An Abstract Approach to Stratification in Linear Logic

Pierre Boudes[a], Damiano Mazza[a], Lorenzo Tortora de Falco[b]

[a]*LIPN, UMR 7030 CNRS-Université Paris 13*
[b]*Dipartimento di Filosofia, Università Roma Tre*

Abstract

We study the notion of stratification, as used in subsystems of linear logic with low complexity bounds on the cut-elimination procedure (the so-called light logics), from an abstract point of view, introducing a logical system in which stratification is handled by a separate modality. This modality, which is a generalization of the paragraph modality of Girard's light linear logic, arises from a general categorical construction applicable to all models of linear logic. We thus learn that stratification may be formulated independently of exponential modalities; when it is forced to be connected to exponential modalities, it yields interesting complexity properties. In particular, from our analysis stem three alternative reformulations of Baillot and Mazza's linear logic by levels: one geometric, one interactive, and one semantic.

Key words: Implicit computational complexity, light linear logics, denotational semantics, categorical semantics.

Introduction

The notion of *stratification* in linear logic may be informally presented as a limitation of the dynamics of cut-elimination: in a stratified subsystem of linear logic, proofs may be seen as partitioned into *strata* which never "communicate" with each other, in the sense that no cut between two dual formulas belonging to different strata will ever appear during cut-elimination. All extant time-bounded subsystems of linear logic (with the exception of Lafont's soft linear logic) use some form of stratification to control the complexity of the cut-elimination procedure, which would otherwise be non-elementary (as a consequence of the well known result of Statman (1979), modulo the translation of intuitionistic logic in linear logic given by Girard (1987)).

In the original systems introduced by Girard (1998), namely elementary and light linear logic, stratification coincided with the *exponential depth*, *i.e.*, the nesting level of the logical rules introducing the exponential modality "of

course". More recently, Baillot and Mazza (2010) introduced a more general form of stratification, still connected with the exponential modalities but no longer coinciding with depth, which keeps ensuring the desired complexity properties.

The present paper originated from a semantic investigation of this more liberal stratification. Our (successful!) attempt to define a denotational semantics for Baillot and Mazza's system naturally revealed that *stratification may actually be formulated independently of exponential modalities; when it is somehow forced to be connected with them, it yields interesting complexity properties*. This is essentially because exponential modalities in linear logic are in control of duplication, the only true source of complexity in cut-elimination.

The above is the main message brought forth by this paper. We shall now proceed to describe its contents more thoroughly.

Computational complexity and proof theory. Ever since its early years, the theory of computational complexity has shown strong and deep connections with mathematical logic. The first major evidence of this came from Fagin's work (1974), which gave birth to a rich research field now known as descriptive complexity, based on finite model theory. Structural proof theory remained out of the picture for some time, but eventually it also proved to be a potentially fruitful tool for the study of complexity classes, as shown by the work of Leivant (1994), Leivant and Marion (1993), and Girard et al. (1992).

The approach originally taken in Girard et al. (1992), and perfected by Girard (1998), is the focus of the present work. At its heart there is the so-called Curry-Howard correspondence, which sees logical proofs as programs, and cut-elimination as their execution. From this perspective, it is not so much the expressiveness of a logical system *as a language* which matters, but the complexity of its cut-elimination procedure: if a logical system has a low-complexity cut-elimination, its proofs will necessarily correspond to low-complexity programs. This approach, which has a marked proof-theoretic nature and, as such, is orthogonal to the model-theoretic methods of descriptive complexity, falls within the larger area of *implicit computational complexity*, whose concrete aim is to define programming languages enjoying intrinsic complexity bounds, *i.e.*, automatically ensured at compile time. Apart from those already mentioned above, other notable examples of work in this field, not necessarily related to logic, are given by Bellantoni and Cook (1992); Jones (1999); Hofmann (2003); Schwichtemberg (2006).

Light linear logics. The key ingredient of Girard's approach to implicit computational complexity is the use of linear logic (Girard, 1987). As a matter of fact, Girard's idea originated from some older results of Grishin (1982), who observed that naive set theory is consistent when formulated in substructural logics where the contraction rule is forbidden. In linear logic, structural rules are managed by the so-called *exponential* modalities. Altering the behavior of these modalities offers a way to define logical systems in which contraction is not forbidden, but in which naive set theory is still consistent.

The control on structural rules is flexible enough as to allow crafting systems with interesting complexity properties: *light linear logic* (**LLL**) exactly captures deterministic polynomial time, and *elementary linear logic* (**ELL**) exactly captures elementary time. The system **LLL** actually uses an additional, self-dual exponential modality, called *paragraph*, which is necessary for reasons of expressiveness (*i.e.*, to represent all polytime functions), but whose logical status has otherwise remained somewhat unclear.

In the ensuing years, a wealth of work has been done to study, extend, and find applications for these logical systems, which are collectively known as *light logics*. For what concerns the refinement and extension of their syntax, and the characterization of further complexity classes (like deterministic logarithmic space and polynomial space), let us mention Asperti and Roversi (2002); Danos and Joinet (2003); Mazza (2006); Schöpp (2007); Baillot and Mazza (2010); Gaboardi et al. (2011). As far as applications are concerned, several type systems guaranteeing bounded-time execution for functional languages (*e.g.*, the pure λ-calculus), together with efficient typing algorithms, have been devised, for instance, in Baillot and Terui (2004); Atassi et al. (2006); Coppola et al. (2005); Coppola and Martini (2006). Last but not least, there have also been efforts for studying **LLL**, **ELL** and related systems from the semantic point view, in the sense of truth semantics (Kanovich et al., 2003), and denotational semantics (Baillot and Pedicini, 2001; Baillot, 2004; Laurent and Tortora de Falco, 2006; Lago and Laurent, 2008; Laurent, 2009).

Stratification. The restriction that Girard imposed on the exponential modalities of linear logic is a form of *stratification*. In his original paper, Girard (1998) justifies the need for, and meaning of stratification by analyzing Russel's antinomy in linear logic (the most well known way of showing that naive set theory is inconsistent). The phenomenon observed by Girard may be reformulated in the λ-calculus, where the term $x \notin x$ of Russel's antinomy has a computational equivalent in the self-application xx. In presence of a type D satisfying $D \Rightarrow D = D$, a type derivation for xx in intuitionistic sequent calculus looks as follows:[1]

$$\frac{\dfrac{}{x_1 : D \vdash x_1 : D} \quad \dfrac{}{y : D \vdash y : D}}{\dfrac{x_1 : D, x_2 : D \Rightarrow D \vdash x_2 x_1 : D}{x : D \vdash xx : D}}$$

where the last rule is a contraction unifying the variables x_1 and x_2 (which have the same type thanks to the equality satisfied by D). The fundamental observation now is that the two occurrences of D are of different nature, and one should not be allowed to contract them. Translating the above proof in

---

[1] We are using the standard sequent-calculus decoration of the left-implication rule:

$$\frac{\Gamma \vdash u : A \quad \Delta, y : B \vdash t : C}{\Gamma, \Delta, x : A \Rightarrow B \vdash t[xu/y] : C}$$

linear logic, one considers a type D satisfying $!D \multimap D = D$, and the derivation becomes

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{x_1 : D \vdash x_1 : D}{x_1 : !D \vdash x_1 : D} \text{ Dereliction}
    }{x_1 : !D \vdash x_1 : !D} \text{ Promotion} \qquad \cfrac{}{y : D \vdash y : D}
  }{
    \cfrac{x_1 : !D, x_2 : !D \multimap D \vdash x_2 x_1 : D}{x_1 : !D, x_2 : !D \vdash x_2 x_1 : D} \text{ Dereliction}
  }
}{x : !D \vdash xx : D}
$$

The different nature of the two occurrences of $!D$ may now be put forward by saying that, while both occurrences are introduced by dereliction rules, the one typing x_1 also crosses the context of a promotion rule, whereas the one typing x_2 does not. In the terminology of linear logic proofs (*i.e.*, proof nets), one says that the two occurrences of $!D$ are not introduced at the same *exponential depth*.

There is a simple way of ensuring that the contraction rule never mixes two occurrences of a formula introduced at different depths: collapsing dereliction and promotion into one single rule, called *functorial promotion*. From the point of view of provability, this ensures that $!A \multimap !B$ may be proved from $A \multimap B$, but that neither $!A \multimap A$ nor $!A \multimap !!A$ are provable; these latter two formulas represent valid linear logic principles, called *dereliction* and *digging*, respectively. In categorical terms, the modality $!$ is a functor, indeed a comonad, with dereliction and digging as its counit and comultiplication, respectively.

If we forget the additive connectives of linear logic, this is precisely the definition of elementary linear logic. This logical system, in which cut-elimination holds even in presence of fixpoints of any type, induces a λ-calculus whose terms are all strongly normalizing and in which, moreover, the functions representable on Church numerals are exactly the elementary functions, *i.e.*, functions which can be computed by a Turing machine in time bounded by a tower of exponentials of fixed height (Danos and Joinet, 2003). To obtain a system characterizing deterministic polynomial time (**LLL**), one must add a further restriction on the context of functorial promotion; in terms of formulas, the provability of $!A \otimes !B \multimap !(A \otimes B)$ is excluded, *i.e.*, the functor $!$ is no longer monoidal.

It must be observed that the stratification principle is not at all new in proof theory or implicit computational complexity. In the latter field, one example is found in the work of Leivant and Marion (1993), in which *tiers*, which are integers assigned to subterms of λ-terms, are used to induce a stratification on the λ-calculus, yielding characterizations of interesting complexity classes. In the former field, stratification was used long ago by Quine (1937) in his axiomatization of set theory known today as "New Foundations" (NF). In NF, naive comprehension, which postulates the existence of the set $\{x \mid A(x)\}$, is restricted to the case where the formula A is *stratified*, which means that there exists a function f from the terms appearing in A to the integers such that, for every atomic subformula of A of the form $t \in u$ (resp. $t = u$), one has $f(u) = f(t) + 1$ (resp. $f(t) = f(u)$). The theory NF escapes Russel's antinomy because the formula $\neg(x \in x)$ is *not* stratified; note the similarity with the

refusal of self-application in **ELL**.

Separating stratification from exponential depth. Recently, Baillot and Mazza (2010) proposed a new subsystem of linear logic corresponding to elementary time, *linear logic by levels* ($\mathbf{L^3}$). This system is also based on a form of stratification, but in this case it is achieved by retaining only those linear logic proofs π for which there exists a function from the occurrences of formulas in π to the integers, called *indexing*, which satisfies certain conditions (this reminds ever more closely of Quine's NF). In a nutshell, these conditions state that axioms introduce dual occurrences of identical level, and that the level of an occurrence of formula is decreased only when it is the principal occurrence of a dereliction or promotion rule.

Interestingly, this stratification not only excludes self-application and guarantees an elementary bound on normalization, but turns out to be a generalization of Girard's stratification: **ELL** is exactly the subsystem of $\mathbf{L^3}$ in which the function assigning to each occurrence its own depth is a valid indexing. This generalization is strict, both in the sense of proofs and provability: there exist **ELL**-provable formulas which admit more proofs in $\mathbf{L^3}$, and there exist $\mathbf{L^3}$-provable formulas which are not provable in **ELL**. Although no concrete use has currently been found for these additional formulas and proofs, $\mathbf{L^3}$ gives us at least one clear, and potentially interesting message: *stratification does not need to coincide with exponential depth.* However, even if separated from the depth, stratification in $\mathbf{L^3}$ is still connected to the exponential modalities; for instance, the formulas corresponding to dereliction and digging are still not provable in $\mathbf{L^3}$.

Denotational semantics: an abstract approach to stratification. Denotational semantics originated in the work of Scott and Strachey (1971) and Scott (1976) as an attempt to interpret in a non-trivial way the quotient induced on λ-terms by β-equivalence. This amounts to finding an invariant of reduction, a question which may be extended to logical systems enjoying cut-elimination. Since its introduction, denotational semantics has proved to be an absolutely essential tool in computer science and proof theory, providing a wealth of information and insights into the nature of computation and formal proofs. A striking example is given by linear logic itself, which arose precisely from a denotational analysis of intuitionistic logic (Girard, 1987).

After the successful introduction of denotational semantics for **ELL** (namely, the work of Baillot (2004) and that of Laurent and Tortora de Falco (2006)), it seemed natural to attempt to analyze the stratification underlying $\mathbf{L^3}$ from the denotational point of view. The result of such an analysis forms the contents of the present paper, whose message broadens that of $\mathbf{L^3}$. Indeed, soon after developing our semantic construction for $\mathbf{L^3}$ (which we present in Sect. 2.3), we realized that it suggested a more general syntax than that of $\mathbf{L^3}$ itself, in which exponential modalities and strata are completely independent. This more general syntax has at least two alternative presentations in terms of sequent calculus, and one in terms of proof nets, all of them shown in Sect. 1. The

induced logical system, which we call *stratified linear logic* ($\mathbf{LL}_\S$), has an additional modality with respect to linear logic, the self-dual *paragraph* $\S$, which is in charge of controlling stratification. In $\mathbf{LL}_\S$, dereliction and digging are provable, $\S A \multimap \S B$ is provable from $A \multimap B$, but $\S A \multimap A$ and $A \multimap \S A$ (or $\S A \multimap \S\S A$) are not provable in general, which is the essence of stratification.

In $\mathbf{LL}_\S$, $\mathbf{L^3}$ appears as a fragment, in which the exponential modalities are forced to be "tied" to paragraph modalities: $!A$ is replaced by $!\S A$, and $?A$ is replaced by $?\S A$. Additionally, the paragraph modality itself is a generalization of the paragraph modality of $\mathbf{LLL}$, which justifies our terminology and notation. This allows to define polytime subsystems within $\mathbf{L^3}$ generalizing $\mathbf{LLL}$, as already shown in Baillot and Mazza (2010).

The paragraph modality. Our denotational analysis brings a new understanding of the exponential modalities of light logics: together with the control of duplication, which is their usual task in linear logic, they are also charged with the additional task of controlling stratification, which is represented by the paragraph modality.

From the semantic point of view, we interpret the paragraph modality in "augmented" models of linear logic. More precisely, we define a categorical construction, represented by a 2-endofunctor $\mathrm{Inv}(-)$ of the 2-category of symmetric monoidal categories, which takes a model of linear logic $\mathcal{L}$ and yields another model of linear logic $\mathrm{Inv}(\mathcal{L})$; this latter model is practically equivalent to the original one, but it has "more space", and this extra space is what allows the definition of a non-trivial paragraph functor.

Concretely, the Inv construction may be understood by looking at its action on categories. Given a category $\mathcal{A}$, we define an *object with involutions* of $\mathcal{A}$ as a pair (A, s), where A is an object of $\mathcal{A}$ and s is a $\mathbb{Z}$-indexed sequence of involutions of A, *i.e.*, automorphisms of A such that $s_k \circ s_k = id_A$ for all $k \in \mathbb{Z}$. Objects with involutions have a natural notion of morphism: a morphism from (A, s) to (B, t) is a morphism $f : A \to B$ such that $t_k \circ f \circ s_k = f$. If the category $\mathcal{A}$ is symmetric monoidal, the objects with involutions of $\mathcal{A}$ and their morphisms may themselves be arranged in a symmetric monoidal category, which is $\mathrm{Inv}(\mathcal{A})$.

It turns out that $\mathrm{Inv}(\mathcal{A})$ has all the structure needed to provide a model of linear logic as soon as $\mathcal{A}$ does (Theorem 13). However, in $\mathrm{Inv}(\mathcal{A})$ we may now define a functor $\S(-)$ which acts on objects by "shifting" the sequence of involutions, *i.e.*, $\S(A, s) = (A, (s_{k-1})_{k\in\mathbb{Z}})$, and which acts as the identity on morphisms. It is possible to show that such a functor is never trivial, *i.e.*, it is never isomorphic to the identity functor, unless the original model of linear logic (in the category $\mathcal{A}$) is itself trivial (Theorem 15).

Alternative formulations of $\mathbf{L^3}$. An important contribution of our work, resulting from an immediate application of the semantic construction described above, is the discovery of alternative formulations of $\mathbf{L^3}$.

The first reformulation (Sect. 3.1) is of geometric nature; it is actually unrelated to the semantics, but we present it here because it fits nicely with the other results. It is a presentation of $\mathbf{L^3}$ in terms of proof nets by means of a

correctness criterion, extending the usual one by Danos and Regnier (1989). Its advantage is to avoid mentioning the notion of indexing, shifting from an existential condition (there exists a function satisfying...) to a universal one (every cycle satisfies...), which is arguably of interest.

The second reformulation (Sect. 3.3) is "interactive", and arises when applying the Inv construction to the syntactic category of formulas and proof nets. Approximatively speaking, we prove that a cut-free linear logic proof net π is in $\mathbf{L^3}$ iff it "interacts well" with all "tests" (Theorem 21), where a test is a certain kind of proof net which may interact with π by means of a cut rule (the interaction being cut-elimination).

Finally, we provide a semantic characterization of $\mathbf{L^3}$ (Sect. 3.4), which may be roughly formulated as follows: given a model of linear logic $\mathcal{L}$ which satisfies a certain condition we call *swap-sensitivity*, we have that a cut-free linear logic proof net π of conclusion A is in $\mathbf{L^3}$ iff its denotational interpretation $[\![\pi]\!]$ in $\mathcal{L}$ is a morphism of $\mathrm{Inv}(\mathcal{L})$ from the tensor unit to a certain object with involutions $(\![A]\!)$, which depends solely on the formula A (Theorem 24). In other words, provided swap-sensitivity holds, the semantics is able to "detect" whether a proof net fails to admit a valid indexing, or fails to satisfy any of the two equivalent conditions mentioned above. The swap-sensitivity condition is a bit technical, but it is very mild: it is satisfied by all models of linear logic we are aware of.

1. Stratified Linear Logic

1.1. The logical system

The formulas of stratifed linear logic ($\mathbf{LL_\S}$) are generated by the following grammar, where $X, X^\perp$ range over a denumerable set of propositional variables:

A, B	::=	$X \mid X^\perp$	propositional atoms
	$\mid$	$1 \mid A \otimes B \mid \perp \mid A \,\invamp\, B$	multiplicative connectives and units
	$\mid$	$\top \mid A \,\&\, B \mid 0 \mid A \oplus B$	additive connectives and units
	$\mid$	$\forall X.A \mid \exists X.A$	second order quantifiers
	$\mid$	$!A \mid ?A \mid \S A$	exponential modalities and paragraph.

Linear negation is defined through De Morgan laws:

$$
\begin{array}{rclcrcl}
(X)^\perp & = & X^\perp & \qquad & (X^\perp)^\perp & = & X \\
(1)^\perp & = & \perp & & (\perp)^\perp & = & 1 \\
(A \otimes B)^\perp & = & A^\perp \,\invamp\, B^\perp & & (A \,\invamp\, B)^\perp & = & A^\perp \otimes B^\perp \\
(\top)^\perp & = & 0 & & (0)^\perp & = & \top \\
(A \,\&\, B)^\perp & = & A^\perp \oplus B^\perp & & (A \oplus B)^\perp & = & A^\perp \,\&\, B^\perp \\
(\forall X.A)^\perp & = & \exists X.A^\perp & & (\exists X.A)^\perp & = & \forall X.A^\perp \\
(!A)^\perp & = & ?A^\perp & & (?A)^\perp & = & !A^\perp \\
\end{array}
$$

$$
(\S A)^\perp = \S A^\perp
$$

Two connectives or modalities exchanged by negation are said to be *dual*; the paragraph modality $\S$ is self-dual.

