing the *non-attractive* occurrence of the cutformula ends with a promotion: in $D^-(\pi)$ this subderivation is an *exponential box* with, say, $\mu D(A)$ as principal door, and the decorated cut is an exponential cut. Reduction of this cut in $D^-(\pi)$ consists in moving it up the principal door's exponential tree (this again is the 'complete exponential reduction' of Regnier(1992)), (possibly) duplicating and/or erasing the box, then cutting it on the derelictions introducing the outermost exponential of ancestors of $(\mu D(A))^\perp$. These derelictions follow introductions in either an axiom or a logical rule. In case of an axiom we proceed by a promotion-dereliction reduction followed by a substition; in case of a logical rule we proceed by a promotion-dereliction reduction, thus obtaining the decoration of a cut of type S2. If in fact the non-attractive occurrence of the cutformula in π was main in a logical rule, we perform one more promotion-dereliction reduction, and find the decoration of a cut of type L. This corresponds precisely to the performance of the S1-step in π.

- If c is a cut on A of type S2, then in the decoration of c in $D^-(\pi)$ it is the decoration of the premise containing the *attractive* occurrence of the cutformula that ends with a promotion: in $D^-(\pi)$ this subderivation is an *exponential box* with, say, $\nu D(A)$ as principal door. Again the decorated cut is an exponential cut and reduction of this cut in $D^-(\pi)$ means (possibly) duplicating and/or erasing the box, then cutting it on the derelictions introducing the outermost exponential of ancestors of $(\nu D(A))^\perp$. These derelictions follow introductions in either an axiom or a logical rule. In case of an axiom we proceed by a substition; in case of a logical rule we proceed by a promotion-dereliction reduction, thus obtaining the decoration of a cut of type L. This corresponds precisely to the performance of the S2-step in π.

We now have obtained a proof net $(D^-(\pi))'$. By corollary 4 the type of residues of all *other* cuts in π did not increase. If the type of some cut did *decrease* we correspondingly lower the number of surface exponentials in $(D^-(\pi))'$ by promotion-dereliction reduction. Thus we obtain $D^-(\pi')$.
$\boxtimes$

Note that the above shows that the simulation of the *structural* part of *tq*-reduction can be treated independently of the specific translation of formulas that is chosen. Hence in what follows we need *only* to consider logical reductions. Of course it is there that the translation D plays a major role. And it is there that we can opt for either an additive or for a multiplicative point of view.

11 Logical reductions involving negation and quantifiers

Let us unveil a first bit of our translation of classical formulas, namely the bit that is common to both the additive and the multiplicative mapping:
D will be 'colour-forgetting' on atomic formula; so if A^ϵ is atomic, we put $D(A^\epsilon) := A$.
Quantified[7] and negated formulas $(\neg A^\epsilon, \forall X A^\epsilon)$ we translate according to the colour of A, as in the following table:

	$\forall X A$
t	$\forall X !?D(A)$
q	$\forall X ?!D(A)$

	$\neg A$
t	$(!?D(A))^\perp$
q	$(?!D(A))^\perp$

A logical cut involving universal quantification and its reduct are

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \vdash A[X^\epsilon]\end{array}}{\vdash \forall Y A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ A[T^\epsilon/X^\epsilon]\vdash\end{array}}{\forall Y A \vdash}}{\vdash} \qquad \text{resp.} \qquad \dfrac{\begin{array}{c}\pi_1[T^\epsilon/X^\epsilon]\\ \vdots\\ \vdash A[T^\epsilon/X^\epsilon]\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ A[T^\epsilon/X^\epsilon]\vdash\end{array}}{\vdash}$$

We leave it to the reader to provide the decorations and verify that they reduce to the decoration of the reduct.
A logical cut in a derivation π involving negation and its reduct π' are

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ A^\epsilon\vdash\end{array}}{\vdash \neg A^\epsilon} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \vdash A^\epsilon\end{array}}{\neg A^\epsilon \vdash}}{\vdash} \qquad \text{resp.} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \vdash A^\epsilon\end{array} \qquad \begin{array}{c}\pi_1\\ \vdots\\ A^\epsilon\vdash\end{array}}{\vdash}$$

In case A is coloured q the decoration of the cut and its reduction are as in:

$$\dfrac{\dfrac{\dfrac{!D(A)\vdash}{?!D(A)\vdash}}{\vdash\ (?!D(A))^\perp} \qquad \dfrac{\vdash ?!D(A)}{(?!D(A))^\perp\vdash}}{\vdash} \qquad \to \qquad \dfrac{\vdash ?!D(A) \qquad \dfrac{!D(A)\vdash}{?!D(A)\vdash}}{\vdash}$$

[7] We will only deal with second order propositional quantifiers. The case of first order quantifiers can be treated similarly.

As *proof nets* these two figures are the same: the operation is *void*. However, in order to obtain $D^-(\pi')$ it might be necessary to proceed by performing one or two promotion/dereliction reductions; hence the simulation will be 'empty' iff the created cut in π' is structural of type S1.
We therefore proved in this section:

Proposition 7 *If $\pi \to_{tq} \pi'$ by a logical reduction step involving a quantifier or negation, then there is a proof net reduction $D^-(\pi) \twoheadrightarrow D^-(\pi')$. However, in the case of negation, it might happen that $D^-(\pi) = D^-(\pi')$.*

12 Logical reductions involving binary connectives

For each of the binary connectives, in free-style $\mathbf{LK}^{tq}$ there are a number of distinct rules to introduce it on the left- and a number of rules to introduce it on the right-hand side of the entailment sign. We correspondingly have to define, for each of the binary connectives, several logical reduction steps. As a logical cut will always be between an occurrence of the cut formula just introduced in a *unary* rule and an occurrence of the cut formula just introduced in a *binary* rule, in free-style $\mathbf{LK}^{tq}$ we distinguish four types of logical cuts for a given connective (as opposed to four types of the connective, as in Joinet et al.(1996)).

$\langle m, m\rangle$ Both the unary and the binary rule are *multiplicative*:

$$\dfrac{\dfrac{A \vdash B}{\vdash A \to B} \qquad \dfrac{\vdash A \qquad B \vdash}{A \to B \vdash}}{\vdash}$$

Now recall that the unary multiplicative introduction rule comes in two guises, determining the order of the descendants of the cuts. Depending on the 'orientation' of the occurrence of the rule in question, we thus have one of the two following reducts:

$$\dfrac{\dfrac{\vdash A \qquad A \vdash B}{\vdash B} \qquad B \vdash}{\vdash} \qquad \text{or} \qquad \dfrac{\vdash A \qquad \dfrac{A \vdash B \qquad B \vdash}{A \vdash}}{\vdash}$$

$\langle m, a\rangle$ The unary rule is *multiplicative* and the binary rule is *additive*:

$$\dfrac{\dfrac{A \vdash B}{\vdash A \to B} \qquad \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}}{\Gamma \vdash \Delta}$$

