The additive multiboxes

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Abstract

We introduce the new notion of additive "multibox" for linear logic proof-nets. Thanks to this notion, we define a cut-elimination procedure which associates with every proof-net of multiplicative and additive linear logic a unique cut-free one.

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1 Introduction

Linear Logic (LL), introduced in [Gir87], is a refinement of classical and intuitionistic logic: the standard connectives ("and" and "or") are split in two classes (additive and multiplicative) and the "exponentials" (the new connectives of LL) give a logical status to the structural rules of classical and intuitionistic sequent calculus.
One of the main consequences of this change of view point on structural rules was the introduction of a new syntax for logic: proof-nets, a graph-theoretic presentation which gives a more geometric account of proofs.

The multiplicatives ($\otimes$ and $\wp$) and the additives ($\oplus$ and $\&$) are the connectives of the "purely logical" fragment of linear logic: $MALL$.
From the point of view of the semantics of proofs (denotational semantics), the four connectives of $MALL$ correspond to four different ways of combining some kind of mathematical structures. In the semantics we will refer to

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in this paper (see appendix B), a multiplicative connective is (essentially) a cartesian product, while an additive one is (essentially) a disjoint union. All these constructions are very simple and standard.

From the syntactical point of view, it is difficult to imagine a greater gap than the one separating the multiplicative from the additive world: we have on the one hand the "perfect" theory of multiplicative proof-nets, developped from all angles, with (almost) half a dozen of correctness criterions (from the longtrip one of [Gir87] to the homological one of [Mét94]), and on the other hand the lack of a satisfying theory of additive proof-nets, and a *very* sparse litterature on the subject.

To deal with additive proof-nets is so difficult, that the additives are usually simply ignored, and when this is impossible, they are often encoded into non additive LL. This is also because the expressive power of LL without additives is already great. However, the additives are the natural candidates to type functions like "predecessor" and instructions like "if...then...else..." (see [Gir87], [Gir95] and [DJ01]). Most notably, in [Gir95] their presence is important to obtain a polynomial complexity of the cut-elimination procedure without loosing the capacity to represent every polytime function (which is the great achievement of that paper).

The presence of additive boxes in [Gir87], entails the existence of an additive commutative elementary reduction step (e.r.s.), a step of cut-elimination which makes the procedure non confluent (even though denotational soundness still holds: two different reducts of the same proof have the same semantics). Following Girard's original definition of cut-elimination for the system PN_2 of second order LL proof-nets, during this step an additive box swallows and duplicates a particular subproof-net having an occurrence of the cut formula among its conclusions (called the empire of this occurrence of formula).

But there is no reason to duplicate precisely the empire, and in [TdF01] we proposed the natural generalization of this e.r.s. (which is, obviously, semantically sound): we allowed *any* subproof-net having this occurrence of formula among its conclusions to be swallowed and duplicated. We still consider such a step in the present paper.

In [TdF01], we proved that every proof-net of second order LL has a unique normal form, *provided* its conclusions have a particular shape. Despite the fact that this result can be applied in some important cases, it does not give a complete solution.

In [Gir96], an improvement of the syntax of the additives is proposed: the additive boxes of [Gir87] are removed, and a correctness criterion using a

notion of “boolean weight” is given. A sequentialization theorem for $MALL$ is proven, but only a lazy cut-elimination procedure is defined (not every *cut*-link is eliminated).
This correctness criterion has been drastically simplified in [Lau99] in the framework of polarized proof-nets. In fact, for these proof-nets, the notion of “positive tree” (coming from [Lau99]) allowed us to give a complete solution to the problems of confluence and strong normalization (see [LQTdF01]). Despite the fact that classical logic can be translated by means of polarized proof-nets (see [LQTdF01]), not all the notable fragments of LL are polarized: for example the system of proof-nets introduced in [Gir95] isn’t. The relashionship between the present work and polarized proof-nets will be discussed at the end of the paper.

Our approach in this paper is more inspired by the search of a better cut-elimination procedure than by the one of a correctness criterion. We tried not to be obsessed by the presence of boxes, but rather to understand their (computational) meaning.
The starting point of this work (dating back to 1997) was the semantical analysis of the basic counter-example to the confluence property (see figure 3 and the beginning of section 3). What is striking, semantically speaking, is that an additive box is nothing but the superimposition of two computational processes. Then, why not superimposing *several* proof-nets instead of two? This led us to introduce the notion of “multibox”, an additive box with *several* “&-doors”, i.e. superimposing *several* computational processes. With this change of syntax is naturally associated a much more flexible (more “asynchronous”, more “parallel”) cut-elimination procedure. We do not obtain a “strong” confluence property, but we get a more satisfying notion of normal form, and we define a cut-elimination procedure which associates with every proof-net of $MALL$ a unique cut-free one. This is the very first time a confluence result (even though a “weak” one) is obtained for unrestricted $MALL$ proof-nets.

Let us now be more precise on the content of the paper. We will only deal with the multiplicative and additive fragment of LL, even though the extension of our results to full LL does not seem problematic (see paragraph 8.1). In section 2, we recall some notions of [TdF01] for the sake of self containment. We present and motivate in section 3 the new notion of “multibox”, and we introduce the system PN_a of multiplicative and additive proof-nets with multiboxes.

Section 4 is devoted to define the cut-elimination steps of PN_a: we generalize to the new framework the usual e.r.s., and we define a new e.r.s. (the step $[r(\&)]$) suggested by the presence of multiboxes. This last step is not (strictly speaking) a cut-elimination step, but a way of "merging" multiboxes. Actually, the true cut-elimination procedure of PN_a is defined in section 7, where we define a new way to compute the commutative steps, combining the usual commutative steps and the steps $[r(\&)]$. This new commutative step is rather complicated, and to define it we first had to define and study the two steps (the usual commutative one and the step $[r(\&)]$) separately.