Linear implication is defined as $A \multimap B = A^\perp \,\invamp\, B$. Multisets of formulas will be ranged over by Γ, Δ. The notations $?\Gamma$ and $\S\Gamma$ will stand for multisets containing formulas all starting with a $?$ or $\S$ modality, respectively.

The most convenient way of formulating a sequent calculus for $\mathbf{LL}_\S$ is using *2-sequents* (Masini, 1992; Guerrini et al., 1998). A 2-sequent is a sort of bidimensional sequent; for our purposes, we define it simply as a sequent in which every occurrence of formula is associated with a relative integer, called its *index*. It is important to note that indexes are not part of the logical language; occurrences of subformulas are not indexed. Moreover, we observe that natural numbers could as well be used as indexes, but it is more natural (and technically simpler) to allow negative indexes.

The 2-sequent rules defining the admissible derivations of $\mathbf{LL}_\S$ are given in Table 1. Indexes are annotated as superscripts; we keep using Γ, Δ to range over multisets of indexed formulas.

The 2-sequent calculus of $\mathbf{LL}_\S$ admits cut-elimination; the cut-elimination rules are those of linear logic, plus a rule transforming the cut

$$
\dfrac{\dfrac{\vdash \Gamma, A^{i+1}}{\vdash \Gamma, \S A^i} \quad \dfrac{\vdash \Delta, A^{\perp\,i+1}}{\vdash \Delta, \S A^{\perp\,i}}}{\vdash \Gamma, \Delta}
$$

into the cut

$$
\dfrac{\vdash \Gamma, A^{i+1} \quad \vdash \Delta, A^{\perp\,i+1}}{\vdash \Gamma, \Delta}
$$

For the propositional fragment, the cut-elimination property may be proved by standard arguments, using the usual notions of degree and rank of cuts. We shall give detailed cut-elimination rules for proof nets (Sect. 1.3), which is why we do not linger any further on this topic here.

The additive group $\mathbb{Z}$ acts on multisets of indexed formulas in the obvious way: given $k \in \mathbb{Z}$ and $\Gamma = A_1^{i_1}, \ldots, A_n^{i_n}$, we set $k \cdot \Gamma = A_1^{i_1+k}, \ldots, A_n^{i_n+k}$. The action can easily be lifted to derivations: if δ is a derivation of $\vdash \Gamma$, we get a derivation $k \cdot \delta$ of $\vdash k \cdot \Gamma$. Then, we obtain

$$\frac{}{\vdash A^{\perp\,i}, A^i}\ \text{Axiom} \qquad\qquad \frac{\vdash \Gamma, A^i \quad \vdash \Delta, A^{\perp\,i}}{\vdash \Gamma, \Delta}\ \text{Cut}$$

$$\frac{}{\vdash 1^i}\ \text{One} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp^i}\ \text{Bottom}$$

$$\frac{\vdash \Gamma, A^i \quad \vdash \Delta, B^i}{\vdash \Gamma, \Delta, A \otimes B^i}\ \text{Tensor} \qquad\qquad \frac{\vdash \Gamma, A^i, B^i}{\vdash \Gamma, A \,\mathfrak{R}\, B^i}\ \text{Par}$$

$$\frac{}{\vdash \Gamma, \top^i}\ \text{Top}$$

$$\frac{\vdash \Gamma, A^i \quad \vdash \Gamma, B^i}{\vdash \Gamma, A \,\&\, B^i}\ \text{With} \qquad\qquad \frac{\vdash \Gamma, A_k^i}{\vdash \Gamma, A_1 \oplus A_2^{\,i}}\ \text{Plus } (k \in \{1,2\})$$

$$\frac{\vdash \Gamma, A^i}{\vdash \Gamma, \forall X.A^i}\ \text{For all } (*) \qquad\qquad \frac{\vdash \Gamma, A^i}{\vdash \Gamma, \exists X.A^i}\ \text{Exists}$$

$$\frac{\vdash ?\Gamma, A^i}{\vdash ?\Gamma, !A^i}\ \text{Promotion} \qquad\qquad \frac{\vdash \Gamma, A^i}{\vdash \Gamma, ?A^i}\ \text{Dereliction}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A^i}\ \text{Weakening} \qquad\qquad \frac{\vdash \Gamma, ?A^i, ?A^i}{\vdash \Gamma, ?A^i}\ \text{Contraction}$$

$$\frac{\vdash \Gamma, A^{i+1}}{\vdash \Gamma, \S A^i}\ \text{Paragraph}$$

Table 1: Stratified sequent calculus. In every rule, $i \in \mathbb{Z}$. The condition $(*)$ is that X is not free in Γ

Lemma 1. *The 2-sequent $\vdash \Gamma$ is derivable iff the 2-sequent $\vdash k \cdot \Gamma$ is derivable, for all $k \in \mathbb{Z}$.* □

Therefore, the "absolute" value of indexes in a 2-sequent does not matter at all; what matters is their relative value, with respect to each other.

A natural property of the sequent calculus of linear logic, which is the linear incarnation of the so-called Deduction Theorem, is that "commas are $\invamp$ connectives", *i.e.*, $\vdash A, B$ is provable iff $\vdash A \invamp B$ is provable. This, which from a semantic point of view is equivalent to asking the closure of the category underlying $\mathbf{LL}_\S$, is obviously false for derivations in the calculus of Table 1. These considerations lead us to define the actual *proofs* of $\mathbf{LL}_\S$ as follows:

Definition 1 ($\mathbf{LL}_\S$ proofs). A *proof* of $\mathbf{LL}_\S$ is a derivation in the 2-sequent calculus of Table 1 such that all occurrences of formulas in the conclusion have the same index (in other words, the 2-sequent derived is actually a sequent).

Observe that second order linear logic is obviously the fragment of $\mathbf{LL}_\S$ in which the modality $\S$ is removed. In that case, indexes are useless, and provability coincides with provability in linear logic. Another, less trivial fragment of $\mathbf{LL}_\S$ will be introduced in the next section. For the moment, let us observe that $\mathbf{LL}_\S$ itself adds virtually nothing to linear logic, because the paragraph modality is close to being trivial. In fact, the reader may check that it commutes to *all* logical connectives, so that, for instance, $\S 1$, $\S(A \otimes B)$, $\S 0$, $\S(A \oplus B)$, $\S \exists X.A$, and $\S!A$ are all provably isomorphic to 1, $\S A \otimes \S B$, 0, $\S A \oplus \S B$, $\exists X.\S A$, and $!\S A$, respectively.

The only purpose of the paragraph modality is to enforce stratification, as will be discussed at the end of Sect. 1.4, when we will have defined cut-elimination more precisely. In terms of provability, the essential meaning of stratification is in the impossibility (which is easy to check in presence of cut-elimination) of deriving any of the implications $X \multimap \S X$ and $\S X \multimap X$, with X a propositional atom. When suitably linked with the exponential modalities, stratification has remarkable complexity effects, as explained in the next section.

The fact that the paragraph modality commutes to all logical operations suggests that $\mathbf{LL}_\S$ may be reformulated by explicitly writing these commutations into its rules. We thus obtain a "usual" sequent calculus, *i.e.*, without indexes on formulas, defined in Table 2. It is not hard to see that this is an equivalent formulation of $\mathbf{LL}_\S$:

Proposition 2. *The 2-sequent $\vdash \Gamma^i$ (i.e., with all occurrences of formulas having the same index) is derivable in the calculus of Table 1 iff $\vdash \Gamma$ is derivable in the calculus of Table 2.*

PROOF. Note that the rules of Table 2 are all derivable in the calculus of Table 1, as soon as we suppose that we translate a sequent $\vdash \Gamma$ with a 2-sequent of the form $\vdash \Gamma^i$. Vice versa, a straightforward induction shows that, if the 2-sequent $\vdash A_1^{i_1}, \ldots, A_n^{i_n}$ is derivable in the calculus of Table 1, then the sequent $\vdash \S^{i_1 - m} A_1, \ldots, \S^{i_n - m} A_n$ is derivable in the calculus of Table 2, with $m = \min_{1 \le k \le n} i_k$. □

10

$$\frac{\vdash \Gamma}{\vdash \Gamma, \S^k \bot} \; \text{Bottom} \qquad\qquad \frac{\vdash \Gamma, \S^k A, \S^k B}{\vdash \Gamma, \S^k (A \,\invamp\, B)} \; \text{Par}$$

$$\frac{}{\vdash \Gamma, \S^k \top} \; \text{Top} \qquad \frac{\vdash \Gamma, \S^k A \quad \vdash \Gamma, \S^k B}{\vdash \Gamma, \S^k (A \,\&\, B)} \; \text{With} \qquad \frac{\vdash \Gamma, \S^k A}{\vdash \Gamma, \S^k \forall X.A} \; \text{For all } (*)$$

$$\frac{\vdash \,?\Gamma, \S^k A}{\vdash \,?\Gamma, \S^k !A} \; \text{Promotion} \qquad \frac{\vdash \Gamma, \S^k A}{\vdash \Gamma, \S^k ?A} \; \text{Dereliction} \qquad \frac{\vdash \Gamma}{\vdash \S\Gamma} \; \text{Paragraph}$$

Table 2: Alternative formulation of $\mathbf{LL}_\S$ in sequent calculus. The rules handling the connectives not mentioned, as well as the axiom, cut, weakening and contraction rules, are the same as those of standard linear logic, *i.e.*, those of Table 1 with no index annotation. In all rules, $\S^k A$ stands for the formula A with $k \geq 0$ paragraph modalities prepended to it (note that A may itself start with a paragraph modality). As usual, the condition $(*)$ is that X is not free in Γ.

The reader may have noticed that, in the alternative formulation of $\mathbf{LL}_\S$, the commutations of the paragraph modality are added only to one "half" of the connectives of linear logic (namely what are known as the *negative* connectives). This is because the implications $1 \multimap \S1$, $\S A \otimes \S B \multimap \S(A \otimes B)$, $0 \multimap \S 0$, $\S A \oplus \S B \multimap \S(A \oplus B)$, and $\exists X.\S A \multimap \S \exists X.A$ are all derivable from the usual rules plus the paragraph rule of Table 2. For the exponential modalities, none of the two directions of the needed equivalences holds naturally, so both promotion and dereliction must be modified.

This alternative formulation of $\mathbf{LL}_\S$ has the drawback of having a more complex cut-elimination procedure, involving even more commutations than usual (we prefer not to give any detail here). However, it is semantically interesting, because it allows the denotational interpretation of a proof to be defined, as customary, by induction on the proof itself. This is impossible if we have only Definition 1 as our definition of proof, because a subderivation of a proof need not be itself a proof. We shall find this alternative formulation of $\mathbf{LL}_\S$ particularly useful in Sect. 2.2.

1.2. Bounded time subsystems

Linear logic by levels ($\mathbf{L^3}$) was introduced by Baillot and Mazza (2010) as a generalization of the ideas which led Girard to the definition of elementary linear logic ($\mathbf{ELL}$). Indeed, $\mathbf{L^3}$ captures elementary time: there is a representation of integers (as proofs) in the system such that the representable functions from integers to integers are exactly the elementary functions, *i.e.*, those that can be computed by a Turing machine whose runtime is bounded by a tower of exponentials of fixed height. However, $\mathbf{L^3}$ is a strict supersystem of $\mathbf{ELL}$, both in terms of proofs and provability. It contains a subsystem, called $\mathbf{L^4}$, which captures deterministic polynomial time computation in the same sense, and which extends Girard's light linear logic ($\mathbf{LLL}$).

The original definition presents $\mathbf{L^3}$ as a subsystem of linear logic, using a notion of indexability (*cf.* Definition 14). Here, we may reformulate it as a *fragment* of $\mathbf{LL}_\S$. In fact, Baillot and Mazza (2010) gives a 2-sequent formulation of $\mathbf{L^3}$ which is nearly identical to that of Table 1; the only difference is in the promotion and dereliction rules, which modify the indexes in the same way as the paragraph modality:

$$\frac{\vdash \, ?\Gamma, A^{i+1}}{\vdash \, ?\Gamma, !A^i} \; \text{Promotion} \qquad\qquad \frac{\vdash \Gamma, A^{i+1}}{\vdash \Gamma, ?A^i} \; \text{Dereliction}$$

Then, it is obvious that $\mathbf{L^3}$ coincides with the fragment of $\mathbf{LL}_\S$ whose formulas are obtained by forcing every exponential modality to be preceded by a paragraph, *i.e.*, if $!_{\mathbf{L^3}}$ and $?_{\mathbf{L^3}}$ denote the exponential modalities of $\mathbf{L^3}$, we have

$$!_{\mathbf{L^3}} = !\S \qquad\qquad \text{and} \qquad\qquad ?_{\mathbf{L^3}} = ?\S.$$

Anticipating on proof nets, since the exponential cut-elimination step of $\mathbf{L^3}$ is defined exactly as in Fig. 7, and because the paragraph step (Fig. 8) is trivial, we obviously have that such equalities are sound also with respect to cut-elimination.

This may all be restated as follows: stratification has an effect on the complexity of cut-elimination as soon as the exponential modalities are linked to it. As a matter of fact, all extant stratification-based, complexity-bounded subsystems of linear logic may be seen as subsystems of the fragment of $\mathbf{LL}_\S$ just introduced.

For instance, $\mathbf{ELL}$ corresponds to the subsystem of the above fragment obtained by restricting to 2-sequents of the form

$$\vdash \S\Delta^i, \Gamma^{i+1},$$

where Γ does not contain paragraph formulas. In other words, only two indexes are allowed at the same time, and all paragraph formulas must have a lower index. For what concerns polynomially-bounded subsystems, we refer the reader to Mazza (2006) for a formulation of $\mathbf{LLL}$ as a subsystem of $\mathbf{ELL}$, and to Baillot and Mazza (2010) for a definition of $\mathbf{L^4}$ as a subsystem of $\mathbf{L^3}$.

The only known complexity-related subsystem of linear logic which is not captured by our notion of stratification is Lafont's (2004) soft linear logic ($\mathbf{SLL}$), a polynomially-bounded system which is based on ideas quite different from those originally put forward by Girard. Of course, as a subsystem of linear logic, $\mathbf{SLL}$ is also a subsystem of $\mathbf{LL}_\S$, but in a completely uninteresting way, *i.e.*, our notion of stratification does not intervene in the definition (the paragraph modality is not used). Furthermore, we immediately see that $\mathbf{SLL}$ rejects the idea that one must link exponentials to stratification in order to achieve complexity effects: in fact, this latter system allows the dereliction principle, *i.e.*, the formula $!A \multimap A$ is provable in $\mathbf{SLL}$, for all A; this is impossible in $\mathbf{LL}_\S$ if the ! modality is in linked in any way to the $\S$ modality. In the end, we may say that linking exponentials to stratification is a sufficient condition for obtaining bounded cut-elimination, but it is by no means necessary.
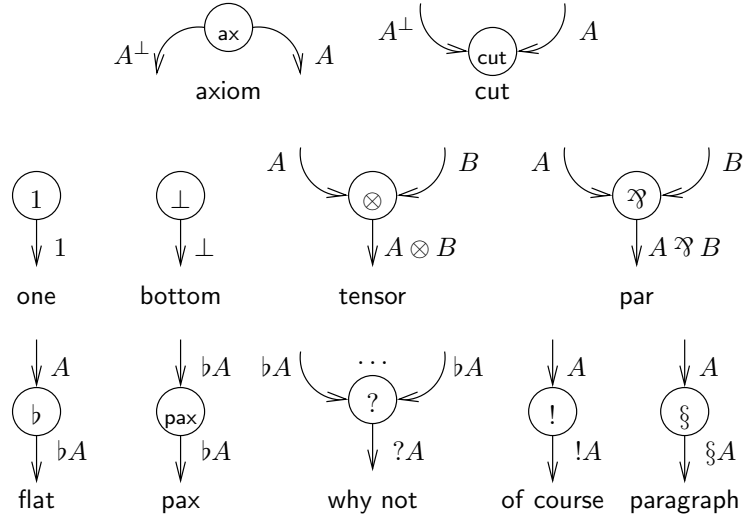
Figure 1: Links.

1.3. Proof nets

The proof theory of $\mathbf{LL}_\S$ may also be developed in terms of proof nets, originally introduced by Girard (1987, 1996), and subsequently reformulated by other authors by means of alternative, but equivalent syntactical definitions. In this paper, we use a combination of the presentations given by Danos and Regnier (1995) and Tortora de Falco (2003), with a slight change in the terminology: the term "proof structure", introduced by Girard (1987) and traditionally used in the literature, is here dismissed in favor of the term *net*. On the contrary, the term proof net, i.e., a net satisfying certain structural conditions (the correctness criterion), retains its usual meaning.

We shall henceforth ignore additive connectives ($\&$, $\oplus$, and their respective units $\top$ and 0) and second order quantifiers, because their representation in proof nets is unnecessarily complex for the purposes of this work.

In the context of proof nets, it is useful to consider $\flat$-*formulas*, which will be denoted by $\flat A$, where A is a formula. Note that $\flat$-formulas are not formulas: they cannot be composed, *i.e.*, they cannot be subformulas of a formula. For instance, $(\flat A) \otimes B$ and $\flat\flat A$ are meaningless expressions.

In the following definition, and throughout the rest of the paper, unless explicitly stated we shall make no distinction between the concepts of *formula* (or $\flat$-formula) and *occurrence of formula*. The same will be done for what we call *links* and their occurrences.

Definition 2 (Net). A *net* is a pair $(\mathcal{G}, \mathsf{B})$, where $\mathcal{G}$ is a finite graph-like object whose nodes are occurrences of what we call *links*, and whose edges are directed and labelled by formulas or $\flat$-formulas; and B is a set of subgraphs of $\mathcal{G}$ called *boxes*.
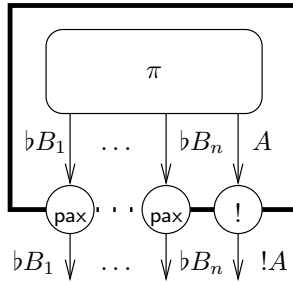
13

Figure 2: A box.

- Links (Fig. 1) are labelled by connectives and constants of $\mathbf{LL}_\S$ (excluding $\&$, $\oplus$, $\top$, and 0), or by one of the labels ax, cut, $\flat$, pax. Two links labelled by dual connectives are said to be *dual*. Each link has an arity and co-arity, which are resp. the number of its incoming and outgoing edges. The arity and co-arity is fixed for all links except why not links, which have co-arity 1 and arbitrary arity. A nullary why not link is also referred to as a weakening link.

- The incoming edges of a link (and the formulas that label them) are referred to as its *premises*; these are ordered (*i.e.*, there is a "left" and a "right" premise) in the case of multiplicative links (par and tensor), and unordered in the case of cut and why not links. The outgoing edges of a link (and the formulas that label them) are referred to as its *conclusions*.