Depending on the 'orientation' of the unary rule, we have one of the two following reducts:

$$
\begin{prooftree}
\AxiomC{$\Gamma \vdash A, \Delta$}
\AxiomC{$A \vdash B$}
\BinaryInfC{$\Gamma \vdash B, \Delta$}
\AxiomC{$\Gamma, B \vdash \Delta$}
\BinaryInfC{$\Gamma, \Gamma \vdash \Delta, \Delta$}
\doubleLine
\UnaryInfC{$\Gamma \vdash \Delta$}
\end{prooftree}
\quad \text{or} \quad
\begin{prooftree}
\AxiomC{$\Gamma \vdash A, \Delta$}
\AxiomC{$A \vdash B$}
\AxiomC{$\Gamma, B \vdash \Delta$}
\BinaryInfC{$\Gamma, A \vdash \Delta$}
\BinaryInfC{$\Gamma, \Gamma \vdash \Delta, \Delta$}
\doubleLine
\UnaryInfC{$\Gamma \vdash \Delta$}
\end{prooftree}
$$

$\langle a, a \rangle$ Both the unary rule and the binary rule are *additive*. As there are two instances of the unary rule, also in this case there are two possibilities. We have

$$
\begin{prooftree}
\AxiomC{$A \vdash$}
\UnaryInfC{$\vdash A \to B$}
\AxiomC{$\Gamma \vdash A, \Delta$}
\AxiomC{$B, \Gamma \vdash \Delta$}
\BinaryInfC{$\Gamma, A \to B \vdash \Delta$}
\BinaryInfC{$\Gamma \vdash \Delta$}
\end{prooftree}
$$

or

$$
\begin{prooftree}
\AxiomC{$\vdash B$}
\UnaryInfC{$\vdash A \to B$}
\AxiomC{$\Gamma \vdash A, \Delta$}
\AxiomC{$B, \Gamma \vdash \Delta$}
\BinaryInfC{$\Gamma, A \to B \vdash \Delta$}
\BinaryInfC{$\Gamma \vdash \Delta$}
\end{prooftree}
$$

which reduce to

$$
\begin{prooftree}
\AxiomC{$\Gamma \vdash A, \Delta$}
\AxiomC{$A \vdash$}
\BinaryInfC{$\Gamma \vdash \Delta$}
\end{prooftree}
\quad \text{resp.} \quad
\begin{prooftree}
\AxiomC{$\vdash B$}
\AxiomC{$B, \Gamma \vdash \Delta$}
\BinaryInfC{$\Gamma \vdash \Delta$}
\end{prooftree}
$$

$\langle a, m \rangle$ The unary rule is *additive* and the binary rule is *multiplicative*. We have:

$$
\begin{prooftree}
\AxiomC{$A \vdash$}
\UnaryInfC{$\vdash A \to B$}
\AxiomC{$\Gamma_1 \vdash A, \Delta_1$}
\AxiomC{$\Gamma_2, B \vdash \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2, A \to B \vdash \Delta_1, \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

or

$$
\begin{prooftree}
\AxiomC{$\vdash B$}
\UnaryInfC{$\vdash A \to B$}
\AxiomC{$\Gamma_1 \vdash A, \Delta_1$}
\AxiomC{$\Gamma_2, B \vdash \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2, A \to B \vdash \Delta_1, \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

which reduce to:

$$
\begin{prooftree}
\AxiomC{$\Gamma_1 \vdash A, \Delta_1$}
\doubleLine
\UnaryInfC{$\Gamma_1, \Gamma_2 \vdash A, \Delta_1, \Delta_2$}
\AxiomC{$A \vdash$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
\quad \text{resp.} \quad
\begin{prooftree}
\AxiomC{$\vdash B$}
\AxiomC{$\Gamma_2, B \vdash \Delta_2$}
\doubleLine
\UnaryInfC{$\Gamma_1, \Gamma_2, B \vdash \Delta_1, \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

(Alternatively, if one prefers, to:

$$\dfrac{\dfrac{\Gamma_1 \vdash A, \Delta_1 \qquad A \vdash}{\Gamma_1 \vdash \Delta_1}}{\overline{\overline{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}}} \qquad \text{resp.} \qquad \dfrac{\dfrac{\vdash B \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_2 \vdash \Delta_2}}{\overline{\overline{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}}}$$

or to:

$$\dfrac{\Gamma_1 \vdash A, \Delta_1 \qquad \dfrac{A \vdash}{\overline{\overline{A, \Gamma_2 \vdash \Delta_2}}}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \qquad \text{resp.} \qquad \dfrac{\dfrac{\vdash B}{\overline{\overline{\Gamma_1 \vdash B, \Delta_1}}} \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

but of course up to weak equivalence these 'alternative' reducts are the same as the previous ones!)

13 The multiplicative interpretation

The multiplicative interpretation maps a classical implication $A^{\epsilon} \rightarrow B^{\epsilon'}$ to a linear implication $\mu D(A) \multimap \nu D(B)$, where μ and ν are modalities that will depend on the colours of the subformulas. There are $2 \times 2 = 4$ possible cases to consider. They are treated as in the following table:

	t	q
t	$!?D(A) \multimap ?!?D(B)$	$!?D(A) \multimap ?!D(B)$
q	$!?!D(A) \multimap ?!?D(B)$	$!?!D(A) \multimap ?!D(B)$

Let us recall the general pattern of our inductive mapping of an $\mathbf{LK}^{tq}$-proof π to a **naLL**-net $D^{-}(\pi)$ in case π ends in an application of a logical introduction-rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D^{-}(\pi_i) \\ \vdots \\ \mu D(\text{premises}) \end{array}}{\text{pre-modalization of active formula(s)}}}{\text{non-additive linear version of the rule}}}{\text{post-modalization of the main formula}}}{\mu D(\text{conclusion})}$$

In defining the multiplicative formula translation D in the above table we were guided by three wishes:

1. we would like to use modalities of the least possible length (minimality of the solution);

2. the decorated unary and binary introduction rules should all lead to the same D-translation of the formula (we have to *unify* the possible minimal decorations of the active formulas in the rules, a matter of syntactic *soundness* of the mapping);

3. in case a unary and a binary rule meet in a cut, the proof net reduction of the decoration should simulate the tq-reduction of the original derivation (dynamic *faithfulness*)

Let us for example try to determine $D(A^t \to B^t)$. Recall that we set out to inductively associate to a derivation of $\Gamma^t, \Lambda^q \Rightarrow \Delta^t, \Sigma^q$ a **naLL**-net $D^-(\pi)$ with conclusions $(!?D(\Gamma))^\perp, (!D(\Lambda))^\perp, ?D(\Delta), ?!D(\Sigma)$.
Hence if our proof ended in an occurrence of a right introduction rule we have by induction hypothesis a premise of the form

$$\ldots, !?D(A) \vdash ?D(B), \ldots \quad \text{or} \ldots, !?D(A) \vdash \ \ldots \quad \text{or} \ldots \vdash ?D(B), \ldots$$

In the left introduction rules the induction hypothesis gives us premises of the form

$$\ldots \vdash ?D(A), \ldots \quad \text{and} \quad \ldots, !?D(B) \vdash \ldots$$

A minimal solution of the unification problem is given by $D(A^t \to B^t) = !?D(A) \multimap !?D(B)$. However this first attempt turns out to be not always faithful with respect to the intended dynamics: we need to be able to simulate 32 logical reduction steps, each of them determined by the parameters: (i) unary rule (four possibilites) (ii) binary rule (2 possibilities) (iii) colour of subformulas (four possibilites) and some of these turn out to be out of reach of the minimal solution.
As an example, consider a cut of type $\langle a, m \rangle$:

$$\dfrac{\dfrac{A^t \vdash}{\vdash A^t \to B^t} \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ \vdash A^t\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ B^t \vdash\end{matrix}}{A^t \to B^t \vdash}}{\vdash} \qquad \text{which reduces to} \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ \vdash A^t\end{matrix} \qquad A^t \vdash}{\vdash}$$