In section 5, we prove that the rewriting rule associated with the e.r.s. previously defined (including the new step $[r(\&)]$) is strongly normalizing (theorem 5.8).

In the last part of the paper (sections 6 and 7), we aim to define a new additive commutative e.r.s., leading to a cut-elimination procedure which associates with every net a unique cut-free one.

In section 6, we analyse the structure of PN_a proof-nets, and show how to decompose every proof-net into "minimal" ones (definition 6.13).

We define in section 7 the normalization procedure of PN_a, recursively with respect to the depth: one first performs cut-elimination at depth 0 until one obtains a net without *cut*-links with depth 0 (except perhaps some additive commutative cuts), one then decomposes the thus obtained net into "minimal" ones (following section 6), and one finally associates with every minimal net its "complete $[ccad]$-reduct" (see definition 7.10), eliminating at once all the commutative cuts with depth 0. This last step is new: it combines the (usual) additive commutative e.r.s. and the step $[r(\&)]$ of section 4 (multiboxes merging). Applying recursively such a procedure, one can associate with every PN_a proof-net a unique cut-free one (theorem 7.14).

The results presented in the paper seem to indicate that a certain control on the normalization procedure in presence of the additives is possible, despite the unbelievably complex notion of superimposition underlying the connective &.

2 Preliminaries: Proof-nets of $MALL$ and their normalization

This section is devoted to set the stage. Proof-nets of LL were introduced by Jean-Yves Girard in [Gir87]. Several simplifications have been discovered

since that first work, mainly by Vincent Danos and Laurent Regnier (see for example [DR89]). Despite the extensive use of the notion of proof-net, after the paper [Gir87], the first work dealing with a notion of proof-net for full second order LL (without constants) is [TdF01] (see also [TdF00b]), at least as far as we know.
We present here the restriction to the fragment $MALL$ of the notions defined in [TdF01]. In fact, the main technical subtleties are due to the presence of weakenings, so that there is nothing new in what follows, which is mainly stated to introduce some notations and conventions, and for the sake of self-containment. The only notable exceptions are the definition of the additive commutative e.r.s. (see figure 2), as mentioned in the introduction, and the separation property (proposition 2.14). Our generalization of the additive commutative step of normalization (already studied in [TdF01] and [TdF00a]) preserves the correctness criterion (ACC) of definition 2.4 and the denotational semantics of definition B.2. Except for these two statements (and some terminology), any reader having some acquaintance with proof-nets can simply skip the present section.

2.1. DEFINITION. A **proof-structure** is an oriented graph whose nodes are called links, and whose edges are labeled by formulas of LL. When drawing a proof-structure we represent edges oriented up-down so that we may speak of moving upwardly or downwardly in the graph. Links are defined together with an arity and a coarity, i.e. a given number of incident edges called the premises of the link and a given number of emergent edges called the conclusions of the link.

- A **hypothesis** or H-link has $n \geq 1$ conclusions, each of them labeled by a formula, and no premise
- an **axiom** or ax-link has no premise and two conclusions labeled by dual formulas
- a **cut** link has two premises labeled by dual formulas (which are also called the active formulas of the cut-link) and no conclusion
- a **par** or $\wp$ (resp. **times** or $\otimes$) link has two premises and one conclusion. If the left premise is labeled by the formula A and the right premise is labeled by the formula B, then the conclusion is labeled by the formula $A \wp B$ (resp. $A \otimes B$)
- a **1-plus** or $\oplus_1$ (resp. a **2-plus** or $\oplus_2$) link has one premise and one conclusion such that if A is the label of the premise, then $A \oplus B$ (resp.

$B \oplus A$) is the label of the conclusion, for some formula B. We also speak of a **plus** or $\oplus$-link when we do not want to specify whether the link is a $\oplus_1$ or a $\oplus_2$-link.

- a **with** or &-link has two premises and one conclusion labeled by the **with** of the two premises. If the left premise is labeled by the formula A and the right premise is labeled by the formula B, then the conclusion is labeled by the formula $A \& B$.

- a **coad** link has two premises and one conclusion, all labeled by the same formula.

Let G be a set of links s.t.:
(α) every edge of G is the conclusion of a unique link
(β) every edge of G is the premise of at most one link.
We say that the edges which are not premise of a link are the conclusions of G.
We will say that G is a proof-structure if the two following conditions are satisfied:

(1) &-box condition:

(1.i) with each &-link n is associated a (unique) subset B of G (satisfying (α) and (β)), s.t. one among the conclusions of B is the conclusion of n and every other conclusion of B (there might be no other conclusion) is the conclusion of a *coad*-link. B is called an additive box, and it is represented by a rectangular frame (see fig. 1). The links of B whose conclusion is a conclusion of B are (n and some *coad*-links and are) called the **doors** of B. The link n is the **front door** (or the &-door) of B.

We ask that there exist two disjoint proof-structures G^1 and G^2 s.t.:

- B is the disjoint union of its doors, G^1 and G^2

- every conclusion of G^1 (resp. G^2) is the premise of a door of B

- every door of B has two premises: a conclusion of G^1 and a conclusion of G^2.

(1.ii) with each *coad*-link n is associated an additive box B, s.t. one among the conclusions of B is the conclusion of n (see fig. 1). The link n is called an **auxiliary door** (or a *coad*-door) of B

(2) nesting condition: two boxes are either disjoint or included one in the other.