- Premises and conclusions of links must respect a precise labeling (which depends on the link itself), given in Fig. 1. In particular, the edges labelled by $\flat$-formulas can only be premises of pax and why not links;

- Each edge must be the conclusion of exactly one link, and the premise of at most one link. The edges that are not premises of any link, and the formulas that label them, are deemed *conclusions* of the net. (Note that the presence of these "pending" edges, together with the fact that some premises are ordered, is why nets are not exactly graphs).

- A box is depicted as in Fig. 2, in which π is a net, said to be *contained* in the box. The links that are explicitly represented in Fig. 2 (i.e., the pax links and the of course link) form the *border* of the box (but are not contained in the box). The unique of course link in the border is called the *principal port* of the box, while the pax links are called *auxiliary ports*. We have the following conditions concerning boxes:

  a. each of course link is the principal port of exactly one box;
  b. each pax link is in the border of exactly one box;
  c. any two distinct boxes are either disjoint or included in one another.

14

If σ is a net, a link (or edge) of σ is said to have *depth* d if it is contained in d (necessarily nested) boxes. The depth of a box of σ is the depth of the links forming its border.

Let $\sigma = (\mathcal{G}, \mathsf{B})$ be a net. If we take $\mathcal{G}$ and ignore the conclusions, the orientation of edges, and the ordering of multiplicative links, we obtain an undirected graph in the usual sense; we call this graph the *underlying graph* of σ, and denote it by $\widetilde{\sigma}$. In what follows, when we speak of a *path* (or *cycle*) of σ, we mean a path (or cycle) in $\widetilde{\sigma}$, in the usual sense of graph theory.

Definition 3 (Switching). Let σ be a net. A *switching* of σ is an undirected graph built from $\widetilde{\sigma}$ as follows:

- for each par and why not node l, exactly one premise is chosen, and all the other premises are erased;

- for each box $\mathcal{B}$ at depth zero of σ, all the nodes and edges of $\widetilde{\sigma}$ coming from nodes and edges of σ belonging to $\mathcal{B}$ are erased, together with all the edges connecting such nodes to the rest of the graph; these are replaced by a new node l and, for any link m of depth zero which was connected to a link of $\mathcal{B}$, a new edge between m and l is added. In other words, the outermost boxes of σ are "collapsed" into single nodes.

Definition 4 (Balanced paths and cycles). Given a path ϕ of $\widetilde{\sigma}$, we may traverse it from one of its extremities to the other in any of the two possible directions (if ϕ is a cycle, it still has two possible orientations). Following ϕ in any of the two directions, we may record the number of times it traverses a paragraph link "upwards", *i.e.*, from conclusion to premise, which we denote by n^-, and the number of times it traverses a paragraph link "downwards", *i.e.*, from premise to conclusion, which we denote by n^+. It is clear that the numbers n^+, n^- are exchanged when we reverse the direction of traversal of ϕ. Therefore, the number $|n^+ - n^-|$ depends only on the path ϕ itself, and we call it its *balance*. In case ϕ is a cycle, the balance is computed by starting from any of the nodes traversed by ϕ and returning to the same node, in any direction.

A *balanced path* (or cycle) of σ is a path (or cycle) whose balance is 0.

In the following, if π is a net of conclusions $A_1, \ldots, A_n$ such that no A_i is a $\flat$-formula, by $\mathscr{P}$-*closure* of π we mean any net π' obtained by adding a tree of par links to the conclusions of π so that π' has exactly one conclusion $A_1 \mathscr{P} \cdots \mathscr{P} A_n$, where the placement of parenthesis depends on the shape of the tree. Of course, a net with more than 2 conclusions has more that one $\mathscr{P}$-closure; however, in all definitions and results below it does not matter which one we chose, whence we abusively speak of "the" $\mathscr{P}$-closure.

Definition 5 (Correctness, DR-net, proof net). A net $(\mathcal{G}, \mathsf{B})$ is said to be *DR-correct* if:

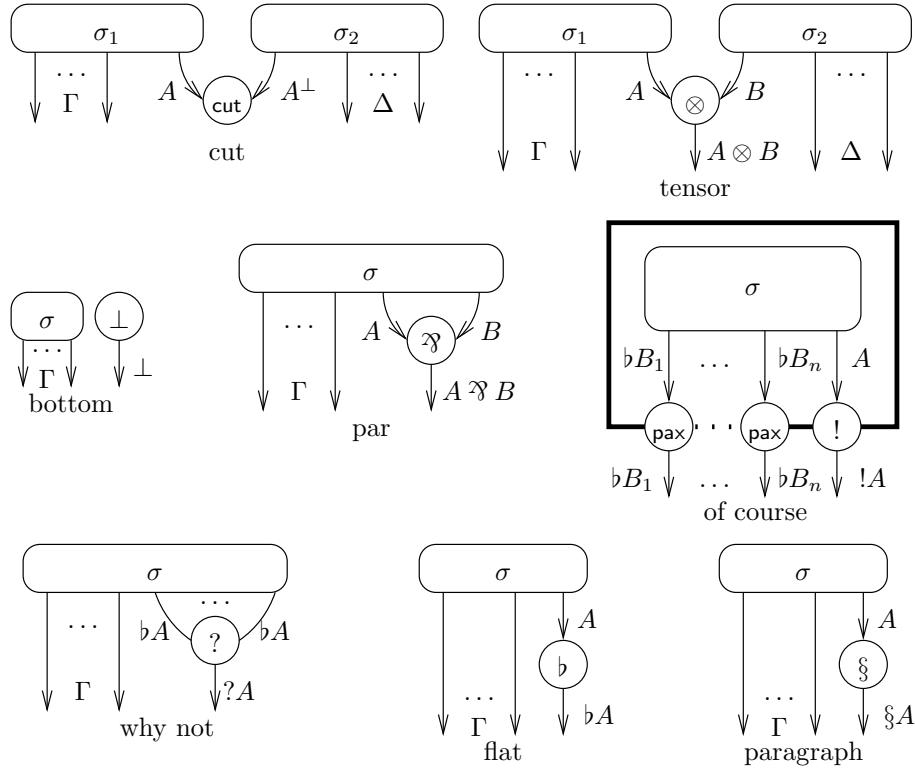- all of its switchings are acyclic;

Figure 3: Rules for building sequentializable nets.

- for all $\mathcal{B} \in \mathsf{B}$, the net contained in $\mathcal{B}$ is DR-correct.

A net is §-*correct* if all of its cycles are balanced.
A *DR-net* is a DR-correct net having no ♭-formulas in its conclusions.
A *proof net* is a DR-net whose ⅋-closure is §-correct.

Note that DR-nets are nothing but "proof nets" in the sense of Girard (1996); Danos and Regnier (1995); Tortora de Falco (2003) ("DR" stands indeed for "Danos-Regnier"), which may however happen to contain paragraph links. We shall prove in the sequel that, instead, our proof nets correspond to $\mathbf{LL}_\S$ proofs, as per Definition 1.

We start by introducing the notion of *sequentializable* net, whose definition mimics the rules of sequent calculus (without indexes):

Definition 6 (Sequentializable net). We define the set of *sequentializable nets* inductively: the empty net, the net consisting of a single axiom link, and the net consisting of a single one link are sequentializable (daimon, axiom, and one); the juxtaposition of two sequentializable nets is sequentializable (mix);

16

finally, if σ, σ_1, σ_2 are sequentializable nets of suitable conclusions, the nets of Fig. 3 are sequentializable.

The reader may object that the rules for building sequentializable nets do not correspond exactly to index-free versions of the rules of Table 1. Indeed, they correspond to an index-free version of that calculus in which, apart from the additive and quantifier rules, also the promotion, dereliction, weakening and contraction rules have been removed and replaced by the rules

$$\frac{\vdash \flat\Gamma, A}{\vdash \flat\Gamma, !A} \; \text{Promotion} \qquad \frac{\vdash \Gamma, \flat A, \ldots, \flat A}{\vdash \Gamma, ?A} \; \text{Why not} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, \flat A} \; \text{Flat}$$

so that $\flat$-formulas may appear in sequents. Moreover, the rules

$$\frac{}{\vdash} \; \text{Daimon} \qquad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \; \text{Mix}$$

are further added to the calculus. However, ignoring daimon and mix, which are introduced for the sole purpose of simplifying the definition of proof nets (otherwise, more conditions concerning *connectedness* of switchings should be added to the formulation of DR-net, requiring the introduction and subsequent handling of *jumps*, a technical complication which we do not need in this paper), this calculus is strictly equivalent to the index-free, additive- and quantifier-free version of that of Table 1. In fact, it only takes a straightforward induction to prove that $\vdash \flat\Gamma, \Delta$ is derivable in the former calculus iff $\vdash ?\Gamma, \Delta$ is derivable in the latter. Hence, derivability coincides for sequents not containing $\flat$-formulas, which are the ones that matter (remember that proof nets have no $\flat$-formula in their conclusions, *cf.* Definition 5).

If sequentializable nets capture the structure of the rules of Table 1, we still need to account for 2-sequents, *i.e.*, the fact that the calculus of $\mathbf{LL}_\S$ manipulates formulas carrying indexes, which influence provability in an essential way. For instance, if we took sequentializable nets as our proofs, both $\S X \multimap X$ and $X \multimap \S X$ would become provable, in striking contrast with what observed about $\mathbf{LL}_\S$. This may be done by resorting to the notion of *indexing*, originally introduced by Baillot and Mazza (2010).

Definition 7 (Indexing, indexable net). Let σ be a net. An *indexing* for σ is a function I from the edges of σ to $\mathbb{Z}$ satisfying the constraints given in Fig. 4. If, furthermore, the indexing satisfies that, for all conclusions e, e' of σ, $I(e) = I(e')$, it is said to be a *strong indexing*.

A net is said to be (strongly) indexable if it admits a (strong) indexing.

Observe that a net with no $\flat$-formulas in its conclusions is strongly indexable iff its $\mathfrak{P}$-closure is indexable (the fact that no conclusion is a $\flat$-formula is only needed to ensure the existence of a $\mathfrak{P}$-closure). This is in accordance with the fact that $\vdash A_1, \ldots, A_n$ is provable in $\mathbf{LL}_\S$ iff $\vdash A_1 \mathfrak{P} \cdots \mathfrak{P} A_n$ is derivable.

The set of indexings of an indexable net σ is easily seen to form an affine space over the $\mathbb{Z}$-module $\mathbb{Z}^n$, where n is the number of connected components of
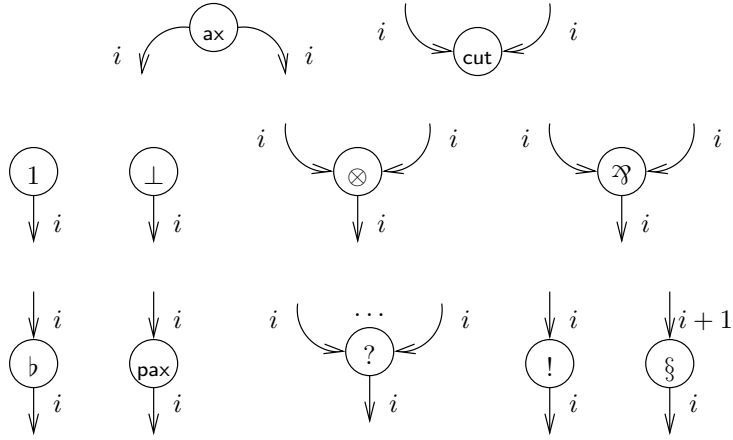
17

Figure 4: Constraints for indexing nets. Next to each edge we represent the integer assigned by the indexing; formulas are omitted, because irrelevant to the indexing.

σ. We refer the reader to Baillot and Mazza (2010) for a proof of this fact; here, we shall only use the following, simpler statement, which is a reformulation of Lemma 1 for nets:

Lemma 3 (Translation). *Let I be an indexing for a net σ, let $\sigma_1, \ldots, \sigma_n$ be distinct connected components of σ, and let $k_1, \ldots, k_n \in \mathbb{Z}$. Then, the function I' defined by*

$$I'(e) = \begin{cases} I(e) + k_i & \text{if } e \text{ is an edge of } \sigma_i, \text{ with } 1 \le i \le n \\ I(e) & \text{otherwise,} \end{cases}$$

is also an indexing for σ.

PROOF. Simply observe that the constraints of Fig. 4 are preserved by translations, and that the indexes of edges in two distinct connected components are completely independent. □

In light of what we said about sequentializable nets, and by looking at how indexes are assigned in Fig. 4, it is obvious that an indexable sequentializable net corresponds to a derivation of the calculus of Table 1. Indeed, for example, the sequentializable nets proving $\S X \multimap X$ and $X \multimap \S X$ are easily seen to admit no indexing. On the other hand, strongly indexable sequentializable nets correspond to proofs (Definition 1), because strong indexability precisely reflects the requirement that all formulas in the conclusion have the same index.

We shall now prove that the proof nets of Definition 5 provide a geometric characterization of strongly indexable sequentializable nets (modulo the absence of $\flat$-formulas in the conclusions).

We start by recalling a classical result of the proof theory of linear logic:

Proposition 4 (Girard (1996)). *A net is sequentializable iff it is DR-correct.*
$\square$

The following is a simple structural property of nets, which will be needed in the sequel. If σ is a net and l a link of σ, we say that l is *terminal* if it is at depth 0 and all of the conclusions of l are conclusions of σ. Then, we have:

**Lemma 5.** *Every non-empty net σ has at least one terminal link which is not an auxiliary port (*pax*).*

PROOF. We start by observing that, because of the typing constraints, σ cannot contain a directed cycle. In fact, a directed cycle would require the existence of a formula which is equal to one of its strict subformulas, which is impossible.

Now, choose any link l_0 at depth 0 of σ (there is at least one since σ is not empty). If l_0 is a cut, we are done, so we may suppose l_0 to be different from a cut; we also suppose, for the time being, that l_0 is not an axiom. If l_0 is a pax link, we let l_0' be the principal port of the unique box to which l_0 belongs; otherwise, we let $l_0' = l_0$. Now, observe that there is a unique maximal directed path ψ "going down" from the conclusion of l_0': the existence is guaranteed by the finiteness of nets and the absence of directed cycles; the uniqueness by the fact that the structure of nets is arborescent. Note also that ψ is entirely at depth 0, because we started at that depth, and the orientation forbids "entering" into a box; in particular, ψ crosses no pax link. Then, by the maximality of ψ, its "last" edge is either the premise of a cut, or a conclusion of σ. In both cases, we found the desired terminal link. We are left with the case in which l_0 is an axiom; but this is either terminal, or induces a terminal link applying the above argument to a conclusion of l_0 which is not a conclusion of σ. $\square$

Lemma 6. *A net is indexable iff it is $\S$-correct.*

PROOF. Let σ be indexable. The fact that all of its cycles are balanced is an immediate consequence of the following remark: if I is an indexing of σ, and if ϕ is a path of σ, the balance of ϕ is obviously equal to $|I(e_1) - I(e_2)|$, where e_1, e_2 are the edges which mark the extremities of ϕ; this is because indexings are defined precisely so as to "count" the traversals of paragraph links.

Let now σ be a net in which all cycles are balanced. We reason by induction on the number of links of σ. If σ is empty, the statement is trivially true. If σ has more than one connected component, the induction hypothesis immediately allows us to conclude. Hence, we may suppose that σ consists of exactly one, non-empty connected component. By Lemma 5, there is a terminal link l of σ which is not a pax. We may immediately discard the cases in which l is an axiom or a weakening link, because the statement is trivially seen to hold. If we ignore for the moment the possibility that l is an of course link, we are in the following situation:
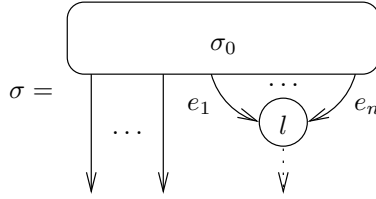
where $n \geq 1$, and the dotted arrow means that l may or may not have a conclusion. Observe that every cycle of σ_0 is a cycle of σ, hence balanced; therefore, the induction hypothesis gives us an indexing I_0 of σ_0. In case $n = 1$, I_0 may obviously be extended into an indexing for σ, so we may actually suppose that $n \geq 2$. Now, if e_{i_1}, e_{i_2} are two distinct premises of l belonging to the same connected component of σ_0, by definition there is path ϕ connecting them. If we had $I_0(e_{i_1}) \neq I_0(e_{i_2})$, by the remark made at the beginning of this proof, the balance of ϕ would be non-null; this would induce a unbalanced cycle in σ, a contradiction. Hence, all premises of l belonging to the same connected component of σ_0 are assigned the same index by I_0. It suffices now to apply the Translation Lemma 3 to find an indexing I_0' for σ_0 which assigns the same integer to $e_1, \ldots, e_n$; such an I_0' obviously extends into an indexing for σ.

We are left with the case in which l is an of course link. Since all other links have been treated, we may assume that σ actually consists of a single box $\mathcal{B}$ whose principal port is l itself. In this case, the induction hypothesis gives us an indexing for the contents of $\mathcal{B}$, from which an indexing for σ may be trivially defined. $\qquad\square$

Composing Lemma 6 with the fact that strong indexability coincides with indexability of the $\invamp$-closure immediately implies the following:

Lemma 7. *Let σ be a net with no $\flat$-formulas in its conclusions. Then, σ is strongly indexable iff its $\invamp$-closure is $\S$-correct.* $\qquad\square$

A combination of Proposition 4 and Lemma 7 gives us the announced result:

Theorem 8 (Sequentialization). *A net with no $\flat$-formulas in its conclusions is strongly indexable and sequentializable iff it is a proof net.* $\qquad\square$

Similarly, one obtains that indexable and sequentializable nets with no $\flat$-formulas in their conclusions (which correspond to *derivations*, instead of proofs) coincide with $\S$-correct DR-nets. Hence, as it is usual in linear logic, the proof theory of $\mathbf{LL}_\S$ may be entirely formulated in terms of proof nets. The advantage is that no notion of index is ever explicitly mentioned.

1.4. Cut-elimination

Formulating the cut-elimination procedure in proof nets is quite simple: there are only five rules (or *steps*, as they are often called), taking the form of the graph-rewriting rules given in Figures 5 through 8. Observe that, to be properly formulated, the exponential step requires that no $\flat$-formula is a conclusion, for otherwise the net on the right-hand side may have more conclusions
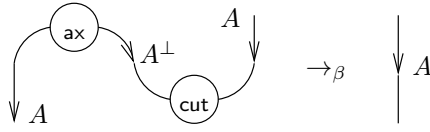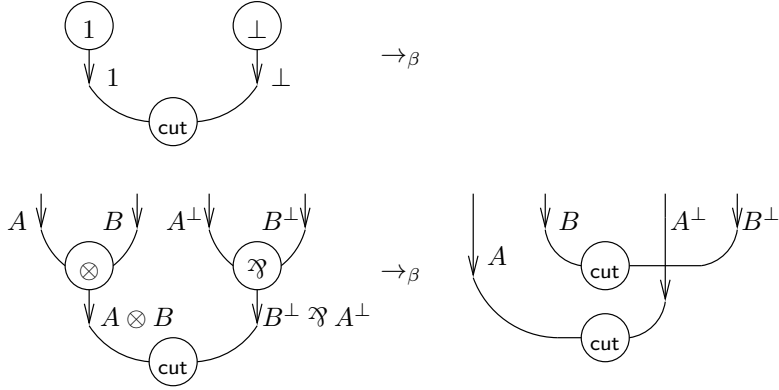
Figure 5: Axiom step.