Using the translation $D(A^t \to B^t) = !?D(A) \multimap !?D(B)$ we (have to) decorate as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{!?D(A) \vdash}{!?D(A) \vdash ?D(B)}}{!?D(A) \vdash !?D(B)}}{\vdash !?D(A) \multimap !?D(B)} \qquad \dfrac{\dfrac{\begin{matrix}D^-(\pi_1)\\ \vdots\\ \vdash ?D(A)\end{matrix}}{\vdash !?D(A)} \qquad \begin{matrix}D^-(\pi_2)\\ \vdots\\ !?D(B) \vdash\end{matrix}}{!?D(A) \multimap !?D(B) \vdash}}{\vdash}$$

which by a (linear) logical reduction step reduces to

$$
\begin{prooftree}
\AxiomC{$D^{-}(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?D(A)$}
\UnaryInfC{$\vdash !?D(A)$}
\AxiomC{$!?D(A) \vdash$}
\UnaryInfC{$!?D(A) \vdash ?D(B)$}
\UnaryInfC{$!?D(A) \vdash !?D(B)$}
\AxiomC{$D^{-}(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$!?D(B) \vdash$}
\TrinaryInfC{$\vdash$}
\end{prooftree}
$$

However, for a simulation of the tq-reduction, which erases *all* of subderivation π_2, we have to be able to erase in the decoration *all* of subderivation $D^{-}(\pi_2)$. In the decoration obtained in general this will *not* be possible. In order to guarantee erasure of $D^{-}(\pi_2)$ we need to 'isolate' it. This means, in proof net terminology, that we should 'box' it. And this is precisely what the additional exponential is going to do. Indeed, using the translation $D(A^t \to B^t) = !?D(A) \multimap ?!?D(B)$ we decorate as follows:

$$
\begin{prooftree}
\AxiomC{$!?D(A) \vdash$}
\UnaryInfC{$!?D(A) \vdash ?!?D(B)$}
\UnaryInfC{$\vdash !?D(A) \multimap ?!?D(B)$}
\AxiomC{$D^{-}(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?D(A)$}
\UnaryInfC{$\vdash !?D(A)$}
\AxiomC{$D^{-}(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$!?D(B) \vdash$}
\UnaryInfC{$?!?D(B) \vdash$}
\BinaryInfC{$!?D(A) \multimap ?!?D(B) \vdash$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

It is *essential* that we weaken with $?!?D(B)$ instead of with $?D(B)$, for then, as wanted, the logical reduct

$$
\begin{prooftree}
\AxiomC{$D^{-}(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?D(A)$}
\UnaryInfC{$\vdash !?D(A)$}
\AxiomC{$!?D(A) \vdash$}
\UnaryInfC{$!?D(A) \vdash ?!?D(B)$}
\AxiomC{$D^{-}(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$!?D(B) \vdash$}
\UnaryInfC{$?!?D(B) \vdash$}
\TrinaryInfC{$\vdash$}
\end{prooftree}
$$

in turn reduces to:

$$
\begin{prooftree}
\AxiomC{$D^{-}(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?D(A)$}
\UnaryInfC{$\vdash !?D(A)$}
\AxiomC{$!?D(A) \vdash$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

Now our image *does* simulate the tq-reduction!
Observe that the fact that we weaken with $?!?D(B)$ indicates *in the decoration* that this is not an appplication of a 'true structural rule' (in the sense that it is not the decoration of a structural rule in the original derivation), but of a structural rule as-part-of the non-additive linear interpretation of an additive logical introduction rule.

So it is dynamic faithfulness that forces us to use a formula translation that is *less economic* than necessary for mere syntactic soundness. The somewhat 'overloaded' translation enables us to more fully use the 'dynamic expressivity' of linear logic, and, by Jove, we get both! In cases where the minimal decoration was already sufficient to model the tq-reduction, the added exponential is dynamically superfluous, as there will never be a structural source to bind it (cf. Danos et al.(1993)). But of course we do need it in all cases, to ensure syntactic soundness.
The above sketches one out of the thirty-two verifications necessary in order to prove:

Proposition 8 *If $\pi \rightarrow_{tq} \pi'$ by a logical reduction step involving an implication, then there is a non-void proof net reduction $D^-(\pi) \twoheadrightarrow D^-(\pi')$.*

proof:
Of course we will not go through all the cases in detail - we'll do some of them, to give the general idea, and leave the rest to the incredulous reader (sunny exercises in linear decoration).

$\langle a, a\rangle \backslash (t, t)$

$$\dfrac{\dfrac{A^t \vdash}{\vdash A^t \rightarrow B^t} \qquad \dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \Gamma \vdash A^t, \Delta\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ B^t, \Gamma \vdash \Delta\end{matrix}}{\Gamma, A^t \rightarrow B^t \vdash \Delta}}{\Gamma \vdash \Delta}$$

reduces to

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \Gamma \vdash A^t, \Delta\end{matrix} \qquad A^t \vdash}{\Gamma \vdash \Delta}$$

Now the linear image will be the following (let us write $[\ldots]$ for the decorated context on the left, $\langle\ldots\rangle$ for the decorated context on the right):

$$\dfrac{\dfrac{\dfrac{!?D(A) \vdash}{!?D(A) \vdash ?!?D(B)}}{\vdash !?D(A) \multimap ?!?D(B)} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}D^-(\pi_1) \\ \vdots \\ [\ldots] \vdash ?D(A), \langle\ldots\rangle\end{matrix}}{[\ldots] \vdash !?D(A), \langle\ldots\rangle} \qquad \dfrac{\begin{matrix}D^-(\pi_2) \\ \vdots \\ [\ldots], !?D(B) \vdash \langle\ldots\rangle\end{matrix}}{[\ldots], ?!?D(B) \vdash \langle\ldots\rangle}}{!?D(A) \multimap ?!?D(B), [\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle}}{!?D(A) \multimap ?!?D(B), [\ldots] \vdash \langle\ldots\rangle}}{\vdash}$$

By a (linear) logical reduction step this reduces to

$$
\dfrac{\dfrac{\dfrac{\dfrac{D^-(\pi_1)}{\vdots}}{[\ldots] \vdash ?D(A), \langle\ldots\rangle}}{[\ldots] \vdash !?D(A), \langle\ldots\rangle} \qquad \dfrac{!?D(A) \vdash}{!?D(A) \vdash ?!?D(B)} \qquad \dfrac{\dfrac{\dfrac{D^-(\pi_2)}{\vdots}}{[\ldots], !?D(B) \vdash \langle\ldots\rangle}}{[\ldots], ?!?D(B) \vdash \langle\ldots\rangle}}{\dfrac{[\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle}{[\ldots] \vdash \langle\ldots\rangle}}
$$

This net in turn first reduces to

$$
\dfrac{\dfrac{\dfrac{\dfrac{D^-(\pi_1)}{\vdots}}{[\ldots] \vdash ?D(A), \langle\ldots\rangle}}{[\ldots] \vdash !?D(A), \langle\ldots\rangle} \qquad \dfrac{!?D(A) \vdash}{[\ldots], !?D(A) \vdash \langle\ldots\rangle}}{\dfrac{[\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle}{[\ldots] \vdash \langle\ldots\rangle}}
$$

and then (remember that it's *a proof net!*), by a certain number of $[cow]$-steps, to

$$
\dfrac{\dfrac{\dfrac{\dfrac{D^-(\pi_1)}{\vdots}}{[\ldots] \vdash ?D(A), \langle\ldots\rangle}}{[\ldots] \vdash !?D(A), \langle\ldots\rangle} \qquad !?D(A) \vdash}{[\ldots] \vdash \langle\ldots\rangle}
$$

This net reduces, by zero or more promotion/dereliction reductions, reduces to the net corresponding to the D^- of the reduct of the original cut.