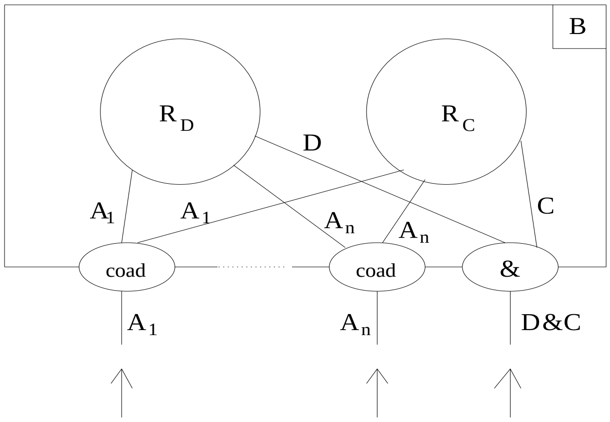


Figure 1. The additive box B of the proof-structure G.

We will often speak of a box, a link or an edge of a proof-structure R contained in a box B of R. In case of links, we will not consider the doors of B as links contained in B. We will also speak of "a link l (resp. an edge a) of a box B" of a given proof-structure, meaning that l (resp. a) is contained in B or it is a door (resp. a conclusion) of B. If B is a box of a proof-structure R, then the biggest (resp. the smallest) box of R containing B is clearly well-defined, thanks to the nesting condition of definition 2.1.
We shall say that a link or an edge of a given proof-structure R has **depth** n in R, if it is contained in exactly n boxes of R. For a box B, we shall say that B has **depth** n in R, if it is contained in exactly n boxes of R, all different from B. When R is a proof-net, the same definition of depth will extend to the case of a subproof-net of R (defined in the sequel).
We will refer to the label A of the edge a of the proof-structure R as "the type of a". In fact, we shall often identify an edge and its type: strictly speaking, this might be ambiguous (remember the premises and the conclusion of the *coad*-links all have the same type): this is nothing but the usual confusion between "a formula A" and "an occurrence of the formula A".

2.2. DEFINITION. **(graph with pairs)** We will say that two edges of an oriented graph are coincident when they have the same target. The couple

$(G, App(G))$ is called a **graph with pairs** when G is an oriented graph and $App(G)$ is a set of pairs of coincident edges.
Let R be a proof-structure and $B_1, \ldots, B_k$ the boxes of R with depth zero. We are going to associate with R a set $App(R)$ and a graph with pairs $R_{ap} = (G_R, App(R))$.
The graph G_R is obtained from R by substituting for each box B_i with p_i conclusions ($i \in \{1, \ldots, k\}$) a link H with p_i conclusions.
The elements of the set $App(R)$ are the pairs of premises of a $\wp$-link of G_R (that is the pairs of premises of a $\wp$-link of R with depth zero).

2.3. DEFINITION. **(correctness graph)** Let R be a proof-structure and $B_1, \ldots, B_k$ the boxes with depth zero in R. Let $R_{ap} = (G_R, App(R))$ be the graph with pairs associated with R by definition 2.2.
A **switching S** of R is the choice of an edge for every pair of $App(R)$.
With each switching S is associated an unoriented graph $S(R)$, called **correctness graph**: for every pair of $App(R)$, erase the edges of G_R which are not selected by S, and then forget the labels and the orientation of the edges of the graph. The correctness graph of R associated with S will be denoted by $S(R)$.

2.4. DEFINITION. **(proof-net)** Let R be a proof-structure and $B_1, \ldots, B_k$ the boxes with depth zero in R. We say that R is a **proof-net** when the following conditions are satisfied:

- R satisfies (ACC): for every switching S of R, the correctness graph $S(R)$ is acyclic and connected (i.e. it is a tree).
- for every box $B_i \in \{B_1, \ldots, B_k\}$, the two disjoint proof-structures R_i^1 and R_i^2 of definition 2.1 are two proof-nets.

A **subproof-net** of a proof-net R is a subgraph of R which is a proof-net.

If B is a box of a proof-net T, then (by definition of proof-net) there is a subproof-structure R of T which is a proof-net and s.t. B has depth zero in R. We can then use the notations of the previous definition: there exists $i \in \{1, \ldots, k\}$ s.t. $B = B_i$. If the conclusion of the front door of B is labeled by $E \& F$, then we will often speak of the first (resp. the second) **component of** B meaning the one among R_i^1 and R_i^2 which has E (resp. F) among its conclusions.

2.5. DEFINITION. Let R be a proof-net and B an additive box of R. Let l_1 and l_2 (resp. a_1 and a_2, B_1 and B_2) be two links (resp. edges, boxes) of R. We shall say that B **separates** l_1 and l_2 (resp. a_1 and a_2, B_1 and B_2), when they are contained in two different components of B. We shall also say that l_1 and l_2 (resp. a_1 and a_2, B_1 and B_2) are **separated by** B, or simply **separated**.

2.6. REMARK. If two links, edges or boxes of a proof-net R are separated, there obviously exists a unique box separating them, to which we can then refer as *the* box which separates them.

Let now $MALLS$ denote the linear sequent calculus of Annex A. It is easy to see that with each proof of $MALLS$ is associated a (unique) proof-net. What is less trivial is that the converse also holds:

2.7. THEOREM. *If* R is a proof-net, then there exists a proof π of $MALLS$ s.t. R is the proof-net associated with π. The proof π is then called a sequentialization of R.

Proof: Straightforward from [DR89]. $\square$

2.8. REMARK. The proof-net associated with a proof of $MALLS$ which does not make use of the rule Hyp does not contain any link H, and conversely. In the sequel, we will restrict to these notions of proof and proof-net, and will speak of "proof-nets with hypothesis" (resp. "proof with hypothesis"), in case some H-links (resp. some Hyp-rules) occur in the graph (resp. in the proof).