Figure 6: Multiplicative steps.

than the net on the left-hand side. This is why such a condition is imposed on DR-nets and proof nets.

When a proof net (or DR-net) π is transformed into π' by the application of one cut-elimination step, we write $\pi \to_\beta \pi'$, and we say that π β-*reduces* to π'. Of course, we have the following:

Proposition 9. *Let π be a proof net (resp. DR-net), and let $\pi \to_\beta \pi'$. Then, the net π' is a proof net (resp. DR-net).*

PROOF. The preservation of DR-nets under cut-elimination is a classical result of linear logic. To see that the condition on cycles is preserved, simply observe, by inspecting every rule, that every cycle of π' induces a cycle of π whose balance is identical. $\qquad\square$

Confluence and strong normalization of reduction of proof nets (and of DR-nets) may be proved by standard arguments, which may be found for instance in Girard (1987); Danos (1989). The key observation is that the paragraph step is trivial, and adds virtually nothing with respect to the dynamics of cut-elimination in propositional multiplicative exponential linear logic.

By looking at cut-elimination in proof nets, we may find a deeper meaning to the concept of stratification. Indeed, as observed in Baillot and Mazza (2010), it is immediate to see that indexings (which exist for proof nets by Theorem 8) are preserved under cut-elimination, in the sense that the residue of a link may be
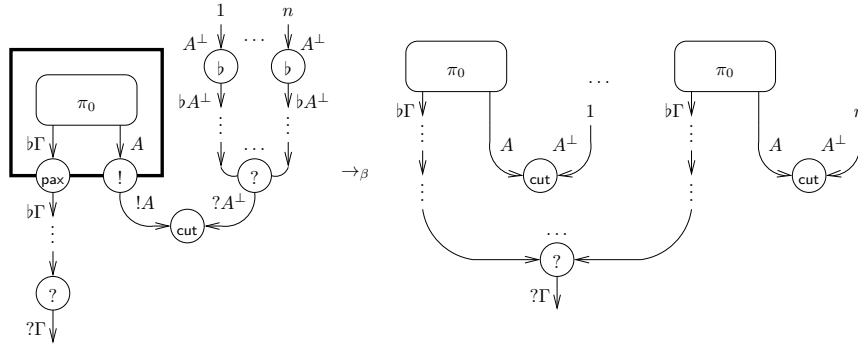
21

Figure 7: Exponential step; $\flat\Gamma$ is a multiset of $\flat$-formulas, so one pax link, why not link, or wire in the picture may in some case stand for several (including zero) pax links, why not links, or wires.
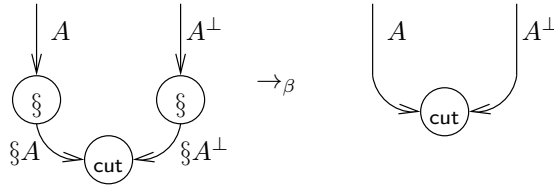

Figure 8: Paragraph step.

given the same index as its ancestor ("residue" and "ancestor" are here meant to take the standard meaning of rewriting theory, *cf.* Definition 21 for a formal definition). This allows one to meaningfully speak of the *level* of a link of a proof net, and to observe that interaction (by means of a cut) may only ever happen between dual links of identical level. A proof net of $\mathbf{LL}_\S$ may therefore be seen as partitioned into *strata*, which do not communicate with each other.

Thanks to the stability of levels, it is possible to consider a cut-elimination protocol which attempts to eliminate cuts at level i before touching any cut at level $i+1$, starting from the minimum level. In full $\mathbf{LL}_\S$, this protocol has no special property; however, in the fragment corresponding to $\mathbf{L^3}$, where the exponential modalities (which control duplication) are forced to modify levels, this protocol enjoys an elementary bound, as proved in Baillot and Mazza (2010) (or a polynomial bound in the case of the subsystem $\mathbf{L^4}$). Furthermore, since in $\mathbf{ELL}$ and $\mathbf{LLL}$ levels coincide with exponential depths (*i.e.*, the nesting of boxes, see Definition 2), this very same protocol is also the one for which the elementary and polynomial bounds were originally proved by Girard.

2. A Categorical Construction for Stratification

2.1. Models of linear logic

Intuitively, a denotational semantics of a logical system is an interpretation $[\![\cdot]\!]$ of proofs as some kind of mathematical objects which:

- is stable under cut-elimination: if $\pi \to_\beta \pi'$, then $[\![\pi]\!] = [\![\pi']\!]$;

- gives rise to a congruence: if $[\![\pi_1]\!] = [\![\pi_2]\!]$ and if π'_1, π'_2 are obtained resp. from π_1, π_2 by applying the same inference rule, then $[\![\pi'_1]\!] = [\![\pi'_2]\!]$.

A denotational semantics is non-trivial if there exist two distinct proofs of the same formula π_1, π_2 such that $[\![\pi_1]\!] \neq [\![\pi_2]\!]$; usually, this is implicitly assumed to be the case.

Category theory is an elegant and powerful framework in which the above intuition can be formalized. The simplest approach is to interpret formulas and proofs as objects and morphisms of a category: given an arbitrary assignment of objects to propositional atoms (such an assignment will be a parameter of the interpretation), axioms are interpreted by identity morphisms, the cut rule by composition, and the other inference rules by functors or natural transformations; the above requirements are thus met automatically (moreover, one also automatically models η-expansion).

At this point, it is quite natural to ask whether there exist categorical structures matching the structure of connectives and inference rules of the logical system under study. This leads to a formal definition of (denotational) *model* of a logical system: a category having enough structure so that a denotational semantics immediately arises from it.

One of the most well-known examples is probably that of propositional[2] intuitionistic logic, whose models correspond to Cartesian closed categories. In the case of propositional linear logic, several authors have proposed their notion of model; let us mention for instance Lafont (1988); Seely (1989); Bierman (1995); Benton et al. (1992); Benton (1994); de Carvalho (2007), and Melliès (2007); Curien et al. (2010) for surveys. Most of these axiomatizations are equivalent, and can be very elegantly synthesized by the notion of *linear-non-linear adjunction*.

Giving the full categorical background needed to formulate linear-non-linear adjunctions is out of the scope of this paper; for the sake of self-containedness, Appendix A recalls the basic definitions and properties, assuming as understood only the concept of symmetric monoidal category. For a more detailed exposition, we refer the reader to Paul-André Melliès excellent survey to be found in Curien et al. (2010).

---

[2]While propositional logical systems admit in many cases satisfactory categorical axiomatizations, there is currently no general, widely accepted categorical formulation of a second order denotational model. This is why, in this paper, we drop second-order quantifiers whenever we deal with denotational semantics. This is consistent with the fact that we only defined propositional proof nets.

Definition 8 (Linear-non-linear adjuction). A *linear-non-linear adjunction* is a symmetric monoidal adjunction between symmetric monoidal functors

$$(\mathcal{M}, \times, T) \underset{(M,m,u)}{\overset{(L,l,t)}{\rightleftarrows}} \bot \quad (\mathcal{L}, \otimes, 1)$$

such that $(\mathcal{M}, \times, T)$ is Cartesian.

As already mentioned, linear-non-linear adjunctions may be used to formulate a categorical axiomatization of linear logic, arguably the most elegant available at present. We just need one further ingredient, namely the notion of $*$-autonomous category (Barr, 1979), which we recall here:

Definition 9 ($*$-autonomous category). A *symmetric monoidal closed category* is a symmetric monoidal category $(\mathcal{A}, \otimes, 1)$ such that, for every object X of $\mathcal{A}$, the functor $- \otimes X$ has a right adjoint R_X. This latter induces a bifunctor $\multimap : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$, such that $R_X = X \multimap -$, and the adjunction corresponds to the existence of a natural isomorphism

$$\Psi : [- \otimes -, -] \Rightarrow [-, - \multimap -] : (\mathcal{A}^{\mathrm{op}} \times \mathcal{A}^{\mathrm{op}}) \times \mathcal{A} \to \mathbf{Set}.$$

Given two objects A, B of $\mathcal{A}$, it is customary to denote by $\mathrm{eval}_{A,B}$ the morphism $\Psi^{-1}_{A \multimap B, A, B}(id_{A \multimap B}) : (A \multimap B) \otimes A \to B$.

A *$*$-autonomous category* is a symmetric monoidal closed category $(\mathcal{A}, \otimes, 1)$ together with a *dualizing object*, *i.e.*, an object $\bot$ of $\mathcal{A}$ such that, for every object A,

$$\Psi_{A, A \multimap \bot, \bot}(\mathrm{eval}_{A, \bot} \circ \gamma_{A, A \multimap \bot}) : A \to (A \multimap \bot) \multimap \bot$$

is an isomorphism, where γ is the symmetry of $\otimes$. In that case, we denote by $(-)^{\bot}$ the contravariant endofunctor $- \multimap \bot$.

Definition 10 (Model of linear logic). A *model* of linear logic is given by the following data:

- a $*$-autonomous category $(\mathcal{L}, \otimes, 1, \bot)$ which has also binary products and a terminal object, denoted by $A \,\&\, B$ and $\top$, respectively;

- a Cartesian category $(\mathcal{M}, \times, T)$;

- a linear-non-linear adjunction $(L, l, t) \dashv (M, m, u)$ between $(\mathcal{M}, \times, T)$ and $(\mathcal{L}, \otimes, 1)$.

The category $\mathcal{L}$ is said to be the *linear* category.

The linear category is where the model "lives"; formulas and proofs are interpreted in it. The motivation behind the structures required by Definition 10 may be spelled out as follows:

- The ∗-autonomous structure is used to interpret the multiplicative layer of linear logic: tensor, linear negation, and par are interpreted by the functors $\otimes$, $(-)^\perp$, and $(-)^\perp \multimap -$, respectively.

- The additive connectives are interpreted by products and coproducts in $\mathcal{L}$. Indeed, since $\mathcal{L}$ has products, for all objects A, B, the object $(A^\perp \,\&\, B^\perp)^\perp$ may be seen to be a coproduct of A and B, and $\top^\perp$ may be seen to be an initial object. We note in passing that right adjoints are *continuous*, *i.e.*, they preserve limits; this implies that M transports products in $\mathcal{L}$ to products in $\mathcal{M}$, *i.e.*, $M(A \,\&\, B) = MA \times MB$ and $M\top = T$.

- The linear-non-linear adjunction offers the structure needed to interpret the exponential layer:

  - we know that the endofunctor $! = L \circ M$ of $\mathcal{L}$ is a comonad: the promotion rule (*i.e.*, boxes in proof nets) is interpreted by $!(-)$ and its comultiplication; dereliction is interpreted by the counit;

  - the structural rules are interpreted thanks to the Cartesian structure of $\mathcal{M}$, using Proposition 25. In fact, from the diagonal maps and the terminal arrows of $\mathcal{M}$, we can define two natural transformations $\kappa : M \Rightarrow M \times M$ and $\omega : M \Rightarrow T$, where by T we denote here the constant functor yielding T and its identity arrow. Now, the fact that (L, l, t) is strong means that l and t are isomorphisms; contraction and weakening are then interpreted by applying L to κ and ω, and by composing with l^{-1} and t^{-1}, respectively:

$$\text{contraction:} \quad !(-) \xRightarrow{L\kappa} L(M(-) \times M(-)) \xRightarrow{l^{-1}} !(-) \otimes !(-)$$

$$\text{weakening:} \quad !(-) \xRightarrow{L\omega} LT \xrightarrow{t^{-1}} 1$$

The terminology *linear* is justified by the fact that $\mathcal{L}$ is a sort of "linear world", where structural rules are forbidden; the functor M allows to transport proofs into a "Cartesian world", where duplication and erasing are available; then, the functor L "linearizes" the proofs back to the linear world. The so-called exponential isomorphisms of linear logic, *i.e.*, $!(A \,\&\, B) \cong !A \otimes !B$ and $!\top \cong 1$, already express this idea; the linear-non-linear adjunction refines it by implementing it as a "return trip" from the linear world to itself, passing through a non-linear stage.

2.2. Models of stratified linear logic

It is fairly straightforward to define a notion of model for stratified linear logic. Essentially, all that is needed is to define how the paragraph modality is interpreted; the remaining connectives will be handled by the same structures as Definition 10.

Definition 11 (Model of stratified linear logic). A *model* of $\mathbf{LL}_\S$ is given by the same data as a model of linear logic, plus an endofunctor $\S$ of the linear category $(\mathcal{L}, \otimes, 1, \perp)$ which commutes to all logical operations, *i.e.*:

- $\S\perp$ is isomorphic to $\perp$;

- $\S \circ \otimes$ is naturally isomorphic to $\otimes \circ (\S \times \S)$;

- $\S \circ \multimap$ is naturally isomorphic to $\multimap \circ (\S^{\mathrm{op}} \times \S)$;

- $\S$ preserves all finite products;

- $\S \circ !$ is naturally isomorphic to $! \circ \S$.

A model of $\mathbf{LL}_\S$ is said to be *degenerate* when $\S$ is isomorphic to the identity functor of $\mathcal{L}$ (which obviously verifies all of the above conditions).

Given a category with the structure of Definition 11, it is immediate to interpet the proofs of propositional $\mathbf{LL}_\S$, provided we use the alternative formulation of Table 2, modulo Proposition 2. The interpretation (which is always parametric in an assignment of objects to propositional atoms) is defined as usual by induction: axioms and cuts are interpreted by identities and composition; the structural rules, and the positive logical rules are identical to those of linear logic, so the interpretation is already defined; for the negative rules and the exponential rules, the standard interpretation of linear logic is composed with the suitable isomorphisms given by Definition 11; finally, the paragraph rule is interpreted by applying the $\S$ functor.

2.3. Objects with involutions

We now introduce the categorical construction at the center of our work. This is based on the notion of *object with involutions*, which we define as an object (of some category) equipped with denumerably many involutions, indexed by $\mathbb{Z}$. Given a category $\mathcal{A}$ and an object A of $\mathcal{A}$, an *involution* of A is an automorphism s of A which is its own inverse, *i.e.*, such that $s \circ s = id_A$.

Definition 12 (Object with involutions). Let $\mathcal{A}$ be a category. An *object with involutions* of $\mathcal{A}$ is a pair (A, s) where A is an object of $\mathcal{A}$ and $s = (s_k)_{k \in \mathbb{Z}}$ is a $\mathbb{Z}$-indexed sequence of involutions of A.

If $(A, s), (B, t)$ are objects with involutions of $\mathcal{A}$, a *morphism* of objects with involutions from the first to the second is an arrow $f : A \to B$ of $\mathcal{A}$ such that, for all $k \in \mathbb{Z}$, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle s_k}\downarrow & & \downarrow{\scriptstyle t_k} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

or, equivalently, $t_k \circ f \circ s_k = f$ for all $k \in \mathbb{Z}$.

We remark that we may restrict the above definition so that, in an object with involutions (A, s), the sequence s is required to be equal to id_A almost everywhere. This restriction, which limits the cardinality of the set of all objects

with involutions, is preserved by all constructions we shall consider, but is by no means necessary.

The composition of two morphisms of objects with involutions is easily seen to be a morphism of objects with involutions, and identity arrows are also easily seen to be morphisms of objects with involutions. Hence, the objects with involutions of a category $\mathcal{A}$ and their morphisms form themselves a category, denoted by $\mathcal{A}^{\mathcal{I}}$.

The reason behind such a notation is the following. Consider the free Coxeter group on $\mathbb{Z}$, which we denote by $\mathcal{I}$: the elements of $\mathcal{I}$ are finite sequences of integers not containing two consecutive equal elements; its group law, denoted by $\circ$, is defined by concatenation modulo the equation $n \circ n = id$ for all $n \in \mathbb{Z}$, where id is the empty sequence. Now, $\mathcal{I}$ can be seen as a groupoid with only one object and whose morphisms are exactly the elements of $\mathcal{I}$ itself; then, we invite the reader to check that the category of objects with involutions of a category $\mathcal{A}$ is exactly $\mathcal{A}^{\mathcal{I}}$, that is, the category of functors from $\mathcal{I}$ to $\mathcal{A}$, with natural transformations as morphisms.

Therefore, $\mathcal{A}^{\mathcal{I}}$ is nothing but the image of $\mathcal{A}$ through the object map of the endofunctor $(-)^{\mathcal{I}}$ of **Cat**, the right adjoint of the product functor $- \times \mathcal{I}$ in the category of categories. Actually, $(-)^{\mathcal{I}}$ is a 2-endofunctor, *i.e.*, it also acts on natural transformations. We may explicitly describe it as follows.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We define a functor $F^{\mathcal{I}} : \mathcal{A}^{\mathcal{I}} \to \mathcal{B}^{\mathcal{I}}$ as follows: if (A, s) is an object of $\mathcal{A}^{\mathcal{I}}$, we set $F^{\mathcal{I}}(A, s) = (FA, Fs)$, where by Fs we mean the sequence $(F(s_k))_{k \in \mathbb{Z}}$; if $f : (A, s) \to (B, t)$ is an arrow of $\mathcal{A}^{\mathcal{I}}$, we set $F^{\mathcal{I}}f = Ff$. Functors preserve involutions, so (FA, Fs) is a object with involutions; moreover, given $k \in \mathbb{Z}$, we have $F(t_k) \circ Ff \circ F(s_k) = F(t_k \circ f \circ s_k) = Ff$, which proves that Ff is a morphism between the objects with involutions $F^{\mathcal{I}}(A, s)$ and $F^{\mathcal{I}}(B, t)$; the fact that $F^{\mathcal{I}}$ preserves composition and identities follows immediately from the fact that F does.

Let now $\varphi : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ be a natural transformation. First of all, we check that, for each object (A, s) of $\mathcal{A}^{\mathcal{I}}$, the arrow $\varphi_A : FA \to GA$ of $\mathcal{B}$ is actually a morphism between the objects with involutions $F^{\mathcal{I}}(A, s)$ and $G^{\mathcal{I}}(A, s)$, *i.e.*, an arrow of $\mathcal{B}^{\mathcal{I}}$. As a matter of fact, given any $k \in \mathbb{Z}$, we have $\varphi_A \circ F(s_k) = G(s_k) \circ \varphi_A$, simply because φ is a natural transformation from F to G. Therefore, if we set $\varphi^{\mathcal{I}}_{(A,s)} = \varphi_A$, we have defined a family of arrows of $\mathcal{B}^{\mathcal{I}}$, whose naturality is an immediate consequence of the naturality of φ. We have thus obtained a natural transformation $\varphi^{\mathcal{I}} : F^{\mathcal{I}} \Rightarrow G^{\mathcal{I}} : \mathcal{A}^{\mathcal{I}} \to \mathcal{B}^{\mathcal{I}}$, as desired.