$\langle m, a\rangle \backslash (q, t)$ Suppose the orientation of the unary rule is 'A-cut under the B-cut':

$$
\dfrac{\dfrac{A^q \vdash B^t}{\vdash A^q \to B^t} \qquad \dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{\Gamma \vdash A^q, \Delta} \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{\Gamma, B^t \vdash \Delta}}{\Gamma, A^q \to B^t \vdash, \Delta}}{\Gamma \vdash \Delta}
$$

reduces to

$$\dfrac{\dfrac{\dfrac{A^q \vdash B^t \qquad \begin{matrix}\pi_2\\ \vdots\\ \Gamma, B^t \vdash \Delta\end{matrix}}{\Gamma, A^q \vdash \Delta} \qquad \begin{matrix}\pi_1\\ \vdots\\ \Gamma \vdash A^q, \Delta\end{matrix}}{\Gamma, \Gamma \vdash \Delta, \Delta}}{\Gamma \vdash \Delta}$$

In this case the linear image will be the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!D(A) \vdash ?D(B)}{!D(A) \vdash !?D(B)}}{!D(A) \vdash ?!?D(B)}}{?!D(A) \vdash ?!?D(B)}}{!?!D(A) \vdash ?!?D(B)}}{\vdash !?!D(A) \multimap ?!?D(B)} \qquad \dfrac{\dfrac{\begin{matrix}D^-(\pi_1)\\ \vdots\\ [\ldots] \vdash ?!D(A), \langle\ldots\rangle\end{matrix}}{[\ldots] \vdash !?!D(A), \langle\ldots\rangle} \qquad \dfrac{\begin{matrix}D^-(\pi_2)\\ \vdots\\ [\ldots], !?D(B) \vdash \langle\ldots\rangle\end{matrix}}{[\ldots], ?!?D(B) \vdash \langle\ldots\rangle}}{[\ldots], !?!D(A) \multimap ?!?D(B) \vdash \langle\ldots\rangle}}{[\ldots] \vdash \langle\ldots\rangle}$$

It reduces to

$$\dfrac{\dfrac{\begin{matrix}D^-(\pi_1)\\ \vdots\\ [\ldots] \vdash ?!D(A), \langle\ldots\rangle\end{matrix} \qquad \dfrac{\dfrac{\dfrac{!D(A) \vdash ?D(B)}{!D(A) \vdash !?D(B)} \qquad \begin{matrix}D^-(\pi_2)\\ \vdots\\ [\ldots], !?D(B) \vdash \langle\ldots\rangle\end{matrix}}{[\ldots], !D(A) \vdash \langle\ldots\rangle}}{[\ldots], ?!D(A) \vdash \langle\ldots\rangle}}{[\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle}}{[\ldots] \vdash \langle\ldots\rangle}$$

This net reduces, by zero or more promotion/dereliction reductions, to the net corresponding to the D^- of the reduct of the original cut.

A general remark to be made as to the decoration of the unary multiplicative rule: in the (q, t)-casse there are *two* ways to decorate an occurrence of the unary introduction-rule. Apart from the one used above, we have the alternative decoration:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!D(A) \vdash ?D(B)}{?!D(A) \vdash ?D(B)}}{!?!D(A) \vdash ?D(B)}}{!?!D(A) \vdash !?D(B)}}{!?!D(A) \vdash ?!?D(B)}}{\vdash !?!D(A) \multimap ?!?D(B)}$$

As the reader might guess, this alternative decoration corresponds to the alternative orientation of the rule. Indeed! For,

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!D(A) \vdash ?D(B)}{?!D(A) \vdash ?D(B)}}{!?!D(A) \vdash ?D(B)}}{!?!D(A) \vdash !?D(B)}}{!?!D(A) \vdash ?!?D(B)}}{\vdash !?!D(A) \multimap ?!?D(B)}
\qquad
\dfrac{
\dfrac{\begin{matrix} D^-(\pi_1) \\ \vdots \\ [\ldots] \vdash ?!D(A), \langle\ldots\rangle \end{matrix}}{[\ldots] \vdash !?!D(A), \langle\ldots\rangle}
\quad
\dfrac{\begin{matrix} D^-(\pi_2) \\ \vdots \\ [\ldots], !?D(B) \vdash \langle\ldots\rangle \end{matrix}}{[\ldots], ?!?D(B) \vdash \langle\ldots\rangle}
}{[\ldots], !?!D(A) \multimap ?!?D(B) \vdash \langle\ldots\rangle}
}{[\ldots] \vdash \langle\ldots\rangle}
$$

reduces to

$$
\dfrac{
\dfrac{
\dfrac{
\begin{matrix} D^-(\pi_1) \\ \vdots \\ [\ldots] \vdash ?!D(A), \langle\ldots\rangle \end{matrix}
\quad
\dfrac{!D(A) \vdash ?D(B)}{?!D(A) \vdash ?D(B)}
}{[\ldots] \vdash ?D(B), \langle\ldots\rangle}
}{[\ldots] \vdash !?D(B), \langle\ldots\rangle}
\qquad
\begin{matrix} D^-(\pi_2) \\ \vdots \\ [\ldots], !?D(B) \vdash \langle\ldots\rangle \end{matrix}
}{\dfrac{[\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle}{\rule{0pt}{0pt}[\ldots] \vdash \langle\ldots\rangle}}
$$

For all the other colourings the decoration is *unique*. The reader should verify that, correspondingly, the reducts corresponding to the two possibilities in fact give rise to *the same* proof net. Cf. the treatment of the following case, and the observations in Danos et al.(1997).

$\langle m, m\rangle \backslash (q, q)$ Suppose the orientation of the unary rule is 'B-cut under the A-cut':

$$
\dfrac{
\dfrac{A^q \vdash B^q}{\vdash A^q \to B^q}
\qquad
\dfrac{\begin{matrix} \pi_1 \\ \vdots \\ \vdash A^q \end{matrix} \quad \begin{matrix} \pi_2 \\ \vdots \\ B^q \vdash \end{matrix}}{A^q \to B^q \vdash}
}{\vdash}
$$

reduces to

$$
\dfrac{
\dfrac{\begin{matrix} \pi_1 \\ \vdots \\ \vdash A^q \end{matrix} \quad A^q \vdash B^q}{\vdash B^q}
\qquad
\begin{matrix} \pi_2 \\ \vdots \\ B^q \vdash \end{matrix}
}{\vdash}
$$

The linear image

$$
\begin{prooftree}
\AxiomC{$!D(A) \vdash ?!D(B)$}
\UnaryInfC{$?!D(A) \vdash ?!D(B)$}
\UnaryInfC{$!?!D(A) \vdash ?!D(B)$}
\UnaryInfC{$\vdash !?!D(A) \multimap ?!D(B)$}
\AxiomC{$D^-(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?!D(A)$}
\UnaryInfC{$\vdash !?!D(A)$}
\AxiomC{$D^-(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$!D(B) \vdash$}
\UnaryInfC{$?!D(B) \vdash$}
\BinaryInfC{$!?!D(A) \multimap ?!D(B) \vdash$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

reduces to

$$
\begin{prooftree}
\AxiomC{$D^-(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?!D(A)$}
\AxiomC{$!D(A) \vdash ?!D(B)$}
\UnaryInfC{$?!D(A) \vdash ?!D(B)$}
\AxiomC{$D^-(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$!D(B) \vdash$}
\UnaryInfC{$?!D(B) \vdash$}
\TrinaryInfC{$\vdash$}
\end{prooftree}
$$

This net reduces, by zero or more promotion/dereliction reductions, to the one corresponding to the D^- of the reduct of the original cut. And, yes, it *also* corresponds to the D^- of the reduct with the other orientation of the unary rule. (Of course, there is but *one* linear image of the original cut.)