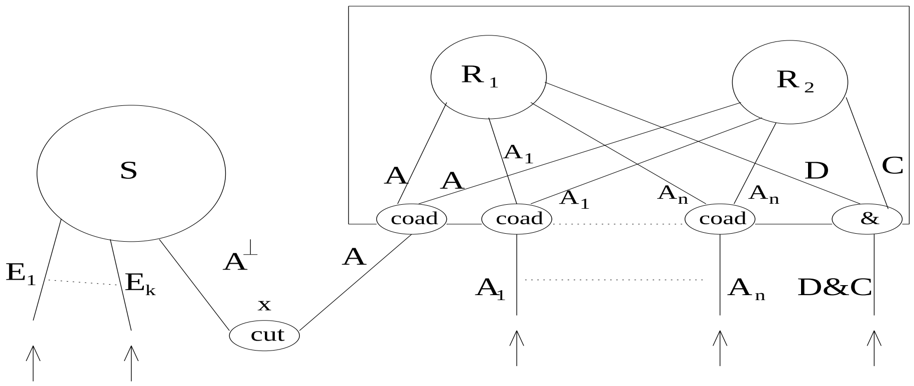
2.9. DEFINITION. Let R be a proof-net and x a *cut*-link of R.
x is called a **logical** *cut*-link when the two premises of the link are conclusions of two (necessarily dual) logical links: we also denote these *cut*-links by $\wp/\otimes$, $\&/\oplus_1$, $\&/\oplus_2$ (or simply $\&/\oplus$ when we do not want to specify which one of the two plus links occurs).
x is called an **axiom** *cut*-link when one of the two premises of x is the conclusion of an axiom link.
x is called an **additive commutative** *cut*-link when one of the two premises of x is the conclusion of a *coad*-link: we also denote this *cut*-link by *ccad*.

2.10. Remark. If x is a *cut*-link of a proof-net, then x is of exactly one of the kinds mentioned in the previous definition, *except in one case*: if one of the premises of x is the conclusion of an axiom link and the other one is the conclusion of a *coad*-link, then x is both an axiom and a *ccad cut*-link.

We are now going to define the elementary reduction steps associated with the different kinds of *cut*-links (except in one case we will actually refer to [Gir87]).

2.11. Definition. Let R be a proof-net and x a *cut*-link of R.
If x is a logical *cut*-link, then we denote by $[x]$ the unique e.r.s. associated with x and defined in [Gir87]. When x is a logical *cut*-link, $[x]$ is called a **logical e.r.s.**.
If x is an axiom *cut*-link, then we denote by $[ax]$ the axiom e.r.s. associated with x defined in [Gir87]. (In case both the premises of x are conclusions of axiom links, we have in fact *two* possible e.r.s.).
If x is a *ccad cut*-link, let n be one of the *coad*-links whose conclusion is a premise of x (there is at least one such link and there are at most two such links) and suppose that the conclusion of n is labeled by the formula A. Let now S be *any* subproof-net of R having the premise of x labeled by $A^{\perp}$ among its conclusions, and let's call B the additive box of R having the premise of x labeled by A among its conclusions. We then denote by $[ccad]$ the e.r.s. defined in figure 2. This e.r.s. is said to be B-**attractive** (or attractive for the box B) because B is the additive box of R that swallows (and duplicates) the subproof-net S.
In the sequel, we will denote by $[x]$ any (given) e.r.s. associated with the *cut*-link x.

The configuration



is replaced by the configuration

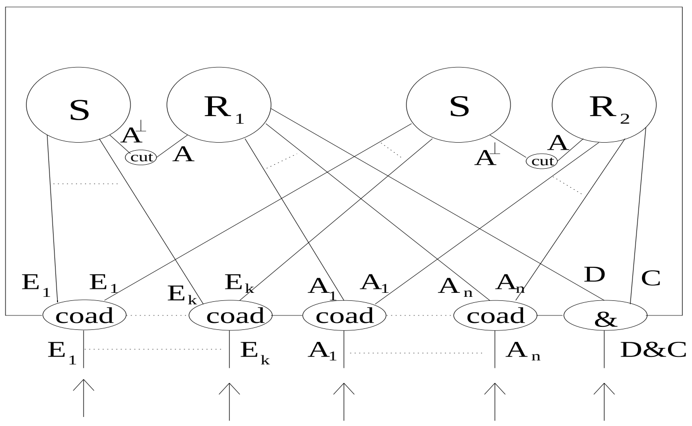


Figure 2. The [*ccad*] elementary reduction step.

2.12. Theorem. *If* R *is a proof-net,* x *is a* cut*-link of* R*, and* $R \xrightarrow{[x]} R'$*, then* R' *is a proof-net.*

proof: See [Dan90], and [TdF00b]. □

We will sometimes denote by σ a sequence of e.r.s., and we will write $T \overset{\sigma}{\twoheadrightarrow} T'$, to indicate that the proof-net T' is obtained from the proof-net T by applying the sequence σ of e.r.s.

2.13. Definition. Let R' be a one step reduct of the proof-net R.
Any logical or axiom link n of R' comes from a unique occurrence of ("the same") logical or axiom link $\overleftarrow{n}$ of R. We call $\overleftarrow{n}$ the **lift of the link** n in R.
Conversely, we define the set of **residues of the** logical (resp. axiom) **link** l of R, as the set of all the occurrences l' of logical (resp. axiom) links in R' such that $\overleftarrow{l'} = l$.
In the particular case where the logical link considered in R' is a *with* link, it is actually a box B, which comes from a unique occurrence of "the same" box $\overleftarrow{B}$ in R (observe, however, that the links contained in the box are not necessarily the same in R as in R'). We call $\overleftarrow{B}$ the **lift of the box** B in R.
Conversely, we define the set of **residues of the box** B_1 of R as the set of all the occurrences B'_1 of boxes in R' such that $\overleftarrow{B'_1} = B_1$.