The following property will be essential in the sequel:

Lemma 10. *Let $\varphi : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ be a natural transformation, let $\mathcal{C}$ be another category and $H : \mathcal{C} \to \mathcal{A}^{\mathcal{I}}$ a functor. Let C be an object of $\mathcal{C}$, and set $\varphi'_C = \varphi_{UH(C)}$, where $U : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$ is the forgetful functor sending an object with involutions (A, s) to A and being the identity on morphisms. Then, φ' is a natural transformation from the functor $F^{\mathcal{I}} \circ H$ to the functor $G^{\mathcal{I}} \circ H$ (which are both of source $\mathcal{C}$ and target $\mathcal{B}^{\mathcal{I}}$).*

PROOF. We first check that, given a generic object C of $\mathcal{C}$, $\varphi_{UH(C)}$ is a morphism of $\mathcal{B}^{\mathcal{I}}$ from $F^{\mathcal{I}}H(C)$ to $G^{\mathcal{I}}H(C)$. Let $H(C) = (A, s)$; then $UH(C) = A$,

$F^{\mathcal{I}}H(C) = (FA, Fs)$, and $G^{\mathcal{I}}H(C) = (GA, Gs)$. Then, it is enough to verify, for all $k \in \mathbb{Z}$, the commutation in $\mathcal{B}$ of

$$
\begin{array}{ccc}
FA & \xrightarrow{\varphi_A} & GA \\
{\scriptstyle Fs_k}\big\downarrow & & \big\downarrow{\scriptstyle Gs_k} \\
FA & \xrightarrow[\varphi_A]{} & GA
\end{array}
$$

which holds by naturality of φ.

Let us now check the naturality of φ'. For this, let $f : C \to C'$ be an arrow of $\mathcal{C}$; we need to verify the commutation in $\mathcal{B}^{\mathcal{I}}$ of the diagram

$$
\begin{array}{ccc}
F^{\mathcal{I}}H(C) & \xrightarrow{\varphi'_C} & G^{\mathcal{I}}H(C) \\
{\scriptstyle F^{\mathcal{I}}H(f)}\big\downarrow & & \big\downarrow{\scriptstyle G^{\mathcal{I}}H(f)} \\
F^{\mathcal{I}}H(C') & \xrightarrow[\varphi'_C]{} & G^{\mathcal{I}}H(C')
\end{array}
$$

But if we set $H(C) = (A, s)$ and $H(C') = (A', s')$, the diagram becomes

$$
\begin{array}{ccc}
(FA, Fs) & \xrightarrow{\varphi_A} & (GA, Gs) \\
{\scriptstyle FH(f)}\big\downarrow & & \big\downarrow{\scriptstyle GH(f)} \\
(FA', Fs') & \xrightarrow[\varphi_A]{} & (GA', Gs')
\end{array}
$$

which commutes because $H(f)$, as an arrow of $\mathcal{A}^{\mathcal{I}}$, is actually an arrow of $\mathcal{A}$ from A to A', φ is natural from F to G, and commutation in $\mathcal{B}$ implies commutation in $\mathcal{B}^{\mathcal{I}}$, because the arrows and composition of this latter category are arrows and composition of the former. $\qquad\square$

With the help of Lemma 10, we now lift $(-)^{\mathcal{I}}$ to a 2-endofunctor of **SMCat**, the 2-category of symmetric monoidal categories, symmetric monoidal functors and monoidal natural transformations. We shall denote such a 2-endofunctor by Inv.

First of all, observe that, if we fix a category $\mathcal{C}$, the covariant endofunctor $(-)^{\mathcal{C}}$ of $(\mathbf{Cat}, \times, \mathbf{1})$ is strong symmetric monoidal, where $\times$ is the product of categories and $\mathbf{1}$ the terminal category (having one object and its identity morphism only). Indeed, for any categories $\mathcal{A}, \mathcal{B}$, we have natural isomorphisms $\Phi^{\mathcal{C}}_{\mathcal{A},\mathcal{B}} : \mathcal{A}^{\mathcal{C}} \times \mathcal{B}^{\mathcal{C}} \to (\mathcal{A} \times \mathcal{B})^{\mathcal{C}}$ and $\Phi^{\mathcal{C}}_0 : \mathbf{1} \to \mathbf{1}^{\mathcal{C}}$, which make $((-)^{\mathcal{C}}, \Phi^{\mathcal{C}}, \Phi^{\mathcal{C}}_0)$ strong symmetric monoidal. In case $\mathcal{C} = \mathcal{I}$, we drop the superscript from notations, and write simply Φ and Φ_0; additionally, we denote by $\Phi_{\mathcal{A}}$ the (invertible) functor $\Phi_{\mathcal{A},\mathcal{A}}$.

We start by defining Inv on objects. Let $(\mathcal{A}, \otimes, 1)$ be a symmetric monoidal category. By definition, $\otimes$ is a bifunctor of $\mathcal{A}$, and 1 an object of $\mathcal{A}$, which may be seen as a functor (which we still denote by 1) from the terminal category $\mathbf{1}$ to

$\mathcal{A}$. If we apply $(-)^{\mathcal{I}}$ to these functors, we obtain a functor $\otimes^{\mathcal{I}} : (\mathcal{A} \times \mathcal{A})^{\mathcal{I}} \to \mathcal{A}^{\mathcal{I}}$ and a functor $1^{\mathcal{I}} : \mathbf{1}^{\mathcal{I}} \to \mathcal{A}^{\mathcal{I}}$. Then, we set $\mathrm{Inv}\,\mathcal{A} = (\mathcal{A}^{\mathcal{I}}, \otimes', 1')$, where

$$
\begin{aligned}
\otimes' &= \otimes^{\mathcal{I}} \circ \Phi_{\mathcal{A}} : \mathcal{A}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}} \to \mathcal{A}^{\mathcal{I}} \\
1' &= 1^{\mathcal{I}} \circ \Phi_0 : \mathbf{1} \to \mathcal{A}^{\mathcal{I}}.
\end{aligned}
$$

As a functor from the terminal category to $\mathcal{A}^{\mathcal{I}}$, $1'$ actually represents an object of $\mathcal{A}^{\mathcal{I}}$, which is readily seen to be $(1, id_1)$, with id_1 representing the constant $\mathbb{Z}$-indexed sequence identically equal to id_1.

We claim that $\mathrm{Inv}\,\mathcal{A}$ is a symmetric monoidal category. To show this, it will be enough to apply Lemma 10 to the natural isomorphisms coming with the monoidal structure of $\mathcal{A}$.

Associator: If we denote by A the invertible functor realizing the isomorphism between $(\mathcal{A} \times \mathcal{A}) \times \mathcal{A}$ and $\mathcal{A} \times (\mathcal{A} \times \mathcal{A})$ (*i.e.*, the associator of the monoidal category $(\mathbf{Cat}, \times, \mathbf{1})$ instantiated on $\mathcal{A}, \mathcal{A}, \mathcal{A}$), we see that the associator α of $\mathcal{A}$ is a natural isomorphism of the following type:

where by $\mathcal{A}$ we also denote the identity functor of $\mathcal{A}$. Then, if we set

$$
\begin{aligned}
F &= \otimes \circ (\otimes \times \mathcal{A}), \\
G &= \otimes \circ (\mathcal{A} \times \otimes) \circ A, \\
H &= \Phi_{\mathcal{A}} \circ (\Phi_{\mathcal{A}} \times \mathcal{A}^{\mathcal{I}}),
\end{aligned}
$$

we may apply Lemma 10 and obtain a natural transformation

$$
\alpha' : F^{\mathcal{I}} H \Rightarrow G^{\mathcal{I}} H : (\mathcal{A}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}}) \times \mathcal{A}^{\mathcal{I}} \to \mathcal{A}^{\mathcal{I}},
$$

which may be seen to have the right type for being the associator of $\mathrm{Inv}\,\mathcal{A}$ by looking at the following diagram:

29

Indeed, diagrams (1) and (3) commute because of the naturality of Φ, and diagram (2) commutes because $(-)^{\mathcal{I}}$ is monoidal (remember that A is an instance of the associator of $(\mathbf{Cat}, \times, \mathbf{1})$). The fact that α' is an isomorphism follows immediately from its definition (*cf.* Lemma 10), because α is an isomorphism and the arrows of $\mathcal{A}^{\mathcal{I}}$ are arrows of $\mathcal{A}$.

Unitors: We proceed as above, checking only the case of the left unitor, the right unitor being completely analogous. If we denote by Λ the invertible functor realizing the isomorphism from $\mathbf{1} \times \mathcal{A}$ to $\mathcal{A}$ (*i.e.*, the left unitor of $(\mathbf{Cat}, \times, \mathbf{1})$ instantiated on $\mathcal{A}$), we see that the left unitor λ of $\mathcal{A}$ is of the following type:

$$
\begin{array}{ccccc}
 & \xrightarrow{1 \times \mathcal{A}} & \mathcal{A} \times \mathcal{A} & \xrightarrow{\otimes} & \\
\mathbf{1} \times \mathcal{A} & & \Downarrow \lambda & & \mathcal{A} \\
 & \xrightarrow{\quad\Lambda\quad} & & &
\end{array}
$$

Then, if we apply Lemma 10 to the data

$$
\begin{aligned}
F &= \otimes \circ (1 \times \mathcal{A}), \\
G &= \Lambda, \\
H &= \Phi_{\mathcal{A}} \circ (\Phi_0 \times \mathcal{A}^{\mathcal{I}}),
\end{aligned}
$$

we obtain the left unitor λ' of $\mathrm{Inv}\,\mathcal{A}$, modulo the commutations shown in the following diagram

$$
\begin{array}{c}
\text{(diagram with vertices } 1^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}},\ \mathcal{A}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}},\ \Phi_{\mathcal{A}},\ (1),\ \Phi_0 \times \mathcal{A}^{\mathcal{I}},\\
\mathbf{1}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}} \xrightarrow{\Phi_{\mathcal{A}}} (\mathbf{1} \times \mathcal{A})^{\mathcal{I}} \xrightarrow{(1\times\mathcal{A})^{\mathcal{I}}} (\mathcal{A} \times \mathcal{A})^{\mathcal{I}},\ \otimes^{\mathcal{I}},\\
\mathbf{1} \times \mathcal{A}^{\mathcal{I}},\ \lambda',\ \Lambda^{\mathcal{I}},\ \mathcal{A}^{\mathcal{I}},\\
\Phi_0 \times \mathcal{A}^{\mathcal{I}},\ \mathbf{1}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}} \xrightarrow{\Phi_{\mathcal{A}}} (\mathbf{1} \times \mathcal{A})^{\mathcal{I}},\ (2),\ \Lambda)
\end{array}
$$

Indeed, (1) holds because of the naturality of Φ, and (2) because $(-)^{\mathcal{I}}$ is monoidal.

Symmetry: Again, if we denote by Γ the endofunctor of $\mathcal{A} \times \mathcal{A}$ which "swaps" the two occurrences of $\mathcal{A}$ (*i.e.*, the involutive braiding of $(\mathbf{Cat}, \times, 1)$ seen as a symmetric monoidal category, instantiated on $\mathcal{A}, \mathcal{A}$), we see that the involutive braiding γ of $\mathcal{A}$ is of the following type:

$$
\begin{array}{ccccc}
 & \xrightarrow{\otimes} & & & \\
\mathcal{A} \times \mathcal{A} & & \Downarrow \gamma & & \mathcal{A} \\
 & \xrightarrow{\Gamma} & \mathcal{A} \times \mathcal{A} & \xrightarrow{\otimes} &
\end{array}
$$

Then, once more, we obtain the involutive braiding γ' of $\mathrm{Inv}\,\mathcal{A}$ by applying Lemma 10 to the following data:

$$
\begin{aligned}
F &= \otimes, \\
G &= \otimes \circ \Gamma, \\
H &= \Phi_{\mathcal{A}},
\end{aligned}
$$

modulo the commutation shown in the following diagram, which holds because $(-)^{\mathcal{I}}$ is symmetric monoidal:

We have just defined the object map of Inv; let us define it on morphisms. Let (F, m, u) be a symmetric monoidal functor from $(\mathcal{A}, \otimes, 1)$ to $(\mathcal{B}, \odot, U)$. We set $\mathrm{Inv}\,F = (F^{\mathcal{I}}, m', u')$, where

$$
\begin{aligned}
m' &= m^{\mathcal{I}} \circ \iota_{\Phi_{\mathcal{A}}} \\
u' &= u^{\mathcal{I}} \circ \iota_{\Phi_0}
\end{aligned}
$$

in which $\Phi_{\mathcal{A}}$ and Φ_0 denote the same functors as above (coming from the monoidality of $(-)^{\mathcal{I}}$), $\iota_{\Phi_{\mathcal{A}}}$ and ι_{Φ_0} their respective identity natural transformations, and $\circ$ is horizontal composition of natural transformations. We can see that m' has the right type as follows: we have $m : \odot \circ (F \times F) \Rightarrow F \circ \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$, so $m^{\mathcal{I}} : \odot^{\mathcal{I}} \circ (F \times F)^{\mathcal{I}} \Rightarrow F^{\mathcal{I}} \circ \otimes^{\mathcal{I}} : (\mathcal{A} \times \mathcal{A})^{\mathcal{I}} \to \mathcal{B}^{\mathcal{I}}$; composing horizontally with $\iota_{\Phi_{\mathcal{A}}} : \Phi_{\mathcal{A}} \Rightarrow \Phi_{\mathcal{A}} : \mathcal{A}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}} \to (\mathcal{A} \times \mathcal{A})^{\mathcal{I}}$ yields

$$
m' : \odot^{\mathcal{I}} \circ (F \times F)^{\mathcal{I}} \circ \Phi_{\mathcal{A}} \Rightarrow F^{\mathcal{I}} \circ \otimes^{\mathcal{I}} \circ \Phi_{\mathcal{A}} : \mathcal{A}^{\mathcal{I}} \times \mathcal{A}^{\mathcal{I}} \to \mathcal{B}^{\mathcal{I}}.
$$

Horizontal source and target are as expected; the vertical target is equal to $F^{\mathcal{I}} \circ \otimes'$, as desired; for what concerns the vertical source, if we remember that $\Phi_{\mathcal{A}}$ is the component of the natural transformation of the monoidality of $(-)^{\mathcal{I}}$, we have $\odot^{\mathcal{I}} \circ (F \times F)^{\mathcal{I}} \circ \Phi_{\mathcal{A}} = \odot^{\mathcal{I}} \circ \Phi_A \circ (F^{\mathcal{I}} \times F^{\mathcal{I}}) = \odot' \circ (F^{\mathcal{I}} \times F^{\mathcal{I}})$, as expected.

If we spell them out, the components of m' and u' are actually extremely simple: if $(A, s), (A', s')$ are objects with involutions of $\mathcal{A}$, we have $m'_{(A,s),(A',s')} = m_{A,A'}$, and u' is actually u itself. This shows immediately the monoidality of $(F^{\mathcal{I}}, m', u')$, because if a diagram commutes in $\mathcal{B}$, it commutes *a fortiori* in $\mathcal{B}^{\mathcal{I}}$ (composition of morphisms is the same).

The definition of Inv on cells is trivial: if $\mathcal{A}, \mathcal{B}$ are symmetric monoidal categories, $F, G : \mathcal{A} \to \mathcal{B}$ symmetric monoidal functors, and $\varphi : F \Rightarrow G$ a

monoidal natural transformation, we simply put $\operatorname{Inv}\varphi = \varphi^{\mathcal{I}}$; the monoidality of $\operatorname{Inv}\varphi$ follows immediately from that of φ and from the definitions of $\operatorname{Inv} F$ and $\operatorname{Inv} G$.

2.4. Building stratified models

We may now state the fundamental properties of the Inv construction:

Lemma 11. *The* Inv *functor preserves linear-non-linear adjunctions.*

PROOF. The fact that Inv is a 2-endofunctor of **SMCat** immediately implies that it preserves monoidal adjunctions (*cf.* Appendix A). The only thing that is left to check is that, whenever a monoidal category $\mathcal{M}$ is Cartesian, $\operatorname{Inv}\mathcal{M}$ is still Cartesian. This is an easy verification, which we leave to the reader. ☐

Lemma 12. *The* Inv *functor preserves* ∗*-autonomous categories.*

PROOF. We start by proving that Inv preserves monoidal closure. Let $(\mathcal{A}, \otimes, 1)$ be a symmetric monoidal closed category, with the right adjoint of $\otimes$ denoted by $\multimap$, as usual. Let $L, R : \mathcal{A}^{\mathrm{op}} \times \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$ denote the hom-functors $\mathcal{A}[- \otimes -, -]$ and $\mathcal{A}[-, - \multimap -]$, respectively. On morphisms, L and R act as follows: if $s : A' \to A$, $t : B' \to B$, and $u : C \to C'$ are morphisms of $\mathcal{A}$, we have

$$
\begin{aligned}
L(s,t,u) &= \lambda f.uf(s \otimes t) : \mathcal{A}[A \otimes B, C] \to \mathcal{A}[A' \otimes B', C'], \\
R(s,t,u) &= \lambda f.(t \multimap u)fs : \mathcal{A}[A, B \multimap C] \to \mathcal{A}[A', B' \multimap C'],
\end{aligned}
$$

where we used λ-calculus notation and we wrote composition in $\mathcal{A}$ by simple juxtaposition. By Definition 9, we have a natural isomorphism $\Psi : L \Rightarrow R$. Naturality means that $\Psi \circ L = R \circ \Psi$; more explicitly, for all $s : A' \to A$, $t : B' \to B$, $u : C \to C'$, and $f \in [A \otimes B, C]$,

$$
\Psi_{A,B,C}(uf(s \otimes t)) = (t \multimap u)\,\Psi_{A,B,C}(f)\,s.
$$

Now, if we define the bifunctor $\multimap'$ of $\operatorname{Inv}\mathcal{A}$ by $(A, s) \multimap' (B, t) = (A \multimap B, (s_k \multimap t_k)_{k \in \mathbb{Z}})$ on objects and $f \multimap' g = f \multimap g$ on morphisms (checking that this is indeed a functor from $(\operatorname{Inv}\mathcal{A})^{\mathrm{op}} \times \operatorname{Inv}\mathcal{A}$ to $\operatorname{Inv}\mathcal{A}$ is straightforward), we claim that the functors $L' = (\operatorname{Inv}\mathcal{A})[- \otimes' -, -]$ and $R' = (\operatorname{Inv}\mathcal{A})[-, - \multimap' -]$ are still naturally isomorphic. In fact, we shall prove that a natural isomorphism is given by Ψ itself, restricted to morphisms of objects with involutions.