As a final case let's have a look at:

$\langle a, m\rangle \backslash (t, q)$

$$
\begin{prooftree}
\AxiomC{$\vdash B^q$}
\UnaryInfC{$\vdash A^t \to B^q$}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_1 \vdash A^t, \Delta_1$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$B^q, \Gamma_2 \vdash \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2, A^t \to B^q \vdash \Delta_1, \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

reduces to

$$
\begin{prooftree}
\AxiomC{$\vdash B^q$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$B^q, \Gamma_2 \vdash \Delta_2$}
\BinaryInfC{$\Gamma_2 \vdash \Delta_2$}
\doubleLine
\UnaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

Now the linear image will be the following (we write $[\ldots]_i$ for the decorated contexts on the left, $\langle \ldots \rangle_i$ for the decorated context on the right):

$$
\dfrac{\dfrac{\dfrac{\vdash ?!D(B)}{!?D(A) \vdash ?!D(B)}}{\vdash !?D(A) \multimap ?!D(B)} \qquad \dfrac{\dfrac{\begin{matrix} D^-(\pi_1) \\ \vdots \\ [\ldots]_1 \vdash ?D(A), \langle\ldots\rangle_1 \end{matrix}}{[\ldots]_1 \vdash !?D(A), \langle\ldots\rangle_1} \qquad \dfrac{\begin{matrix} D^-(\pi_2) \\ \vdots \\ !D(B), [\ldots]_2 \vdash \langle\ldots\rangle_2 \end{matrix}}{?!D(B), [\ldots]_2 \vdash \langle\ldots\rangle_2}}{!?D(A) \multimap ?!D(B), [\ldots]_1, [\ldots]_2 \vdash \langle\ldots\rangle_1, \langle\ldots\rangle_2}}{[\ldots]_1, [\ldots]_2 \vdash \langle\ldots\rangle_1, \langle\ldots\rangle_2}
$$

which reduces to

$$
\dfrac{\dfrac{\vdash ?!D(B) \qquad \dfrac{\begin{matrix} D^-(\pi_2) \\ \vdots \\ !D(B), [\ldots]_2 \vdash \langle\ldots\rangle_2 \end{matrix}}{?!D(B), [\ldots]_2 \vdash \langle\ldots\rangle_2}}{[\ldots]_2 \vdash \langle\langle\ldots\rangle_2}}{\overline{[\ldots]_1, [\ldots]_2 \vdash \langle\ldots\rangle_1, \langle\ldots\rangle_2}}
$$

This net reduces, by zero or more promotion/dereliction reductions, to the one corresponding to the D^- of the reduct of the original cut.

Verification of the other cases is left to the reader. ⊠

We are now able to prove:

Theorem 9 (SN) *The tq-protocol applied to free-style* $\mathbf{LK}^{tq}$ *is strongly normalizing.*

proof: Because of propositions 6, 7 and 8, this is almost immediate from the strong normalization theorem for non-additive proof net reductions. For, given an infinite *tq*-reduction sequence

$$\pi_0 \rightarrow_{tq} \pi_1 \rightarrow_{tq} \pi_2 \rightarrow_{tq} \pi_3 \rightarrow_{tq} \pi_4 \rightarrow_{tq} \pi_5 \rightarrow_{tq} \cdots$$

we know from these propositions that there is a corresponding proof net reduction sequence

$$D^-(\pi_0) \twoheadrightarrow D^-(\pi_1) \twoheadrightarrow D^-(\pi_2) \twoheadrightarrow D^-(\pi_3) \twoheadrightarrow D^-(\pi_4) \twoheadrightarrow D^-(\pi_5) \twoheadrightarrow \ldots$$

By noetherianity of proof net reductions, this sequence has to be finite, which means that from a certain point onwards there are only ‘empty steps’. But such an infinite sequence of ‘empty steps’ would be the image of an infinite sequence of logical negation reduction steps in the *tq*-reduction sequence, and obviously such an infinite sequence is impossible. ⊠

Theorem 10 (CR) *Two tq-normal forms of a free-style* $\mathbf{LK}^{tq}$*-proof are equal up to permutations of structural rules.*

proof: The trick is the same as that of Danos et al.(1997): because CR holds for **naLL**, D^- maps all normal forms of a given free-style $\mathbf{LK}^{tq}$-proof to the same proof net. But this net can have a whole lotta sequentializations. We therefore try to 'rigidify' the nets, by defining a decoration fD (f as in 'fat') that uses more exponentials. The 'fat decoration' inductively associates to a free-style proof π of $\Gamma^t, \Lambda^q \Rightarrow \Delta^t, \Sigma^q$ a **naLL**-net $fD^-(\pi)$ with conclusions $(!?!fD(\Gamma))^\perp, (!?fD(\Lambda))^\perp, ?!fD(\Delta), ?!?fD(\Sigma)$. It is straightforward to derive the corresponding translation of formulas (a mere 'shifting' of the old one). As the additional exponentials are *superfluous* (in the sense of Danos et al.(1993)), the simulation propositions will continue to hold. When applying fD, *each* formula main in a logical rule will be promoted during post-modalization. This suffices to completely determine a cut-free net's sequentialization, modulo the order of applications of structural rules. ⊠

14 The additive interpretation

The additive interpretation maps a classical implication $A^\epsilon \rightarrow B^{\epsilon'}$ to a linear implication $?(\mu D(A) \leadsto \nu D(B))$, where μ and ν are *modalities* (sequences of occurrences of !, ?) that will depend on the colours of the subformulas. The outer exponential '?' is necessary, because, when additively simulating a multiplicative unary introduction rule, we need to *contract* the main formula. As in the multiplicative case there are $2 \times 2 = 4$ possible cases to consider. They are treated as in the following table:

	t	q
t	$?(!?D(A) \leadsto !?D(B))$	$?(!?D(A) \leadsto ?!D(B))$
q	$?(!?!D(A) \leadsto ?!?D(B))$	$?(?!D(A) \leadsto ?!D(B))$

Here's the general pattern of our inductive mapping of an $\mathbf{LK}^{tq}$-proof π to a non-multiplicative (**nmLL**)-net.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mu D(\text{premises})}{\text{pre-mod. of active formulas}}}{\text{non multiplicative linear version(s) of the rule}}}{\text{derelictions} \parallel \text{promotion}}}{\text{contraction of main formulas}}}{\text{post-mod. of the main formula}}}{\mu D(\text{conclusion})}$$

As in the multiplicative case we can show:

Proposition 11 *If $\pi \to_{tq} \pi'$ by a logical reduction step involving an implication, then there is a non-void proof net reduction $D^-(\pi) \twoheadrightarrow D^-(\pi')$.*

proof:
The proof of course again consists in a verification of many cases, most of which we leave as exercises in additive decoration. Let us give a few examples.