The following proposition (proven in a much more general framework in [TdF01]) is a simple but important property of the cut-elimination procedure in presence of the additives.

2.14. Proposition. **(separation property).** Let T, T' be two proof-nets s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$, and let $\mathbb{B}$ be an additive box of T.
If $\mathbb{B}'$ and $\mathbb{B}''$ are two (different) residues of $\mathbb{B}$ in T', then $\mathbb{B}'$ and $\mathbb{B}''$ are separated.

3 Non confluence and multiboxes

One of the main drawbacks of the procedure of cut-elimination for the system PN_2 of second order LL proof-nets defined in [Gir87], is that it is not confluent. In [Dan90], the multiplicative and exponential fragment of PN_2 is shown to enjoy the confluence property, and the proof can clearly be extended to second order quantifiers. I proved in [TdF01] that the confluence result can be extended to full PN_2, *provided* the conclusions of the proof-nets satisfy certain conditions (roughly speaking, they have to be &-free).
This lack of confluence (in the general case) is due to the way we handle the connective &: the presence of boxes entails the presence of a "commutative e.r.s." (the step [*ccad*]) which drammatically complicates the matter. The

non-confluence phenomenon is already present in the fragment $MALL$ of PN_2, and "a fortiori" in the system of proof-nets we have previously defined.

Figure 3 illustrates the standard counter-example to the confluence of PN_2. Let us start with a semantical analysis of this counterexample. We use here appendix B. The interpretation of all the proof-nets of figure 3 is the disjoint union of the interpretation of four axiom links. More precisely, the set associated with the two cut-free proof-nets of figure 3 (which obviously means "of all the proof-nets of figure 3") is

$$\{(x,0),(x,0) : x \in |\mathcal{A}|\} \cup \{(y,1),(y,0) : y \in |\mathcal{A}|\} \cup \{(z,0),(z,1) : z \in |\mathcal{A}|\} \cup \{(t,1),(t,1) : t \in |\mathcal{A}|\}.$$

This clearly shows that the only "semantically visible" feature of these proof-nets is the position of the axiom links inside the boxes: 00, 10, 01, 11. The interpretation gives an account of the proof-nets in figure 3 as a "superimposition" of four axiom links, each of them coming with its position. There is no trace of the nesting of the two boxes B_1 and B_2 (or, better said, of the nesting of their residues in the normal forms). This suggests that the two &-rules associated with these two boxes can be performed "at the same time".
Our idea was to express this "parallelism" of the semantics within the syntax: this is rather attractive, because it might yield a unique normal form for the proof-net (with cuts) of figure 3.
We introduce the notion of "multibox": a box with several doors &, which seems interesting for several reasons. On the one hand it corroborates the intuition that a(n additive) box indicates the superimposition of several computational processes, and on the other hand one might hope, by means of this notion, to represent the "atemporality" of the rules & (which is striking in the semantics). The presence of boxes makes the sequentialization theorem unproblematic, and the modification of the e.r.s. associated with the change of syntax looks manageable. Well, on we go...

3.1. DEFINITION. **(Proof-structure)** A proof-structure G is defined like in 2.1 with some differences:
- every &-link has 2^n premises (for some $n \geq 1$) and one conclusion. 2^{n-1} among these premises have the same type (say C), and the other 2^{n-1} have also the same type (say D). The conclusion of the &-link will be of type $C\&D$ iff all the edges of type C (resp. D) are the left (resp. right) premises of the link.
- every *coad*-link has 2^n premises (for some $n \geq 1$) and one conclusion, all of the same type