For all objects $(A, s), (B, t), (C, u)$ of $\operatorname{Inv}\mathcal{A}$, by definition, $(\operatorname{Inv}\mathcal{A})[(A, s) \otimes' (B, t), (C, u)]$ and $(\operatorname{Inv}\mathcal{A})[(A, s), (B, t) \multimap' (C, u)]$ are subsets of $\mathcal{A}[A \otimes B, C]$ and $\mathcal{A}[A, B \multimap C]$, respectively. Then, given $f \in (\operatorname{Inv}\mathcal{A})[(A, s) \otimes' (B, t), (C, u)]$, we define $\Psi'_{(A,s),(B,t),(C,u)}(f) = \Psi_{A,B,C}(f)$. It is now enough to prove that $\Psi_{A,B,C}(f) \in (\operatorname{Inv}\mathcal{A})[(A, s), (B, t) \multimap' (C, u)]$. In other words, since $(B, t) \multimap' (C, u)$ is by definition equal to $(B \multimap C, (s_k \multimap t_k)_{k \in \mathbb{Z}})$, we need to check that, for all $k \in \mathbb{Z}$, $\Psi_{A,B,C}(f) = (t_k \multimap u_k) \circ \Psi_{A,B,C}(f) \circ s_k$. But this is an immediate consequence of the fact that f is a morphism of objects with involutions, which means that $f = u_k f(s_k \otimes t_k)$, and of the naturality of Ψ as described above.

32

Let now $(\mathcal{A}, \otimes, 1, \perp)$ be a $*$-autonomous category. We have just proved that $\mathrm{Inv}(\mathcal{A}, \otimes, 1)$ is symmetric monoidal closed; we claim that $\perp' = (\perp, id_\perp)$, where $id_\perp$ is the $\mathbb{Z}$-indexed sequence identically equal to $id_\perp$, is a dualizing object. By the definitions given above, it is actually easy to check that, for all objects $(A, s), (B, t)$ of $\mathrm{Inv}\,\mathcal{A}$, we have $\Psi'_{(A,s),(A,s)\multimap'(B,t),(B,t)}(\mathrm{eval}_{(A,s),(B,t)} \circ \gamma'_{(A,s),(A,s)\multimap'(B,t)}) = \Psi_{A,A\multimap B,B}(\mathrm{eval}_{A,B} \circ \gamma_{A,A\multimap B})$. Therefore, if we set, for a given object A, $h_A = \mathrm{eval}_{A,\perp} \circ \gamma_{A,A\multimap\perp}$, it is enough to check that $\Psi_{A,A\multimap\perp,\perp}(h_A)$ is a morphism of objects with involutions, because the fact that it is an isomorphism in $\mathrm{Inv}\,\mathcal{A}$ follows immediately from the fact that it is an isomorphism in $\mathcal{A}$. For this, we use the fact that h_A is a morphism of objects with involutions, which means that $h_A = h_A(s_k \otimes (s_k \multimap id_\perp))$ for all $k \in \mathbb{Z}$, and the naturality of Ψ, as above. $\qquad\square$

As a consequence, we have

Theorem 13 (Preservation of models). *Models of linear logic are preserved by* Inv*, i.e., if $\mathcal{L}$ is the linear category of a model of linear logic, then $\mathrm{Inv}\,\mathcal{L}$ is also the linear category of a model of linear logic.* $\qquad\square$

Observe that, up to and including Lemma 11, we never used the fact that the endomorphisms s_k in the objects (A, s) are involutions. Indeed, we may define more generally a notion of "object with endomorphisms" as a pair (A, s) where s is a $\mathbb{Z}$-indexed sequence of endomorphisms of A, not necessarily involutive, or even invertible. The notion of morphism for these objects is the same as that of Definition 12, *i.e.*, an arrow f of the original category such that $fs_k = t_k f$ for all $k \in \mathbb{Z}$, where s and t are the sequences of endomorphisms of the source and target object, respectively. This gives again a category, of objects with endomorphisms and their morphisms (the free Coxeter groupoid $\mathcal{I}$ is replaced by the free monoid on $\mathbb{Z}$). From this, a 2-endofunctor of **SMCat**, call it $\mathcal{F}$, may be defined, which shares with Inv every property mentioned in this section, except Lemma 12. Indeed, this last result uses in a somewhat hidden but crucial way the fact that, given objects with involutions $(A, s), (B, t)$, a morphism between them is an arrow $f : A \to B$ such that $f = t_k f s_k$, for all $k \in \mathbb{Z}$, which is equivalent to $fs_k = t_k f$ because s_k (or t_k) is an involution.

It is easy to see that requiring the endomorphisms to be involutions is in general necessary for $\mathcal{F}$ to preserve $*$-autonomous categories (*i.e.*, to have Lemma 12), or even monoidal closure, which is an essential requirement for a model of linear logic. In fact, consider the category $(\mathbf{Set}, \times, 1)$ of sets and functions, with the usual Cartesian product, and where 1 is a singleton. This is the prototypical Cartesian closed category, which is therefore symmetric monoidal closed. We claim that $\mathcal{F}(\mathbf{Set})$, defined mimicking the definition of $\mathrm{Inv}(\mathbf{Set})$, is not closed. If it were, we would have that the set $\mathcal{F}(\mathbf{Set})[(A, s), (B, t)]$, where A, B are sets and s, t $\mathbb{Z}$-indexed sequences of functions on them, would be in bijection with the set $\mathcal{F}(\mathbf{Set})[(1, id_1), (B^A, t^s)]$, where B^A is the set of all functions from A to B, and, given $f \in B^A$ and $k \in \mathbb{Z}$, $(t^s)_k(f) = t_k f s_k$. Now, $\mathcal{F}(\mathbf{Set})[(1, id_1), (B^A, t^s)]$ is in bijection with the set of all functions $f : A \to B$ such that $f = t_k f s_k$ for all $k \in \mathbb{Z}$. But this set is too "small"; indeed, given

some $g \in \mathcal{F}(\mathbf{Set})[(A, s), (B, t)]$, we do know that $gs_k = t_k g$ for all $k \in \mathbb{Z}$, but there is no reason for the equality $g = t_k gs_k$ to hold as well, if s_k and t_k are arbitrary functions from A and B to themselves.

The model resulting from the Inv construction is in some sense "isomorphic" to the original one. Nevertheless, the construction is of interest because it yields a model of *stratified* linear logic which is never degenerate, unless the original model is trivial. In order to prove this, we first need to recall a basic categorical result concerning models of linear logic.

Lemma 14. *Let $(\mathcal{L}, \otimes, 1)$ be the linear category of a model of linear logic, with the symmetry of $\otimes$ denoted by γ, and suppose that X is an object of $\mathcal{L}$ such that $\gamma_{X,X} = id_{X \otimes X}$. Then, there is at most one arrow in $\mathcal{L}$ from 1 to X.*

PROOF. Let X be an object of $\mathcal{L}$. Consider the first projection $\pi_1 : X \& X \rightarrow X$; pre-composing $!\pi_1$ with the exponential isomorphism (of type $!X \otimes !X \rightarrow !(X \& X)$) and post-composing it with the counit of the comonad (of type $!X \rightarrow X$), we obtain a morphism $h : !X \otimes !X \rightarrow X$. Take now any two morphisms $f, f' : 1 \rightarrow X$, and consider the morphisms $g = h \circ (!f \otimes !f')$ and $g' = h \circ \gamma_{B,B} \circ (!f \otimes !f')$. Elementary computations show that $g = f$ and $g' = f'$. But under the hypothesis $\gamma_{X,X} = id_{X \otimes X}$ we obviously have $g = g'$, and therefore $f = f'$. $\qquad\square$

Theorem 15. *Every non-trivial model of linear logic yields, through the Inv construction, a non-degenerate model of $\mathbf{LL}_\S$.*

PROOF. Let $\mathcal{L}$ be the linear category of a non-trivial model of linear logic. By Theorem 13, $\mathrm{Inv}\,\mathcal{L}$ also hosts a model of linear of logic; by Definition 11, in order to define a model of $\mathbf{LL}_\S$ we only need to find an interpretation of the paragraph modality. This may be done by fixing some $n \in \mathbb{Z}$ and considering the endofunctor $\S_n$ of $\mathrm{Inv}\,\mathcal{L}$ defined as follows. If (A, s) is an object of $\mathrm{Inv}\,\mathcal{L}$, we set $\S_n(A, s) = (A, (s_{k-n})_{k \in \mathbb{Z}})$, *i.e.*, the image of (A, s) through $\S_n$ has the same underlying object, but the sequence of involutions is "shifted" by n; on morphisms, $\S_n$ is the identity. For convenience, we set $\S = \S_1$, but what we shall say holds for all $n \in \mathbb{Z}$.

It is immediate to see that the functor $\S$ satisfies all of the requirements of Definition 11; we leave the verification to the reader. The fact that the model is non-degenerate is a consequence of Lemma 14. Indeed, composed with the non-triviality of the original model, that result ensures us that in $\mathcal{L}$ there exists an object X such that $\gamma_{X,X} \neq id_{X \otimes X}$. Consider then the object with involutions $(X \otimes X, \xi)$, where $\xi_0 = \gamma_{X,X}$ and $\xi_k = id_{X \otimes X}$ for all $k \neq 0$. Let $f : (X \otimes X, \xi) \rightarrow \S(X \otimes X, \xi)$ in the category $\mathrm{Inv}\,\mathcal{L}$. By definition, f is an endomorphism of $X \otimes X$ in $\mathcal{L}$, which must further satisfy $f \circ \gamma_{X,X} = f = f \circ id_{X \otimes X}$ (and also $\gamma_{X,X} \circ f = f$). But then f is not monic (nor epic), and thus cannot be an isomorphism. $\qquad\square$

3. Applications to Bounded Complexity

3.1. A geometric definition of $\mathbf{L^3}$

Recall how in Sect. 1.2 we reformulated linear logic by levels ($\mathbf{L^3}$), which is in some sense the largest extant stratification-based, bounded-complexity subsystem of linear logic, as the fragment of $\mathbf{LL_\S}$ in which the exponential modalities are forced to be "tied" to the paragraph modality. In light of this, using the results of Sect. 1.3, we may give a new, geometric definition of $\mathbf{L^3}$, equivalent to the original one given in Baillot and Mazza (2010) and based on indexings.

In the following, when we speak of a DR-net *of linear logic*, we mean a DR-net as in Definition 5, but containing no paragraph link. Under this assumption, DR-nets and proof nets actually coincide, but we use the term DR-net to stress that §-correctness is void. Then, we shall impose on linear logic DR-nets a criterion adapted from §-correctness, in order to characterize $\mathbf{L^3}$ within linear logic.

We start by recalling the definition of $\mathbf{L^3}$, as given in Baillot and Mazza (2010).

Definition 13 (Exponential indexing). An *exponential indexing* (called *indexing* in Baillot and Mazza (2010)) for a linear logic DR-net π is a function I from the edges of π to $\mathbb{Z}$ which:

- satisfies the constraints of Fig. 4, except for of course and why not links, for which, if $e_1, \ldots, e_n$ and e' are the premises and conclusion of such a link, we have $I(e_1) = \cdots = I(e_n) = I(e') + 1$;

- satisfies $I(e) = I(e')$ for all conclusions e, e' of π.

In other words, an exponential indexing is just as a strong indexing of Definition 7 in which exponential links behave like paragraph links.

Definition 14 (Linear logic by levels (Baillot and Mazza, 2010)). $\mathbf{L^3}$ is the subsystem of linear logic consisting of all linear logic DR-nets admitting an exponential indexing.

Let us now give the geometric definition.

Definition 15 (!?-balanced cycle). We define a !?-balanced cycle in a linear logic DR-net just as a balanced cycle (Definition 4), except that instead of counting paragraph links, we count exponential links (*i.e.*, of course and why not links alike).

Definition 16 (Geometric definition of $\mathbf{L^3}$). $\mathbf{L^3}$ is the subsystem of linear logic consisting of all linear logic DR-nets in which every cycle is !?-balanced.

That Definition 16 is equivalent to Definition 14 is an immediate consequence of Theorem 8, modulo the fact that, in the fragment of $\mathbf{LL_\S}$ corresponding to $\mathbf{L^3}$, every of course and why not link is preceded by a paragraph link, so that exponential indexings and !?-balanced cycles exactly match strong indexings and balanced cycles, respectively.

3.2. Denotational semantics of $\mathbf{L^3}$

As a subsystem of linear logic, $\mathbf{L^3}$ may be semantically interpreted in any model of linear logic. However, such a denotational semantics is absolutely uninformative by itself, because it is incapable of distinguishing a linear logic proof which is part of $\mathbf{L^3}$ from one that is not.

We shall see how the Inv construction may be used to build more informative denotational semantics of $\mathbf{L^3}$. For this, we shall use the models defined in the proof of Theorem 15, living in the category Inv $\mathcal{L}$ where $\mathcal{L}$ is the linear category of a model of linear logic. We remind that such models interpret the paragraph modality with the functor defined by $\S(A, s) = (A, (s_{k-1})_{k \in \mathbb{Z}})$ on objects and acting as the identity on morphisms. To be able to fully use Theorem 15, in the sequel we shall assume that the starting model of linear logic is non-trivial.

Recall that denotational interpretations are always parametric in an assignment of objects to propositional atoms. For all of our future purposes, it will be enough to consider what we shall call the *default assignment*, which was already defined in the proof of Theorem 15, and which we give again here.

Definition 17 (Default assignment). Let $\mathcal{L}$ be the linear category of a non-trivial model of linear logic. By Lemma 14, there exists an object X of $\mathcal{L}$ such that $\gamma_{X,X} \neq id_{X \otimes X}$. A *default assignment* maps every propositional atom to the object with involutions $(X \otimes X, \xi)$, where $\xi_0 = \gamma_{X,X}$ and $\xi_k = id_{X \otimes X}$ for all $k \neq 0$.

Of course, there is one default assignment for each object X of $\mathcal{L}$ such that $\gamma_{X,X} \neq id_{X \otimes X}$; however, since the choice of a particular X is irrelevant, we shall abusively speak of "the" default assignment, and denote by $[\![\cdot]\!]$ the interpretation map (from formulas and proofs of $\mathbf{LL_\S}$ to objects and morphisms of Inv $\mathcal{L}$, respectively) under that default assignment.

Observe that, even in non-degenerate models, the $\S$ functor still acts as the identity on morphisms; this means that, if π^- denotes the proof obtained from the $\mathbf{LL_\S}$ proof π by removing all paragraph rules, we have $[\![\pi]\!] = [\![\pi^-]\!]$. In other words, paragraphs only matter in formulas, not in proofs.

Definition 18 ($\mathbf{L^3}$-interpretation of formulas). Given a non-trivial model of linear logic whose linear category is $\mathcal{L}$, we define the $\mathbf{L^3}$-*interpretation map* $(\!|\cdot|\!)$ from linear logic formulas (*i.e.*, containing no paragraph) to objects of Inv $\mathcal{L}$ by $(\!|A|\!) = [\![A^+]\!]$, where A^+ is the $\mathbf{LL_\S}$ formula obtained by adding a paragraph modality after every exponential modality appearing in A (*i.e.*, every occurrence of ! and ? is replaced by !$\S$ and ?$\S$, respectively).

If we are given a linear logic proof π of the formula A, since linear logic is a fragment of $\mathbf{LL_\S}$, the interpretation $[\![\pi]\!]$ is always well defined, as a morphism of Inv $\mathcal{L}[(1, id_1), [\![A]\!]]$. We shall be interested in the following question:

$$\text{do we have } [\![\pi]\!] \in \text{Inv } \mathcal{L}[(1, id_1), (\!|A|\!)]?$$

This is meaningful, because A^+ and A differ only for the addition of paragraph modalities and thus, by definition of the $\S$ functor, if $[\![A]\!] = (S, \sigma)$, then $(\!|A|\!) =$

36

(S, σ'), *i.e.*, the underlying object S is the same. Therefore, a morphism of type $(1, id_1) \to (S, \sigma)$ in $\mathrm{Inv}(\mathcal{L})$, which is first of all a morphism of type $1 \to S$ in $\mathcal{L}$, may *a priori* be also a morphism of type $(1, id_1) \to (S, \sigma')$ in $\mathrm{Inv}(\mathcal{L})$.

To state the question more concisely, we shall use the notation $[\![\pi]\!] \in (\!|A|\!)$ as a short hand for $[\![\pi]\!] \in \mathrm{Inv}\,\mathcal{L}[(1, id_1), (\!|A|\!)]$. There is one interesting case in which the answer is always positive:

Theorem 16 (Soundness). *If a proof π of A belongs to $\mathbf{L^3}$, then $[\![\pi]\!] \in (\!|A|\!)$.*

PROOF. We know that π may be embedded in $\mathbf{LL}_\S$ by adding a paragraph rule before every promotion and dereliction rule (in DR-nets, we add a paragraph link above every of course and flat link), obtaining the $\mathbf{LL}_\S$ proof π^+ of conclusion A^+. We also know that the Inv construction yields models of $\mathbf{LL}_\S$, so we have that $[\![\pi^+]\!] \in \mathrm{Inv}\,\mathcal{L}[(1, id_1), [\![A^+]\!]]$, which means $[\![\pi^+]\!] \in (\!|A|\!)$. But $(\pi^+)^- = \pi$, hence, as remarked after Definition 17, we have $[\![\pi^+]\!] = [\![\pi]\!]$. $\qquad\square$

Therefore, testing whether $[\![\pi]\!] \in (\!|A|\!)$ is always a way to obtain negative information about the membership of a linear logic proof π to $\mathbf{L^3}$; if the answer is negative, then π cannot belong to $\mathbf{L^3}$.

There is no hope for the converse of Theorem 16 to hold in general, for the simple reason that, while $\mathbf{L^3}$ is of course stable under cut-elimination, *non*-membership to $\mathbf{L^3}$ is *not* itself preserved under cut-elimination. It is indeed easy to exhibit a non-cut-free linear logic DR-net π violating the conditions for belonging to $\mathbf{L^3}$, such that $\pi \to_\beta^* \pi'$ with π' in $\mathbf{L^3}$. Since denotational semantics "sees through" cut-elimination, we would have $[\![\pi]\!] \in (\!|A|\!)$ whilst having $\pi \notin \mathbf{L^3}$.

In Sect. 3.4 we shall give, under the hypothesis of absence of cuts, a sufficient condition for the converse of Theorem 16 to hold. For the moment, we observe that, for some cut-free proofs, the converse is always realized. It is the case, for example, of the proof of the dereliction principle, $!A \multimap A$, which does not belong to $\mathbf{L^3}$. Indeed, if π is such a proof in the case in which A is a propositional atom, we have $[\![\pi]\!] = \varepsilon_{X \otimes X} : !(X \otimes X) \to X \otimes X$, *i.e.*, the counit of the comonad $!(-)$. Then, if we had $[\![\pi]\!] \in \mathrm{Inv}\,\mathcal{L}[(\!|!A|\!), (\!|A|\!)]$, we would have in particular $\gamma_{X,X} \circ \varepsilon_{X \otimes X} = \varepsilon_{X \otimes X}$, which elementary computations show to be inconsistent with the non-triviality of the original model (*i.e.*, we would obtain $\gamma_{X,X} = id_{X \otimes X}$).

3.3. Interactive characterization of $\mathbf{L^3}$

A particularly simple, and indeed rather uninformative model of linear logic is the so-called *syntactic model*. The linear category of this model, which we denote by $\mathbf{PN}$, is defined as follows:

- its objects are the formulas of propositional multiplicative exponential linear logic (note that there is no paragraph modality);

- a morphism from a formula A to a formula B is a cut-free DR-net (or proof net, the two coincide in the absence of the paragraph modality) with atomic axioms (*i.e.*, η-expanded), of conclusions $A^\perp, B$;

- composition is given by forming a cut and computing the cut-free form. This operation is well defined because cut-elimination always terminates and, in the propositional case, preserves atomic axioms; it is associative thanks to confluence. The identities of the category are η-expansions of axioms.