$\langle m, m\rangle\backslash(t,t)$ Suppose the orientation of the unary rule is 'A-cut under the B-cut':

$$\dfrac{\dfrac{A^t \vdash B^t}{\vdash A^t \to B^t} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Gamma_1 \vdash A^t, \Delta_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots \\ \Gamma_2, B^t \vdash \Delta_2\end{array}}{A^t \to B^t, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

reduces to

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Gamma_1 \vdash A^t, \Delta_1\end{array} \qquad \dfrac{A^t \vdash B^t \qquad \begin{array}{c}\pi_2\\ \vdots \\ B^t, \Gamma_2 \vdash \Delta_2\end{array}}{A^t, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

The non-multiplicative linear image of the cut consists in a cut between

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!?D(A) \vdash ?D(B)}{!?D(A) \vdash !?D(B)}}{!?D(A) \vdash !?D(A) \leadsto !?D(B)}}{\vdash !?D(A) \leadsto !?D(B), !?D(A) \leadsto !?D(B)}}{\vdash ?(!?D(A) \leadsto !?D(B)), !?D(A) \leadsto !?D(B)}}{\vdash ?(!?D(A) \leadsto !?D(B)), ?(!?D(A) \leadsto !?D(B))}}{\vdash ?(!?D(A) \leadsto !?D(B))}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D^-(\pi_1)\\ \vdots \\ [\ldots]_1 \vdash ?D(A), \langle\ldots\rangle_1\end{array}}{[\ldots]_1 \vdash !?D(A), \langle\ldots\rangle_1}}{[\ldots]_1, [\ldots]_2 \vdash !?D(A), \langle\ldots\rangle_1, \langle\ldots\rangle_2} \qquad \dfrac{\begin{array}{c}D^-(\pi_2)\\ \vdots \\ [\ldots]_2, !?D(B) \vdash \langle\ldots\rangle_2\end{array}}{[\ldots]_1, [\ldots]_2, !?D(B) \vdash \langle\ldots\rangle_1, \langle\ldots\rangle_2}}{[\ldots]_1, [\ldots]_2, (!?D(A) \leadsto !?D(B)) \vdash \langle\ldots\rangle_1, \langle\ldots\rangle_2}}{[\ldots]_1, [\ldots]_2, ?(!?D(A) \leadsto !?D(B)) \vdash \langle\ldots\rangle_1, \langle\ldots\rangle_2}$$

which (writing $[\ldots]$ for $[\ldots]_1, [\ldots]_2$ and $\langle\ldots\rangle$ for $\langle\ldots\rangle_1, \langle\ldots\rangle_2$) reduces to

$$
\begin{prooftree}
\AxiomC{$D^-(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$[\ldots]_1 \vdash ?D(A), \langle\ldots\rangle_1$}
\UnaryInfC{$[\ldots]_1 \vdash !?D(A), \langle\ldots\rangle_1$}
\doubleLine
\UnaryInfC{$[\ldots] \vdash !?D(A), \langle\ldots\rangle$}
\AxiomC{$!?D(A) \vdash ?D(B)$}
\UnaryInfC{$!?D(A) \vdash !?D(B)$}
\AxiomC{$D^-(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$[\ldots]_2, !?D(B) \vdash \langle\ldots\rangle_2$}
\doubleLine
\UnaryInfC{$[\ldots], !?D(B) \vdash \langle\ldots\rangle$}
\TrinaryInfC{$[\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle$}
\doubleLine
\UnaryInfC{$[\ldots] \vdash \langle\ldots\rangle$}
\end{prooftree}
$$

This proof net $[cow]$-reduces to

$$
\begin{prooftree}
\AxiomC{$D^-(\pi_1)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$[\ldots]_1 \vdash ?D(A), \langle\ldots\rangle_1$}
\UnaryInfC{$[\ldots]_1 \vdash !?D(A), \langle\ldots\rangle_1$}
\AxiomC{$!?D(A) \vdash ?D(B)$}
\UnaryInfC{$!?D(A) \vdash !?D(B)$}
\AxiomC{$D^-(\pi_2)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$[\ldots]_2, !?D(B) \vdash \langle\ldots\rangle_2$}
\TrinaryInfC{$[\ldots] \vdash \langle\ldots\rangle$}
\end{prooftree}
$$

This $[cow]$-reduct in turn reduces, by zero or more promotion/dereliction reductions, to the net corresponding to the D^- of the reduct of the original cut. It *also* corresponds to the D^- of the reduct with the other orientation of the unary rule. (As before: there is but *one* linear image of the original cut. The orientation is effective *only* in the presence of a multiplicative unary rule and a colouring (q, t).)

$\langle a, m\rangle \backslash (q, t)$

$$
\begin{prooftree}
\AxiomC{$A^q \vdash$}
\UnaryInfC{$\vdash A^q \to B^t$}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_1 \vdash A^q, \Delta_1$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_2, B^t \vdash \Delta_2$}
\BinaryInfC{$A^q \to B^t, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\BinaryInfC{$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

reduces to

$$
\begin{prooftree}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_1 \vdash A^q, \Delta_1$}
\AxiomC{$A^q \vdash$}
\BinaryInfC{$\Gamma_1 \vdash \Delta_1$}
\doubleLine
\UnaryInfC{$\Gamma_, \Gamma_2 \vdash \Delta_1, \Delta_2$}
\end{prooftree}
$$

The non-multiplicative linear image is

$$
\dfrac{\dfrac{\dfrac{\dfrac{!D(A) \vdash}{?!D(A) \vdash}}{\vdash ?!D(A) \rightsquigarrow ?!D(B)}}{\vdash ?(?!D(A) \rightsquigarrow ?!D(B))} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c} D^-(\pi_1) \\ \vdots \\ [\ldots]_1 \vdash ?!D(A), \langle\ldots\rangle_1 \end{array}}{[\ldots] \vdash ?!D(A), \langle\ldots\rangle} \quad \dfrac{\begin{array}{c} D^-(\pi_2) \\ \vdots \\ [\ldots]_2, !?D(B) \vdash \langle\ldots\rangle_1 \end{array}}{[\ldots], !?D(B) \vdash \langle\ldots\rangle}}{?!D(A) \rightsquigarrow !?D(B), [\ldots] \vdash \langle\ldots\rangle}}{?(?!D(A) \rightsquigarrow !?D(B)), [\ldots] \vdash \langle\ldots\rangle}}{[\ldots] \vdash \langle\ldots\rangle}
$$

and reduces to

$$
\dfrac{\dfrac{\begin{array}{c} D^-(\pi_1) \\ \vdots \\ [\ldots]_1 \vdash ?!D(A), \langle\ldots\rangle_1 \end{array}}{[\ldots] \vdash ?!D(A), \langle\ldots\rangle} \qquad \dfrac{!D(A) \vdash}{?!D(A) \vdash}}{[\ldots] \vdash \langle\ldots\rangle}
$$

which, as a proof net, is identical to

$$
\dfrac{\dfrac{\begin{array}{c} D^-(\pi_1) \\ \vdots \\ [\ldots]_1 \vdash ?!D(A), \langle\ldots\rangle_1 \end{array} \qquad \dfrac{!D(A) \vdash}{?!D(A) \vdash}}{[\ldots]_1 \vdash \langle\ldots\rangle_1}}{[\ldots] \vdash \langle\ldots\rangle}
$$

This net in turn reduces, by zero or more promotion/dereliction reductions, to the net corresponding to the D^- of the reduct of the original cut.