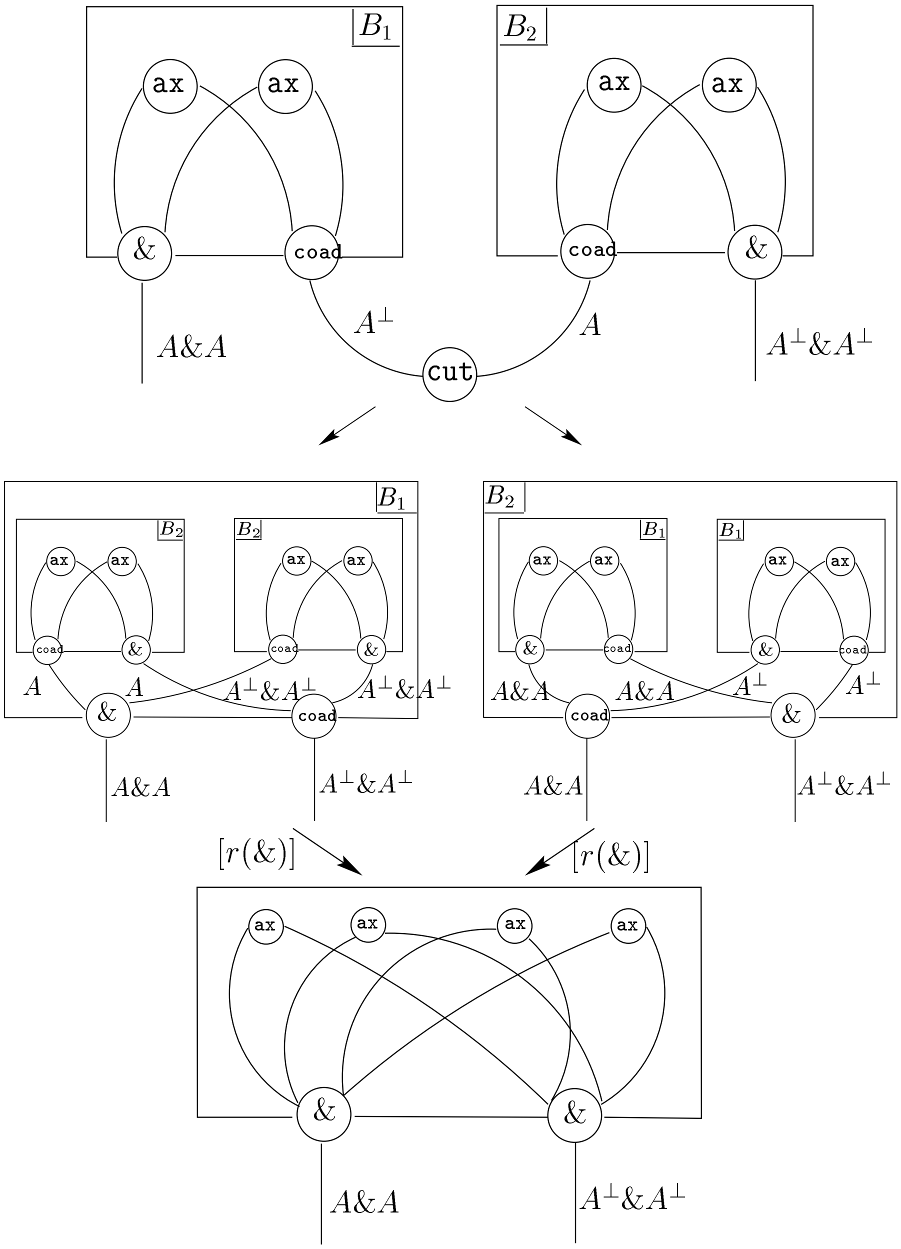


Figure 3. PN_2 is not confluent.

- with every &-link m with 2^n premises is associated a (unique) subset B of G, s.t. one among the conclusions of B is the conclusion of m and every other conclusion of B (there might be no other conclusions) is the conclusion of a *coad*-link with 2^n premises or of another &-link with 2^n premises. B is still called a (multi)box; m is one of the front doors (or &-doors) of B. The multibox B is now the disjoint union of its doors and of 2^n disjoint proof-structures
- with every *coad*-link m with 2^n premises is associated a box B, s.t. one among the conclusions of B is the conclusion of m; m is an auxiliary door (or a *coad*-door) of B.

The nesting condition is always the same: two boxes are either disjoint or included one in the other.

3.2. DEFINITION. **(Proof-net)** A proof-net is defined like in 2.4, with the straightforward necessary modifications. We denote by PN_a the set of these proof-nets.

3.3. REMARK. Every proof-net in the sense of definition 2.4, is a proof-net of PN_a. And of course the converse does not hold: the last graph of figure 3 is an element of PN_a which is not a proof-net in the sense of definition 2.4. Observe that one is naturally tempted to say that this proof-net is "the (unique) normal form" of the proof-net with cuts of figure 3.

In the sequel, we will restrict the term "proof-net" to the elements of PN_a. All the notations, the terminology and the conventions of section 2 can be straightforwardly extended to PN_a.

Let us now turn our attention to the generalization of theorem 2.7. In order to obtain a sequentialization theorem for PN_a, we clearly have to extend the sequent calculus of appendix A by means of a generalized &-rule (with 2^n premises, $n \geq 1$).

Let $D_1, \ldots, D_n$ (resp. $E_1, \ldots, E_n$) be (occurrences of) formulas. There is a bijection between the integers $l \in \{1, \ldots, 2^n\}$ and the multisets $C_1^l, \ldots, C_n^l$, where $C_i^l \in \{D_i, E_i\}$ (and $i \in \{1, \ldots, n\}$). This bijection associates with each integer l a multiset $C_1^l, \ldots, C_n^l$.

For every $l \in \{1, \ldots, 2^n\}$, let α_l be a proof whose sequent conclusion is $\vdash \Gamma, C_1^l, \ldots, C_n^l$. The &-rule of our "new" calculus allows to obtain from these 2^n proofs a proof α with conclusion $\vdash \Gamma, D_1 \& E_1, \ldots, D_n \& E_n$. This rule has the following shape:

$$\frac{\ldots \quad \vdash \Gamma, C_1^l, \ldots, C_n^l \quad \ldots}{\vdash \Gamma, D_1 \& E_1, \ldots, D_n \& E_n} \&$$

In case $n = 2$, one has the following rule:

$$\frac{\vdash \Gamma, D_1, D_2 \qquad \vdash \Gamma, D_1, E_2 \qquad \vdash \Gamma, E_1, D_2 \qquad \vdash \Gamma, E_1, E_2}{\vdash \Gamma, D_1 \& E_1, D_2 \& E_2} \&$$

where $C_1^1 = D_1, C_2^1 = D_2$, $C_1^2 = D_1, C_2^2 = E_2$, $C_1^3 = E_1, C_2^3 = D_2$, $C_1^4 = E_1, C_2^4 = E_2$.

The same remark as before applies: every proof of $MALLS$ is a proof of the new calculus, and the converse does not hold: there clearly exists a sequentialization of the last proof-net of figure 3 which is a proof of the generalized calculus, without being a proof of $MALLS$.
We will refer in the sequel only to this (generalized) version of sequent calculus, which will still be denoted by $MALLS$.
Theorem 2.7 can now be straighforwardly generalized to our new framework.