Note that, in this paper, we did not define proof nets for additive connectives, so technically speaking **PN** is not a linear category because it lacks products and coproducts (and we cannot properly define the Cartesian category of the linear-non-linear adjunction). This is not a concern here, as we shall not need all of the structure of the model in what follows.

All semantic constructions (monoidal structure, exponentials) are defined in **PN** using the very syntactic constructions they are supposed to model, so everything works without surprises.

Although of little value *per se*, the syntactic model lends itself to an interesting application of the Inv construction. Let us describe the category Inv(**PN**) in detail:

- its objects are pairs (A, σ) where A is a formula of propositional multiplicative exponential linear logic (*i.e.*, still with no paragraph modality) and σ is a $\mathbb{Z}$-indexed sequence of cut-free, η-expanded DR-nets of conclusions $A^\perp, A$, such that when σ_k is cut with itself, it reduces to id_A (we may assume that σ is almost everywhere equal to id_A, so the set of objects of Inv(**PN**) is still denumerable);

- a morphism from (A, σ) to (B, τ) is a cut-free, η-expanded DR-net π of conclusions $A^\perp, B$ such that, for all $k \in \mathbb{Z}$, the DR-net
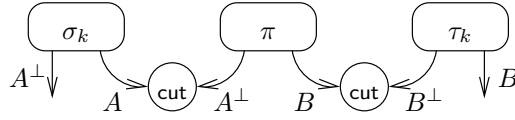
reduces to π itself. Composition and identity morphisms are the same as **PN**.

The default interpretations, which, in the case of the syntactic model, we denote by $[\![\cdot]\!]_s$ and $(\!|\cdot|\!)_s$, may be described as follows.

Definition 19 (Default atomic substitution). if A is a formula, we denote by $A^\bullet$ the formula obtained from A by replacing every atom with $X \otimes X$ (and its dual with $X^\perp \,\mathfrak{N}\, X^\perp$).

Similarly, If π is an η-expanded net of conclusion A, we denote by $\pi^\bullet$ the net of conclusion $A^\bullet$ obtained by replacing every atomic axiom of π with the net $id_{X \otimes X}$ (Fig. 9, left).

Definition 20 (Test). Let A be a formula of linear logic (*i.e.*, without paragraph modalities), and let $k \in \mathbb{Z}$. The *test* of index k of type A, denoted by θ_k^A,
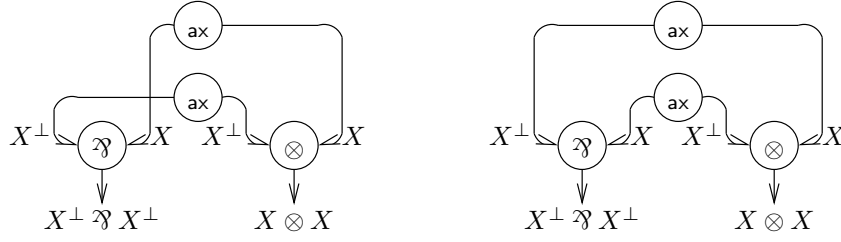
Figure 9: The morphisms $id_{X \otimes X}$ (left) and $\gamma_{X,X}$ (right) in **PN**.
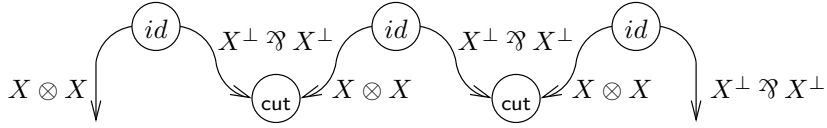

Figure 10: A foot. The node *id* represents the net $id_{X \otimes X}$.

is the DR-net of conclusions $(A^\bullet)^\perp, A^\bullet$ obtained from $id_{A^\bullet}$ by replacing to *all* occurrences of $id_{X \otimes X}$ appearing at exponential depth k (*cf.* Definition 2) the net $\gamma_{X,X}$ (Fig. 9, right). We set $\theta^A = (\theta_k^A)_{k \in \mathbb{Z}}$.

Lemma 17. *Let A be a formula of linear logic. Then, $(\!|A|\!)_s = (A^\bullet, \theta^A)$.*

PROOF. A straightforward induction on A. □

The interpretation of proofs is particularly easy to describe. If π is a linear logic DR-net of conclusion A, whose cut-free and η-expanded form is π_0, $[\![\pi]\!]_s$ is simply $\pi_0^\bullet$ with a bottom link juxtaposed to it. The bottom link is technically necessary to obtain a morphism of Inv(**PN**) whose source is $(1, id_1)$.

We shall now prove that, for cut-free proofs, the converse of Theorem 16 holds in the syntactic model. We start by proving a property of the reduction of DR-nets involving identity nets. We first need a couple of preliminary definitions; the first one is completely standard in rewriting theory, specialized here to proof nets following Tortora de Falco (2003).

Definition 21 (Lift, residue). Whenever $\pi \to_\beta \pi'$, by simple inspection of the cut-elimination rules it is clear that any edge or link (except cut) x' of π', comes from a unique ("the same") edge or link x of π; we say that x is the *lift* of x', denoted by $\overleftarrow{x'}$, and that x' is a *residue* of x. When it makes sense, we define in the same way the lift and residues of a subnet.

Definition 22 (Foot). A *foot* is a net composed of a chain of three $id_{X \otimes X}$ nets cut together as in Fig. 10. The $id_{X \otimes X}$ net whose both conclusions are premises of cuts is called the *inner toe* of the foot; the other two are called *outer toes*.
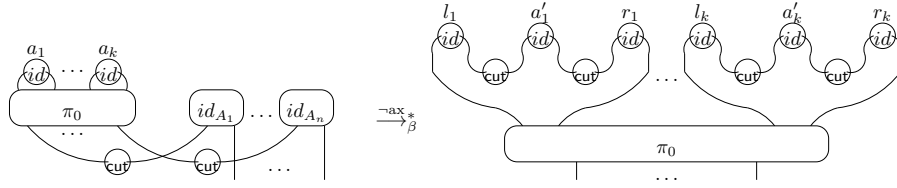
Figure 11: Graphical representation of Lemma 18. Formulas and the orientation of edges are omitted. We suppose that π contains k axioms, which become k subnets of the form $id_{X \otimes X}$ in $\pi^\bullet$, represented here by the id nodes $a_1, \ldots, a_k$; the boxes that may be present in π are not drawn in the picture, and π_0 represents "π without the axioms". For each $1 \le i \le k$, a_i' is a residue of a_i, and l_i, r_i are residues of $id_{X \otimes X}$ subnets of id_{A_i}.

In the following, if π is a net of conclusions $A_1, \ldots, A_n$, none of which is a $\flat$-formula, and if $\pi_1, \ldots, \pi_n$ are nets such that, for all $1 \le i \le n$, π_i has a unique occurrence of $A_i^\perp$ among its conclusions, we denote by $\text{cut}(\pi, \pi_1, \ldots, \pi_n)$ the net obtained by juxtaposing all of the nets $\pi, \pi_1, \ldots, \pi_n$ and adding n cut links of premises $A_i, A_i^\perp$, where A_i ranges over the conclusions of π.

Furthermore, we write $\pi \xrightarrow{\neg\text{ax}}_{\beta}^* \pi'$ when $\pi \to_\beta^* \pi'$ without using axiom steps (Fig. 5).

Lemma 18. *Let π be a cut-free, η-expanded net of conclusions $A_1, \ldots, A_n$, none of which is a $\flat$-formula. Then, $\text{cut}(\pi^\bullet, id_{A_1}, \ldots, id_{A_n}) \xrightarrow{\neg\text{ax}}_{\beta}^* \pi'$, where π' is obtained from $\pi^\bullet$ by replacing all occurrences of the subnet $id_{X \otimes X}$ with feet such that, for each foot, the external toes are residues of an $id_{X \otimes X}$ net of id_{A_i} for some i, and the inner toe is the residue of the $id_{X \otimes X}$ subnet of $\pi^\bullet$ which is replaced by that foot (see Fig. 11).*

PROOF. A straightforward induction on the size of π. □

We now recur once more to indexings, but we modify them so that every net becomes indexable. These *quasi-indexings* of course have a mere technical value: they are preserved under cut-elimination not involving axiom steps.

Definition 23 (Exponential quasi-indexing). An *exponential quasi-indexing* is defined just as an exponential indexing (Definition 13), except that it need not satisfy any constraint on axioms, *i.e.*, if σ is net, Q an exponential quasi-indexing for it, and e, e' the conclusions of an axiom link of σ, we may have $Q(e) \ne Q(e')$.

Let σ be a cut-free net. We may assign an index to all of its edges by proceeding as follows: we assign 0 to all conclusions, then we "go up", incrementing indexes whenever we cross an exponential link (of course or why not). This obviously yields an exponential quasi-indexing for σ (which is an exponential indexing iff σ is in $\mathbf{L}^3$); we call it the *default exponential quasi-indexing*.

Lemma 19. *Let π be a DR-net, let Q be an exponential quasi-indexing for π, and let $\pi \to_\beta \pi'$ by means of a non-axiom step. Then, there exists an exponential*

quasi-indexing Q' for π' such that, for every edge e of π, whenever e' is a residue of e, $Q'(e') = Q(e)$.

PROOF. We define Q' by composing the lift function l, which maps edges of π' to edges of π, with Q: $Q' = Q \circ l$. The claim may then be immediately verified by inspecting Figures 5 to 7. □

Definition 24 (Swapping relation). We define the relation $\prec$ on nets as follows: $\pi' \prec \pi$ iff π' is obtained from π by replacing a non-null number of $id_{X \otimes X}$ subnets with $\gamma_{X,X}$.

We may at last prove the main result of this section:

Lemma 20 (Swapping). *Let π be a cut-free DR-net of conclusion A. Suppose π is not in $\mathbf{L^3}$; then, there exists $k \in \mathbb{N}$ such that $\theta_k^A \circ [\![\pi]\!]_s \prec [\![\pi]\!]_s$.*

PROOF. First of all, we remark that, modulo the irrelevant addition of a bottom link, we have $[\![\pi]\!]_s = \pi_\eta^\bullet$, where π_η is the η-expanded form of π. Observe that $\mathrm{cut}(\pi_\eta^\bullet, \theta_k^A)$ admits an exponential quasi-indexing Q such that, if e is the conclusion of an $id_{X \otimes X}$ or $\gamma_{X,X}$ subnet of θ_k^A, then $Q(e)$ is equal to the exponential depth of such subnet: this is obtained by joining the default exponential quasi-indexings of $\pi_\eta^\bullet$ and θ_k^A, which are compatible because they assign the same index (that is, 0) to both premises of the cut link introduced in $\mathrm{cut}(\pi_\eta^\bullet, \theta_k^A)$.

Now, the fact that π is not in $\mathbf{L^3}$ implies that there exists an axiom of π whose conclusions are assigned different indexes by Q. The η-expansion and substitution with $id_{X \otimes X}$ propagate the mismatch, so in $\pi_\eta^\bullet$ there is a subnet $id_{X \otimes X}$, call it a, which is is "wrongly" indexed, *i.e.*, its conclusions e_1, e_2 are such that $Q(e_1) \neq Q(e_2)$.

Let $i = Q(e_1)$ and $j = Q(e_2)$, and consider the test θ_i^A (the reader is invited to check that the same arguments would apply if we chose θ_j^A); by Lemma 18, we have $\mathrm{cut}(\pi_\eta^\bullet, \theta_k^A) \xrightarrow{\neg\mathrm{ax}}^*_\beta \pi'$ as in Fig. 11, modulo the fact that the outer toes of the feet present in π' may now be $\gamma_{X,X}$ nets.

Consider the $id_{X \otimes X}$ subnet a' of π' which is the residue of a (the "wrongly" indexed subnet $id_{X \otimes X}$ of $\pi_\eta^\bullet$), and let e_1', e_2' be its conclusions. By Lemma 19, we have an indexing Q' for π' such that $Q'(e_1') = i$ and $Q'(e_2') = j$. Moreover, using again Lemma 18, a' is the inner toe of a foot whose outer toes, which we denote by t_1, t_2, are residues of $id_{X \otimes X}$ or $\gamma_{X,X}$ subnets $\overleftarrow{t_1}, \overleftarrow{t_2}$ of θ_i^A. We claim that t_1 is a $\gamma_{X,X}$ net, while t_2 is an $id_{X \otimes X}$ net; this is enough to conclude, because the foot composed of t_1, a', t_2 reduces to a $\gamma_{X,X}$ net, and therefore the cut-free form π_1 of π' is obtained from $\pi_\eta^\bullet$ by replacing at least one $id_{X \otimes X}$ subnet with $\gamma_{X,X}$, which proves $\pi_1 \prec \pi_\eta^\bullet$.

So let d_1, d_1' and d_2, d_2' be the conclusions of t_1 and t_2, respectively. Again by Lemma 19, and by the fact that Q is actually an indexing for θ_i^A, we have $Q'(d_1) = Q'(d_1')$ and $Q'(d_2) = Q'(d_2')$. But even in exponential quasi-indexings the premises of cut links must have the same index, so $Q'(d_1) = i$ and $Q'(d_2) = j$, which means that the exponential depths of $\overleftarrow{t_1}$ and $\overleftarrow{t_2}$ are i and j, respectively. Then, t_1 is of the form $\gamma_{X,X}$, while t_2 is of the form $id_{X \otimes X}$, as claimed. □

Since everything in the syntactic model is computed through cut-elimination, the Swapping Lemma 20, together with Theorem 16 and Lemma 17, actually give us yet another definition of $\mathbf{L^3}$, which is of interactive nature, *i.e.*, a net is in $\mathbf{L^3}$ iff it "passes all tests". Of course, this only works for cut-free nets.

Theorem 21 (Interactive characterization of $\mathbf{L^3}$). *Let π be a cut-free linear logic DR-net of conclusion A, whose η-expansion we denote by π_η. Then, π is in $\mathbf{L^3}$ iff, for all $k \in \mathbb{N}$, $\mathrm{cut}(\pi_\eta^\bullet, \theta_k^A) \to_\beta^* \pi_\eta^\bullet$.* □

3.4. Semantic characterization of $\mathbf{L^3}$

Basically, the syntactic model interprets proofs with cut-free, η-expanded proofs. These latter may still be interpreted by means of a non-syntactic model, into a category $\mathcal{L}$. An easy but important remark is that the default interpretation of a non-syntactic model always factors through the default interpretation of the syntactic model, via a canonical interpretation of cut-free, η-expanded proofs:

$$
\begin{array}{ccc}
\text{linear logic} & \xrightarrow{\;\llbracket \cdot \rrbracket_s\;} & \mathbf{PN} \\
& \searrow_{\llbracket \cdot \rrbracket} & \downarrow \\
& & \mathcal{L}
\end{array}
$$

Consider a non-trivial model of linear logic, whose linear category is $\mathcal{L}$, and let Ξ be an object of $\mathcal{L}$ such that $\gamma_{\Xi,\Xi} \neq id_{\Xi \otimes \Xi}$, *i.e.*, the object used by the default assignment. We define an assignment mapping all propositional atoms to Ξ; then, the interpretation map of the model under this assignment yields a functor from $\mathbf{PN}$ to $\mathcal{L}$, which we denote by $\mathcal{X}$.

Proposition 22 (Factoring). *For every non-trivial model of linear logic, we have $\llbracket \cdot \rrbracket = \mathrm{Inv}(\mathcal{X}) \circ \llbracket \cdot \rrbracket_s$.*

PROOF. For clarity, we shall denote by $\odot$ the semantic tensor, *i.e.*, the functor interpreting in $\mathcal{L}$ the multiplicative conjunction $\otimes$ of linear logic.

In the case of formulas, the proof is by induction. Let Z be a propositional atom; we have $\mathrm{Inv}(\mathcal{X})(\llbracket Z \rrbracket_s) = \mathrm{Inv}(\mathcal{X})(X \otimes X, \xi) = (\Xi \odot \Xi, (\mathcal{X}(\xi_k))_{k \in \mathbb{Z}})$, where ξ_0 is the $\gamma_{X,X}$ net and ξ_k is the identity net everywhere else. But then $\mathcal{X}(\xi_0) = \gamma_{\Xi,\Xi}$ and $\mathcal{X}(\xi_k)$ is the identity everywhere else, so $\mathrm{Inv}(\mathcal{X})(\llbracket Z \rrbracket_s) = \llbracket Z \rrbracket$. The inductive cases are straightforward.

For proofs, recall that $\mathrm{Inv}(\mathcal{X})$ acts like $\mathcal{X}$ on morphisms. Then, for every DR-net π whose cut-free, η-expanded form is π_0, we have $\mathcal{X}(\llbracket \pi \rrbracket_s) = \mathcal{X}(\pi_0^\bullet) = \llbracket \pi_0 \rrbracket = \llbracket \pi \rrbracket$. □

Lemma 23. *Consider a non-trivial model of linear logic, let A be a linear logic formula, and let $\llbracket A \rrbracket = (S, s)$. Then, for all $k \in \mathbb{Z}$, $s_k = \mathcal{X}(\theta_k^A)$, where the nets θ_k^A are the tests introduced in Definition 20.*

PROOF. Again, by induction on A. □

Thanks to Proposition 22 and Lemma 23, the interactive characterization of Theorem 21 may be lifted from the syntax to any non-syntactic model, provided the model does not "blur out" too many differences in proofs.

Definition 25 (Swap-sensitivity). A non-trivial model of linear logic is said to be *swap-sensitive* if, for every cut-free, η-expanded DR-net π, $\pi' \prec \pi$ implies $\mathcal{X}(\pi') \neq \mathcal{X}(\pi)$.

Swap-sensitivity is a very mild condition; it is enjoyed by all models of linear logic which we are aware of. For instance, coherence spaces are swap-sensitive, which implies the swap-sensitivity of all web-based models of linear logic (relational model, finiteness spaces, etc.).

We may now give a converse of Theorem 16:

Theorem 24 (Semantic characterization of $\mathbf{L^3}$). *Let π be a cut-free linear logic DR-net of conclusion A. Then, for every swap-sensitive model of linear logic, π is in $\mathbf{L^3}$ iff $[\![\pi]\!] \in (\!|A|\!)$.*

PROOF. The forward implication is Theorem 16. For what concerns the converse, let π be a cut-free linear logic DR-net of conclusion A, whose η-expansion we denote by π_η, such that $\pi \notin \mathbf{L^3}$. By Lemma 20, there exists $k \in \mathbb{N}$ such that $\mathrm{cut}(\theta_k^A, \pi_\eta^\bullet)$ reduces to a net π' such that $\pi' \prec \pi_\eta^\bullet$. Let $(\!|A|\!) = (S, s)$. Then, using Lemma 23, Proposition 22, and the swap-sensitivity of the model, we have

$$s_k \circ [\![\pi]\!] = \mathcal{X}(\theta_k^A) \circ \mathcal{X}([\![\pi]\!]_s) = \mathcal{X}(\theta_k^A \circ [\![\pi]\!]_s) \neq \mathcal{X}([\![\pi]\!]_s) = [\![\pi]\!],$$

which proves that $[\![\pi]\!] \notin (\!|A|\!)$. $\qquad\square$

References

Asperti, A., Roversi, L., 2002. Intuitionistic Light Affine Logic. ACM Transactions on Computational Logic 3 (1), 137–175.