As a final example we consider

$\langle m, a\rangle \backslash (t, t)$

$$
\dfrac{\dfrac{A^t \vdash B^t}{\vdash A^t \to B^t} \qquad \dfrac{\begin{array}{c} \pi_1 \\ \vdots \\ \Gamma \vdash A^t, \Delta \end{array} \qquad \begin{array}{c} \pi_2 \\ \vdots \\ \Gamma, B^t \vdash \Delta \end{array}}{A^t \to B^t, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta}
$$

reduces to

$$
\cfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma \vdash A^t, \Delta\end{matrix} \qquad \cfrac{A^t \vdash B^t \qquad \begin{matrix}\pi_2\\ \vdots\\ \Gamma, B^t \vdash \Delta\end{matrix}}{\Gamma, A^t \vdash \Delta}}{\cfrac{\Gamma, \Gamma \vdash \Delta, \Delta}{\Gamma \vdash \Delta}}
$$

or

$$
\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma \vdash A^t, \Delta\end{matrix} \qquad A^t \vdash B^t}{\Gamma \vdash B^t, \Delta} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Gamma, B^t \vdash \Delta\end{matrix}}{\cfrac{\Gamma, \Gamma \vdash \Delta, \Delta}{\Gamma \vdash \Delta}}
$$

The linear image of the cut here will consist in a cut between the image of the multiplicative rule, as given on page 34, and

$$
\cfrac{\cfrac{\cfrac{\begin{matrix}D^-(\pi_1)\\ \vdots\\ [\ldots] \vdash ?D(A), \langle\ldots\rangle\end{matrix}}{[\ldots] \vdash !?D(A), \langle\ldots\rangle} \qquad \begin{matrix}D^-(\pi_2)\\ \vdots\\ [\ldots], !?D(B) \vdash \langle\ldots\rangle\end{matrix}}{!?D(A) \rightsquigarrow !?D(B), [\ldots] \vdash \langle\ldots\rangle}}{?(!?D(A) \rightsquigarrow !?D(B)), [\ldots] \vdash \langle\ldots\rangle}
$$

which reduces to

$$
\cfrac{\cfrac{\begin{matrix}D^-(\pi_1)\\ \vdots\\ [\ldots] \vdash ?D(A), \langle\ldots\rangle\end{matrix}}{[\ldots] \vdash !?D(A), \langle\ldots\rangle} \qquad \cfrac{\cfrac{!?D(A) \vdash ?D(B)}{!?D(A) \vdash !?D(B)} \qquad \begin{matrix}D^-(\pi_2)\\ \vdots\\ [\ldots], !?D(B) \vdash \langle\ldots\rangle\end{matrix}}{[\ldots], !?D(A) \vdash \langle\ldots\rangle}}{\cfrac{[\ldots], [\ldots] \vdash \langle\ldots\rangle, \langle\ldots\rangle}{[\ldots] \vdash \langle\ldots\rangle}}
$$

This net reduces, by zero or more promotion/dereliction reductions, to the net corresponding to the D^- of the "A-cut under B-cut"-reduct of the original cut.

By a commutative reduction, it becomes

$$
\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c} D^-(\pi_1) \\ \vdots \\ [\ldots] \vdash ?D(A), \langle \ldots \rangle \end{array}}{[\ldots] \vdash\, !?D(A), \langle \ldots \rangle} \qquad !?D(A) \vdash ?D(B)}{[\ldots] \vdash ?D(B), \langle \ldots \rangle}}{[\ldots] \vdash\, !?D(B), \langle \ldots \rangle} \qquad \begin{array}{c} D^-(\pi_2) \\ \vdots \\ [\ldots], !?D(B) \vdash \langle \ldots \rangle \end{array}}{\cfrac{[\ldots], [\ldots] \vdash \langle \ldots \rangle, \langle \ldots \rangle}{[\ldots] \vdash \langle \ldots \rangle}}
$$

which in turn reduces, by zero or more promotion/dereliction reductions, to the net corresponding to the D^- of the "B-cut under A-cut"-reduct of the original cut. ⊠

15 taLL

When naively checking homoreductivity of the mapping with respect to the reduction of the non-multiplicative fragment of $\mathbf{PN}_2$ (defined in Girard(1987)), one finds that it fails for the 'additive commutative reduction step': to impose, as does Girard, that the subnet 'swallowed' by an additive box be the 'empire' associated with the cut-formula, is too narrow a restriction for *tq*-reductions steps to be simulable. The point is that in sequent calculus the *entire* subproof determined by the relevant premiss of the cut rule has to 'commute'. It is however in general not the case that this subproof corresponds to the empire of the cut formula in the associated proofnet.

There are, however, ways out. One consists in allowing more general additive commutative reduction steps in $\mathbf{PN}_2$, namely steps that allow *any* subnet to commute. A proof of SN for this 'revised' $\mathbf{PN}_2$ should combine Girard's original proof of strong normalization, the main result of Tortora de Falco(2000a) and a generalization of the standardization lemma for the non-additive fragment of **LL** (Danos(1990)) to full linear logic (see also Tortora de Falco(2000b)). (The details of such a proof have not all been checked yet. Moreover full $\mathbf{PN}_2$ does *not* enjoy the CR property. The approach of the problem via the alternative embedding hence will anyway only allow us to deduce noetherianity.)

A second option is to make use of the fact that the additive boxes that are in the image of our mapping are of a rather special sort (following Danos et al.(1997) we speak of *tamed* additives), which allows the definition of a proper fragment of the non-multiplicative fragment of $\mathbf{PN}_2$ that can be embedded homoreductively into... **naLL**. Hence by a detour we end up by simulating once more in the *non-additive* fragment of linear logic ...

Does this point at something typically additive missing from our 'additive' steps - maybe even from classical logic?

16 On the dynamical pertinence of style

We have seen above that the different formulations of the logical introduction rules have two possible uniform interpretations in linear logic: we can interpret them either in the non-additive or in the non-multiplicative fragment. Of course this is reminiscent of the well-known fact that by means of the structural rules different formulations of the introduction rules for the binary connectives are interderivable in classical (and intuitionistic) sequent calculus. The decorations however provide us with more than this: we also have simulability of the dynamics. And it is precisely in order to achieve this dynamical faithfulness that we exploited the greater expressiveness of linear logic: in a pure classical setting, in general derived rules will fail to simulate the dynamics of the original ones. In order to explain why, let us have a closer look at this 'sex-change' of introduction rules in classical (intuitionistic) sequent-calculus.

- *from multiplicative to additive*

Depending on the orientation, the additivization of the multiplicative unary rule

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

is

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma, A \vdash A \to B, \Delta}}{\Gamma \vdash A \to B, A \to B, \Delta}}{\Gamma \vdash A \to B, \Delta} \qquad \text{or} \qquad \dfrac{\dfrac{\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, B, \Delta}}{\Gamma \vdash A \to B, A \to B, \Delta}}{\Gamma \vdash A \to B, \Delta}$$

Note that we introduce structural rules on the main formula. This means that in case the main formula is *attractive* and active in a structural cut, the cut's type may change (L can become S2, or, worse, S2 can become S1); consequently, in general we will not be able to simulate the reduction of the original proof by a reduction of the additivized derivation. (The reader may try his hand at finding a concrete example - the problem, of course, is with the *structural* steps.)