3.4 Internalizing distributivity

The present subsection is not that formal. We give an example which suggests that multiboxes are, in some sense, a way to "internalize", within the syntax, the distributivity isomorphism of the connective $\wp$ over the connective &.
We will speak of the semantics of a proof of sequent calculus always meaning the semantics of the proof-net associated with the proof, following appendix B. As usual, the semantics of the elements of PN_a which are not proof-nets in the sense of definition 2.4, is obtained straightforwardly by generalizing in the unique obvious way appendix B. We will also refer to cut-elimination for proof-nets that is not yet completly defined: the reader can either refer to its own intuition, or first read section 4.
If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are mathematical structures interpreting linear formulas in a given denotational semantics (the reader can think for example at coherent spaces), then one has the following isomorhism of mathematical structures (at least for "good" semantics):

$$\mathcal{A} \wp (\mathcal{B} \& \mathcal{C}) \simeq (\mathcal{A} \wp \mathcal{B}) \& (\mathcal{A} \wp \mathcal{C}).$$

This has a syntactical meaning, which we now explain. Let π_1 (resp. π_2) be the canonical proof of $\vdash A \wp (B\&C), (A^\perp \otimes B^\perp) \oplus (A^\perp \otimes C^\perp)$ (resp. $\vdash (A \wp B)\&(A \wp C), A^\perp \otimes (B^\perp \oplus C^\perp)$).
The proof $cut(\pi_1, \pi_2)$ (resp. $cut(\pi_2, \pi_1)$) of $\vdash A \wp (B\&C), A^\perp \otimes (B^\perp \oplus C^\perp)$ (resp. $\vdash (A \wp B)\&(A \wp C), (A^\perp \otimes B^\perp) \oplus (A^\perp \otimes C^\perp)$) obtained by performing a cut rule with premises π_1 and π_2, is a semantical identity, i.e. it is interpreted by the identity on the space $\mathcal{A} \wp (\mathcal{B}\&\mathcal{C})$ (resp. $(\mathcal{A} \wp \mathcal{B})\&(\mathcal{A} \wp \mathcal{C})$). We write for short $[cut(\pi_1, \pi_2)] = id$ and $[cut(\pi_2, \pi_1)] = id$.
Because we are in a linear framework, $[cut(\pi_1, \pi_2)] = id$ and $[cut(\pi_2, \pi_1)] = id$ (which can be expressed in purely syntactical terms) is actually equivalent to say that $\mathcal{A} \wp (\mathcal{B}\&\mathcal{C}) \simeq (\mathcal{A} \wp \mathcal{B})\&(\mathcal{A} \wp \mathcal{C})$ (see [LQTdF01] for more details).

Our aim is now to show that one can associate with every proof-net T without multiboxes and with conclusion $\Gamma, (C\&D) \wp (G\&H)$, a proof-net T_{rev} which has "the same computational content as T", and which is a multibox with two front doors (with conclusions $C\&D$ and $G\&H$ respectively), and some auxiliary doors (with conclusions the (occurrences of) formulas of Γ). The distributivity isomorphism states that:

$$(\mathcal{C}\&\mathcal{D}) \wp (\mathcal{G}\&\mathcal{H}) \simeq (\mathcal{C} \wp \mathcal{G})\&(\mathcal{C} \wp \mathcal{H})\&(\mathcal{D} \wp \mathcal{G})\&(\mathcal{D} \wp \mathcal{H}).$$

Let us consider, like before, the canonical proofs π_1 of
$\vdash (C \wp G)\&(C \wp H)\&(D \wp G)\&(D \wp H), (C^\perp \oplus D^\perp) \otimes (G^\perp \oplus H^\perp)$ and π_2 of
$\vdash (C\&D) \wp (G\&H), (C^\perp \otimes G^\perp) \oplus (C^\perp \otimes H^\perp) \oplus (D^\perp \otimes G^\perp) \oplus (D^\perp \otimes H^\perp)$.
We actually have several possibilities for π_1 and π_2, depending on the order of some &-rules. Let's choose any of these possibilities for π_1. Concerning π_2, observe that its last rules are either two &-rules introducing $C\&D$ and a rule introducing $G\&H$ followed by the $\wp$-rule introducing $(C\&D) \wp (G\&H)$, or two &-rules introducing $G\&H$ and a rule introducing $C\&D$ followed by the $\wp$-rule introducing $(C\&D) \wp (G\&H)$. Let us call R_1, R_2, R_3, R_4 any of these two possible sequences of rules, and π_2 any of these two possible proofs. We have $[cut(\pi_2, \pi_1)] = id$, and it is easy to see that the proof $cut(\pi_2, \pi_1)$ reduces to a cut-free proof ending precisely with the sequence of rules R_1, R_2, R_3, R_4. In the new version of sequent calculus we have previously introduced, the rules R_1, R_2 and R_3 can be replaced by a unique rule introducing both $C\&D$ and $G\&H$: let π be the proof obtained from the normal form of $cut(\pi_2, \pi_1)$ after this substitution. The proof(-net associated with) π is a multibox, and $[\pi] = id$.

Let T be any proof-net without multiboxes and with conclusion $\Gamma, (C\&D) \wp (G\&H)$. The proof(-net) $cut(T, \pi)$ reduces to a proof-net T_{rev} which is a multibox with two front doors (with conclusions $C\&D$ and $G\&H$ respectively), and some auxiliary doors (with conclusions the (occurrences of) formulas of Γ). Because $[\pi] = id$, one has $[T] = [T_{rev}]$, which is a way to express that T_{rev} has "the same computational content as T".