Atassi, V., Baillot, P., Terui, K., 2006. Verification of Ptime reducibility for System F terms via Dual Light Affine Logic. In: Proceedings of CSL 2006. Vol. 4207 of LNCS. Springer, pp. 150–166.

Baillot, P., 2004. Stratified coherence spaces: a denotational semantics for Light Linear Logic. Theoretical Computer Science 318 (1-2), 29–55.

Baillot, P., Mazza, D., 2010. Linear logic by levels and bounded time complexity. Theoretical Computer Science 411 (2), 470–503.

Baillot, P., Pedicini, M., 2001. Elementary complexity and geometry of interaction. Fundamenta Informaticae 45 (1-2), 1–31.

Baillot, P., Terui, K., 2004. Light types for polynomial time computation in lambda-calculus. In: Proceedings of LICS'04. IEEE Computer Society Press, pp. 266–275.

Barr, M., 1979. *-Autonomous Categories. Vol. 752 of Lecture Notes in Mathematics. Springer-Verlag.

Bellantoni, S., Cook, S., 1992. New recursion-theoretic characterization of the polytime functions. Computational Complexity 2, 97–110.

Benton, N., 1994. A mixed linear and non-linear logic: Proofs, terms and models (extended abstract). In: Pacholski, L., Tiuryn, J. (Eds.), Proceedings of CSL '94. Vol. 933 of Lecture Notes in Computer Science. Springer, pp. 121–135.

Benton, N., Bierman, G., de Paiva, V., Hyland, M., 1992. Term assignment for intuitionistic linear logic. Technical Report 262, Computer Laboratory, University of Cambridge.

Bierman, G., 1995. What is a categorical model of intuitionistic linear logic? In: Dezani-Ciancaglini, M., Plotkin, G. (Eds.), Proceedings of TLCA '95. Vol. 902 of Lecture Notes in Computer Science. Springer, pp. 78–93.

Coppola, P., Dal Lago, U., Ronchi Della Rocca, S., 2005. Elementary affine logic and the call-by-value lambda calculus. In: Proc. of TLCA 2005. Vol. 3461 of LNCS. Springer, pp. 131–145.

Coppola, P., Martini, S., 2006. Optimizing optimal reduction. A type inference algorithm for elementary affine logic. ACM Transactions on Computational Logic 7 (2), 219–260.

Curien, P.-L., Herbelin, H., Krivine, J.-L., Melliès, P.-A., 2010. Interactive Models of Computation and Program Behavior. American Mathematical Society.

Danos, V., 1989. Logique linéaire, etude statique et dynamique. Ph.D. Thesis, Université Paris 7.

Danos, V., Joinet, J.-B., 2003. Linear logic & elementary time. Inform. and Comput. 183, 123–137.

Danos, V., Regnier, L., 1989. The structure of multiplicatives. Archive for Mathematical Logic 28, 181–203.

Danos, V., Regnier, L., 1995. Proof nets and the Hilbert space. In: Girard, J.-Y., Lafont, Y., Regnier, L. (Eds.), Advances in Linear Logic. Cambridge University Press, pp. 307–328.

de Carvalho, D., 2007. Sémantiques de la logique linéaire et temps de calcul. Ph.D. Thesis, Universitée de la Méditerrannée.

Fagin, R., 1974. Generalized first-order spectra and polynomial-time recognizable sets. In: Karp, R. (Ed.), Complexity and Computation. Vol. 7. SIAM-AMS Proceedings, pp. 43–73.

Gaboardi, M., Marion, J.-Y., Rocca, S. R. D., 2011. A implicit characterization of PSPACE. ACM Transactions on Computational LogicTo appear.

Girard, J.-Y., 1987. Linear logic. Theoretical Computer Science 50 (1), 1–102.

Girard, J.-Y., 1996. Proof-nets: the parallel syntax for proof-theory. In: Agliano, P., Ursini, A. (Eds.), Logic and Algebra. Marcel Dekker, pp. 97–124.

Girard, J.-Y., 1998. Light linear logic. Inform. and Comput. 14 (3), 175–204.

Girard, J.-Y., Scedrov, A., Scott, P., 1992. Bounded linear logic: A modular approach to polynomial time computability. Theoretical Computer Science 97, 1–66.

Grishin, V. N., 1982. Predicate and set-theoretic calculi based on a logic without contractions. Math. of USSR, Izvestiya 18 (1), 41–59.

Guerrini, S., Martini, S., Masini, A., 1998. An Analysis of (Linear) Exponentials Based on Extended Sequents. Logic Journal of the IGPL 6 (5), 735–753.

Hofmann, M., 2003. Linear Types and Non-Size-Increasing Polynomial Time Computation. Information and Computation 183 (1), 57–85.

Jones, N. D., 1999. LOGSPACE and PTIME characterized by programming languages. Theoretical Computer Science 228 (1-2), 151–174.

Kanovich, M. I., Okada, M., Scedrov, A., 2003. Phase semantics for light linear logic. Theoretical Computer Science 294 (3), 525–549.

Lafont, Y., 1988. Logiques, catégories et machines. Ph.D. Thesis, Universitée Paris 7.

Lafont, Y., 2004. Soft linear logic and polynomial time. Theoret. Comput. Sci. 318 (1–2), 163–180.

Lago, U. D., Laurent, O., 2008. Quantitative game semantics for linear logic. In: Kaminski, M., Martini, S. (Eds.), Proceedings of CSL 2008. Vol. 5213 of Lecture Notes in Computer Science. Springer, pp. 230–245.

Laurent, O., Jun. 2009. On the categorical semantics of elementary linear logic. Theory and Applications of Categories 22 (10), 269–301.

Laurent, O., Tortora de Falco, L., 2006. Obsessional cliques: a semantic characterization of bounded time complexity. In: Proceedings of LICS'06. IEEE Computer Society Press, pp. 179–188.

Leivant, D., 1994. A foundational delineation of poly-time. Information and Computation 110 (2), 391–420.

Leivant, D., Marion, J.-Y., 1993. Lambda-calculus characterisations of polytime. Fundamenta Informaticae 19, 167–184.

Masini, A., 1992. 2-sequent calculus: A proof-theory of modalities. Annals of Pure and Applied Logic 58, 229–246.

Mazza, D., 2006. Linear logic and polynomial time. Mathematical Structures in Computer Science 16 (6), 947–988.

Melliès, P.-A., Jun. 2007. Categorical models of linear logic revisited. HAL technical report, CNRS-Université Paris 7.

Quine, W. V. O., 1937. New foundations for mathematical logic. The American mathematical monthly 44, 70–80.

Schöpp, U., 2007. Stratified bounded affine logic for logarithmic space. In: Proceedings of Symposium on Logic in Computer Science (LICS'07). IEEE Computer Society, pp. 411–420.

Schwichtemberg, H., 2006. An arithmetic for polynomial-time computation. Theoretical Computer Science 357 (1–3), 202–214.

Scott, D., 1976. Data types as lattices. SIAM Journal on Computing 5 (3), 522–587.

Scott, D., Strachey, C., 1971. Toward a mathematical semantics for computer languages. In: Fox, J. (Ed.), Proc. Symp. Computers and Automata. Polytechnic Inst. of Brooklyn Press, also Technical Monograph PRG-6, Programming Research Group, Oxford University.

Seely, R., 1989. Linear logic, ∗-autonomous categories, and cofree coalgebras. In: Categories in Computer Science and Logic. Vol. 92 of Contemporary Mathematics. AMS Press, pp. 371–382.

Statman, R., 1979. The typed lambda calculus is not elementary recursive. Theoretical Computer Science 9, 73–81.

Tortora de Falco, L., 2003. Additives of linear logic and normalization – Part I: a (restricted) Church-Rosser property. Theoret. Comput. Sci. 294 (3), 489–524.

A. Some Categorical Background

Monoidal functors. In the following, when declaring a category to be (symmetric) monoidal, we shall only specify the bifunctor and the unit object of the monoidal structure, leaving implicit the natural transformations accounting for associativity, left and right unit, and—if present—symmetry; these will always be denoted by $\alpha, \lambda, \rho, \gamma$, respectively. In case there is more than one monoidal structure involved, we shall use superscripts to denote to which structure these natural transformations belong; for instance, if we have two monoidal structures given by the bifunctors $\otimes$ and $\odot$, then their associativities will be denoted by $\alpha^{\otimes}$ and $\alpha^{\odot}$, respectively.

Let $(\mathcal{A}, \otimes, 1)$ and $(\mathcal{B}, \odot, U)$ be two (symmetric) monoidal categories. A *monoidal functor* from $\mathcal{A}$ to $\mathcal{B}$ is a triple (F, m, u) where $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor, $m : F(-) \odot F(-) \Rightarrow F(- \otimes -)$ is a natural transformation, and $u : U \rightarrow F1$ is

an arrow of $\mathcal{B}$, such that the following diagrams commute in $\mathcal{B}$, for all objects A, B, C of $\mathcal{A}$:

$$
\begin{array}{ccc}
(FA \odot FB) \odot FC & \xrightarrow{\;\alpha^{\odot}\;} & FA \odot (FB \odot FC) \\
{\scriptstyle m \odot id_{FC}} \downarrow & & \downarrow {\scriptstyle id_{FA} \odot m} \\
F(A \otimes B) \odot FC & & FA \odot F(B \otimes C) \\
{\scriptstyle m} \downarrow & & \downarrow {\scriptstyle m} \\
F((A \otimes B) \otimes C) & \xrightarrow{\;F\alpha^{\otimes}\;} & F(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
U \odot FB & \xrightarrow{\;\lambda^{\odot}\;} & FB \\
{\scriptstyle u \odot id_{FB}} \downarrow & & \uparrow {\scriptstyle F\lambda^{\odot}} \\
F1 \odot FB & \xrightarrow{\;m\;} & F(1 \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
FA \odot U & \xrightarrow{\;\rho^{\odot}\;} & FA \\
{\scriptstyle id_{FA} \odot u} \downarrow & & \uparrow {\scriptstyle F\rho^{\odot}} \\
FA \odot F1 & \xrightarrow{\;m\;} & F(A \otimes 1)
\end{array}
$$

In case the natural transformation m and the arrow u are actually isomorphisms, we speak of a *strong* monoidal functor.

If the monoidal structures on $\mathcal{A}$ and $\mathcal{B}$ are symmetric, one further speaks of a *symmetric monoidal functor* whenever the following additional diagram commutes in $\mathcal{B}$, for all objects A, B of $\mathcal{A}$:

$$
\begin{array}{ccc}
FA \odot FB & \xrightarrow{\;m\;} & F(A \otimes B) \\
{\scriptstyle \gamma^{\odot}} \downarrow & & \downarrow {\scriptstyle F\gamma^{\otimes}} \\
FB \odot FA & \xrightarrow{\;m\;} & F(B \otimes A)
\end{array}
$$

Monoidal natural transformations. Let $(\mathcal{A}, \otimes, 1), (\mathcal{B}, \odot, U)$ be (symmetric) monoidal categories, let $(F, m, u), (G, n, v)$ be two (symmetric) monoidal functors between them, and let $\varphi : F \Rightarrow G$ be a natural transformation. We say that φ is *monoidal* if the following diagrams commute in $\mathcal{B}$, for all objects A, B of $\mathcal{A}$:

$$
\begin{array}{ccc}
FA \odot FB & \xrightarrow{\;\varphi_A \odot \varphi_B\;} & GA \odot GB \\
{\scriptstyle m} \downarrow & & \downarrow {\scriptstyle n} \\
F(A \otimes B) & \xrightarrow{\;\varphi_{A \otimes B}\;} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
 & U & \\
{\scriptstyle u} \swarrow & & \searrow {\scriptstyle v} \\
F1 & \xrightarrow{\;\varphi_1\;} & G1
\end{array}
$$

2-categories and 2-functors. Basically, a 2-category is a category in which the homsets are themselves categories, and in which the two compositions (the one induced by the category and the one present in the homsets) interact in a certain way.

More formally, a 2-category $\mathcal{A}$ is given by following data:

- a collection of *objects* and *morphisms* between them, forming a usual category;

- for each ordered pair of morphisms $f, g : A \to B$, a collection of *cells* which are said to be of *horizontal source A*, *vertical source f*, *horizontal target B*, and *vertical target g*; if φ is such a cell, we write $\varphi : f \Rightarrow g : A \to B$;

- for each morphism $f : A \to B$, an *identity cell* $\iota_f : f \Rightarrow f : A \to B$;

- a notion of *horizontal composition* of cells, denoted by $\circ$ (just like the usual composition of morphisms), giving, for all cells $\varphi : f \Rightarrow g : A \to B$ and $\psi : h \Rightarrow k : B \to C$, a cell $\psi \circ \varphi : h \circ f \Rightarrow k \circ g : A \to C$;

- a notion of *vertical composition* of cells, denoted by $*$, giving, for all cells $\varphi : f \Rightarrow g : A \to B$ and $\psi : g \Rightarrow h : A \to B$, a cell $\psi * \varphi : f \Rightarrow h : A \to B$;

- both notions of composition are required to be associative;

- identity cells of identity morphisms are required to be neutral elements for horizontal composition, that is, for all $\varphi : f \Rightarrow g : A \to B$, we have $\varphi \circ \iota_{id_A} = \iota_{id_B} \circ \varphi = \varphi$;

- identity cells are required to be neutral elements for vertical composition, that is, for all $\varphi : f \Rightarrow g : A \to B$, we have $\varphi * \iota_f = \iota_g * \varphi = \varphi$;

- the two compositions must satisfy the following *exchange law*, for all $\varphi : f \Rightarrow g : A \to B$, $\psi : g \Rightarrow h : A \to B$, $\theta : i \Rightarrow j : B \to C$, and $\zeta : j \Rightarrow k : B \to C$:

$$(\zeta \circ \psi) * (\theta \circ \varphi) = (\zeta * \theta) \circ (\psi * \varphi).$$

In what follows, we shall often omit horizontal source and target of cells, *i.e.*, we shall simply write $\varphi : f \Rightarrow g$ instead of $\varphi : f \Rightarrow g : A \to B$ whenever A and B are clear from the context. Moreover, if f is a morphism, in the context of an equation concerning cells we shall abusively denote its identity cell ι_f simply by f.

The prototypical example of 2-category is **Cat**, whose objects, morphisms and cells are categories, functors and natural transformations, respectively. One can check that symmetric monoidal categories, symmetric monoidal functors and monoidal natural transformations also form a 2-category, which is of special interest in this work, and which we denote by **SMCat**. A further, albeit more trivial example is given by *strict* monoidal categories: each such category induces a 2-category with one object, whose morphisms and cells are respectively the objects and arrows of the original category.

Just as functors are the natural notion of morphism for categories, 2-functors are the corresponding notion for 2-categories. Given two 2-categories $\mathcal{A}, \mathcal{B}$, a *2-functor* F from $\mathcal{A}$ to $\mathcal{B}$ is a map associating with each object A, morphism f, and cell φ of $\mathcal{A}$ an object FA, a morphism Ff, and a cell $F\varphi$ of $\mathcal{B}$, in such a way that:

- when restricting to objects and morphisms, F is a functor in the usual sense;

- if $\varphi : f \Rightarrow g : A \to B$ is a cell of $\mathcal{A}$, then $F\varphi : Ff \Rightarrow Fg : FA \to FB$;

- for every morphism f of $\mathcal{A}$, we have $F(\iota_f) = \iota_{Ff}$;

- for every horizontally-composable cells φ, ψ of $\mathcal{A}$, we have $F(\psi \circ \varphi) = F\psi \circ F\varphi$;

- for every vertically-composable cells φ, ψ of $\mathcal{A}$, we have $F(\psi * \varphi) = F\psi * F\varphi$.

Adjunctions in 2-categories. The concept of adjunction, fundamental in category theory, can be nicely reformulated using the language of 2-categories. Let $\mathcal{A}$ be a 2-category. An *adjunction* in $\mathcal{A}$ is a quadruple $(f, g, \eta, \varepsilon)$ where $f : A \to B$, $g : B \to A$ are morphisms (A, B being objects of $\mathcal{A}$) and $\eta : id_A \Rightarrow g \circ f$, $\varepsilon : f \circ g \Rightarrow id_B$ are cells, satisfying the so-called *triangle identities* (or *zig-zag identities*)

$$
\begin{aligned}
(\varepsilon \circ f) * (f \circ \eta) &= f, \\
(g \circ \varepsilon) * (\eta \circ g) &= g.
\end{aligned}
$$

The morphisms f and g are called the left and right adjoint of the adjunction, respectively, and one writes $f \dashv g$; the endomorphisms $g \circ f$ and $f \circ g$ are called the *monad* and *comonad* of the adjunction; η is called the *unit* of the monad, and ε the counit of the comonad.

If we apply the above formulation to **Cat**, we obtain the usual notion of adjunction; moreover, the endofunctor $g \circ f$ (resp. $f \circ g$) is a monad in the category A (resp. a comonad in the category B) in the usual sense, with η as its unit (resp. ε as its counit).

However, we can also apply this formulation to other 2-categories; in particular, here we are interested in *symmetric monoidal adjunctions*, which are adjunctions in **SMCat**. Spelled out, given two symmetric monoidal categories $\mathcal{A}, \mathcal{B}$, a symmetric monoidal adjunction is a quadruple $((F, m, u), (G, n, v), \eta, \varepsilon)$ such that $(F, m, u) : \mathcal{A} \to \mathcal{B}$, $(G, n, v) : \mathcal{B} \to \mathcal{A}$ are symmetric monoidal functors, $F \dashv G$ in the usual sense, but the natural transformations $\eta : Id_{\mathcal{A}} \Rightarrow G \circ F$, $\varepsilon : F \circ G \Rightarrow Id_{\mathcal{B}}$ are required to be monoidal.

An important fact that we use in our work is that 2-functors preserve adjunctions: if $\mathcal{A}, \mathcal{B}$ are 2-categories, F is a 2-functor from $\mathcal{A}$ to $\mathcal{B}$, and $(f, g, \eta, \varepsilon)$ is an adjunction in $\mathcal{A}$, then $(Ff, Fg, F\eta, F\varepsilon)$ is an adjunction in $\mathcal{B}$, as can be immediately verified by applying F to both sides of the triangle identities, and using the properties defining a 2-functor.

Another useful result, which we shall only state, is the following characterization of symmetric monoidal adjunctions among usual adjunctions (*i.e.*, adjunctions in **SMCat** among adjunctions in **Cat**):

Proposition 25. *Let $\mathcal{A}, \mathcal{B}$ be symmetric monoidal categories, and let (F, m, u) be a symmetric monoidal functor from $\mathcal{A}$ to $\mathcal{B}$ such that F admits a right adjoint G. Then, the adjunction $F \dashv G$ lifts to a symmetric monoidal adjunction $(F, m, u) \dashv (G, n, v)$ iff (F, m, u) is strong.*