- *from additive to multiplicative*

We multiplicativize the additive unary rules as follows:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A \to B, \Delta} \qquad \text{and} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

become

$$\dfrac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash A \to B, \Delta} \qquad \text{resp.} \qquad \dfrac{\dfrac{\Gamma \vdash B, \Delta}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash A \to B, \Delta}$$

Here the problem is that only when the weakened formula is *attractive*, we can guarantee that in case of a logical reduction the appropriate subderivation will disappear. So again we will in general not be able to simulate the reduction of the original proof by a reduction of the multiplicativized derivation - now the problem is with the *logical* step.
So: style-change of a *unary* introduction rule involves the introduction of either a contraction of the *main* formula in the rule, or a weakening of a formula *active* in the rule, and, whereas in linear logic we can, by a subtle exponentialization, *isolate* this additional use of structural rules for simulation from the structural rules that are simply present already in the derivation, in classical logic *per se* this is impossible.
The distinction is that between the decoration of e.g. an additive unary rule and the decoration of the *multiplicativization* of an additive unary rule in **naLL**. Indeed, this was the point made already in section 13!
On the other hand, style-change of a *binary* introduction rule involves only the addition of structural rules on contextformulas. Therefore all goes well for the binary rules... we can take either the *multiplicative* or the *binary* one, and simulate the other. It is the *unary* rules that are important from the dynamical point of view. And it is with them that a certain pertinence of style, also in *classical* logic, resides.

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Appendix 1: LK, classical logic

Identity axiom and cut rule:

$$\text{(Ax)}\quad A \Rightarrow A \qquad\qquad \text{(cut)}\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Negation rules:

$$\text{(L}\neg\text{)}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad\qquad \text{(R}\neg\text{)}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

Multiplicative logical rules:

$$\text{(L}_m \rightarrow\text{)}\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \text{(R}_m \rightarrow\text{)}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

$$\text{(L}_m\vee\text{)}\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \text{(R}_m\vee\text{)}\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$\text{(R}_m\wedge\text{)}\ \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B, \Delta_1, \Delta_2} \qquad\qquad \text{(L}_m\wedge\text{)}\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

Additive logical rules:

$$\text{(R}_a \rightarrow\text{)}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \qquad\qquad \text{(L}_a \rightarrow\text{)}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

$$\text{(R}_a\vee\text{)}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad\qquad \text{(L}_a\vee\text{)}\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$\text{(L}_a\wedge\text{)}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad\qquad \text{(R}_a\wedge\text{)}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

Rules for the quantifiers (y, Y not free in Γ, Δ):

$$\text{(L}\forall\text{)}\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \quad \text{(R}\forall\text{)}\ \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta} \quad \text{(L}\forall_2\text{)}\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \quad \text{(R}\forall_2\text{)}\ \frac{\Gamma \Rightarrow \Delta, A[Y/X]}{\Gamma \Rightarrow \Delta, \forall X\, A}$$

$$\text{(L}\exists\text{)}\ \frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta} \quad \text{(R}\exists\text{)}\ \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta} \quad \text{(L}\exists_2\text{)}\ \frac{\Gamma, A[Y/X] \Rightarrow \Delta}{\Gamma, \exists X\, A \Rightarrow \Delta} \quad \text{(R}\exists_2\text{)}\ \frac{\Gamma \Rightarrow \Delta, A[T/X]}{\Gamma \Rightarrow \Delta, \exists X\, A}$$

Structural rules:
Weakening and contraction, left and right.

Appendix 2: LL, classical linear logic

Identity axiom and cut rule:

$$(\mathrm{Ax})\quad A \Rightarrow A \qquad\qquad (\mathrm{cut})\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Rules and axioms for the constants:

$$(\mathrm{L}\mathbf{1})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathbf{1} \Rightarrow \Delta} \qquad\qquad (\mathrm{R}\mathbf{1})\quad \Rightarrow \mathbf{1}$$

$$(\text{no L}\top) \qquad\qquad (\mathrm{R}\top)\quad \Gamma \Rightarrow \top, \Delta$$

$$(\mathrm{L}\bot)\quad \bot \Rightarrow \qquad\qquad (\mathrm{R}\bot)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot}$$

$$(\mathrm{L}\mathbf{0})\quad \Gamma, \mathbf{0} \Rightarrow \Delta \qquad\qquad (\text{no R}\mathbf{0})$$

Negation rules:

$$(\mathrm{L}^{\bot})\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, A^{\bot} \Rightarrow \Delta} \qquad\qquad (\mathrm{R}^{\bot})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A^{\bot}, \Delta}$$

Multiplicative logical rules:

$$(\mathrm{L}\multimap)\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}\multimap)\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}$$

$$(\mathrm{L}\wp)\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \wp B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}\wp)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \wp B, \Delta}$$

$$(\mathrm{R}\otimes)\ \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B, \Delta_1, \Delta_2} \qquad\qquad (\mathrm{L}\otimes)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta}$$

Additive logical rules:

$$(\mathrm{R}\rightsquigarrow)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \rightsquigarrow B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightsquigarrow B, \Delta} \qquad\qquad (\mathrm{L}\rightsquigarrow)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \rightsquigarrow B \Rightarrow \Delta}$$

$$(\mathrm{R}\oplus)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \oplus B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \oplus B, \Delta} \qquad\qquad (\mathrm{L}\oplus)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \oplus B \Rightarrow \Delta}$$

$$(\mathrm{L}\&)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \,\&\, B \Rightarrow \Delta} \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \,\&\, B \Rightarrow \Delta} \qquad\qquad (\mathrm{R}\&)\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \,\&\, B, \Delta}$$

Rules for the first-order quantifiers (y not free in Γ, Δ):

$$(\mathrm{L}\forall)\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \qquad (\mathrm{R}\forall)\ \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}$$

$$(\mathrm{L}\exists)\ \frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta} \qquad (\mathrm{R}\exists)\ \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta}$$

Rules for the second-order quantifiers (Y not free in Γ, Δ):

$$(\mathrm{L}\forall_2)\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \qquad (\mathrm{R}\forall_2)\ \frac{\Gamma \Rightarrow \Delta, A[Y/X]}{\Gamma \Rightarrow \Delta, \forall X\, A}$$

$$(\mathrm{L}\exists_2)\ \frac{\Gamma, A[Y/X] \Rightarrow \Delta}{\Gamma, \exists X\, A \Rightarrow \Delta} \qquad (\mathrm{R}\exists_2)\ \frac{\Gamma \Rightarrow \Delta, A[T/X]}{\Gamma \Rightarrow \Delta, \exists X\, A}$$

Exponential structural rules:

$$(\mathrm{W}!)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad (\mathrm{W}?)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

$$(\mathrm{C}!)\ \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad (\mathrm{C}?)\ \frac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

Exponential contextual or box rules:

$$(\mathrm{L}?)\ \frac{!\Gamma, A \Rightarrow ?\Delta}{!\Gamma, ?A \Rightarrow ?\Delta} \qquad (\mathrm{R}!)\ \frac{!\Gamma \Rightarrow A, ?\Delta}{!\Gamma \Rightarrow !A, ?\Delta}$$

Exponential dereliction rules:

$$(\mathrm{R}?)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow ?A, \Delta} \qquad (\mathrm{L}!)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$