4 Cut-elimination in PN_a

We come now to the crucial definition of the cut-elimination steps for PN_a proof-nets. In case of figure 3, the first idea is to modify the $[ccad]$ e.r.s. of definition 2.11, in such a way that the $[ccad]$-reduct of the proof-net with cut is the proof-net with the multibox. This is what we eventually do in section 7 (when we compute the "complete $[ccad]$-reduct"). However, in the general case, this turns out to be extremely complicated. We then decided to split this step in two steps: a generalization of the $[ccad]$ step allowing to copy 2^n times a given subproof-net (in case of figure 3, $n = 1$ and this step produces one of the two intermidiate proof-nets), and a new e.r.s., denoted by $[r(\&)]$ (in case of figure 3 this step applied to any of the two intermidiate proof-nets produces the proof-net with the multibox).
Is this new step "correct"? Well, it certainly preseves denotational semantics (by construction!), it still satisfies the separation property (see proposition 5.4), and we will show that adding it to the usual e.r.s. yields a rewriting rule on PN_a proof-nets which still enjoys the strong normalization property (theorem 5.8).

These remarks led us to the following definition.

4.1. Definition. **(cut-elimination steps of PN_a).** The kind of cuts of PN_a are the ones of definition 2.9. We are going to define five e.r.s.: $[ax]$, $[\wp/\otimes]$, $[\&/\oplus]$, $[ccad]$, $[r(\&)]$.
The steps $[ax]$ and $[\wp/\otimes]$ are defined like in 2.11 (that is like in [Gir87]).
The step $[\&/\oplus]$ can be either "special" (when the &-link has only two premises), in which case it is the usual reduction step, or "non special" (when the &-link has 2^n premises, with $n \geq 2$) like in figure 4.
The step $[ccad]$ is defined like in 2.11, except the fact that the proof-net S of figure 2 will be copied 2^n times, if the box B for which the reduced cut is B-attractive contains 2^n components.
The step $[r(\&)]$ is not associated with a *cut*-link. It applies when there exists a *coad*-link n, every premise of which is the conclusion of a &-link.

We actually require that the &-links whose conclusion is a premise of n are all doors of (different) boxes with the same number of &-doors. This case is easier to handle (at least for notations!) and enough for our purposes, but no theoretical reason prevents us from a generalization of this step.
This step can be either “special” (like in figure 5) or “non special” (like in figure 6). It generally has the (rather unpleasant from a practical point of view) effect of increasing the number of links of the proof-net.

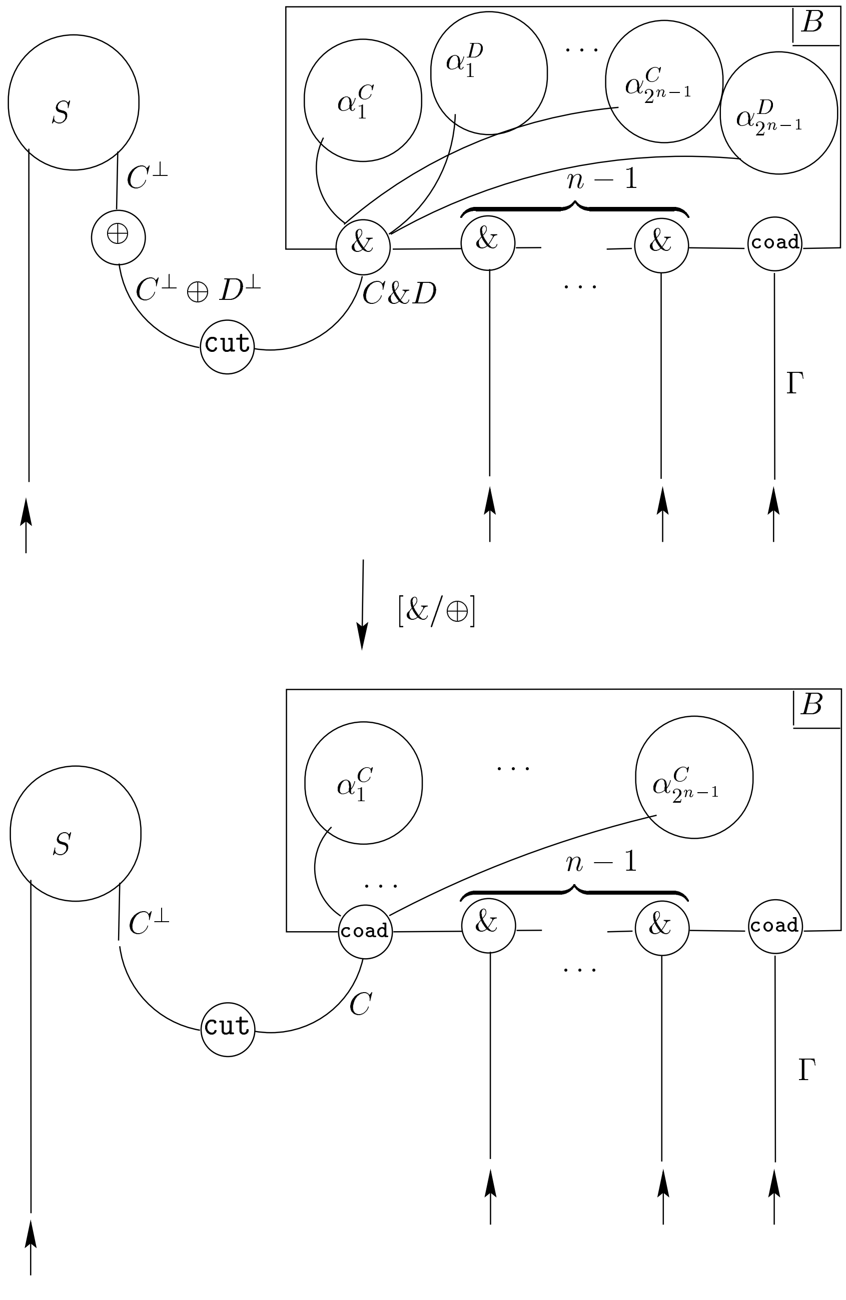


Figure 4. The non special $[\&/\oplus]$ e.r.s. $(n \geq 2)$

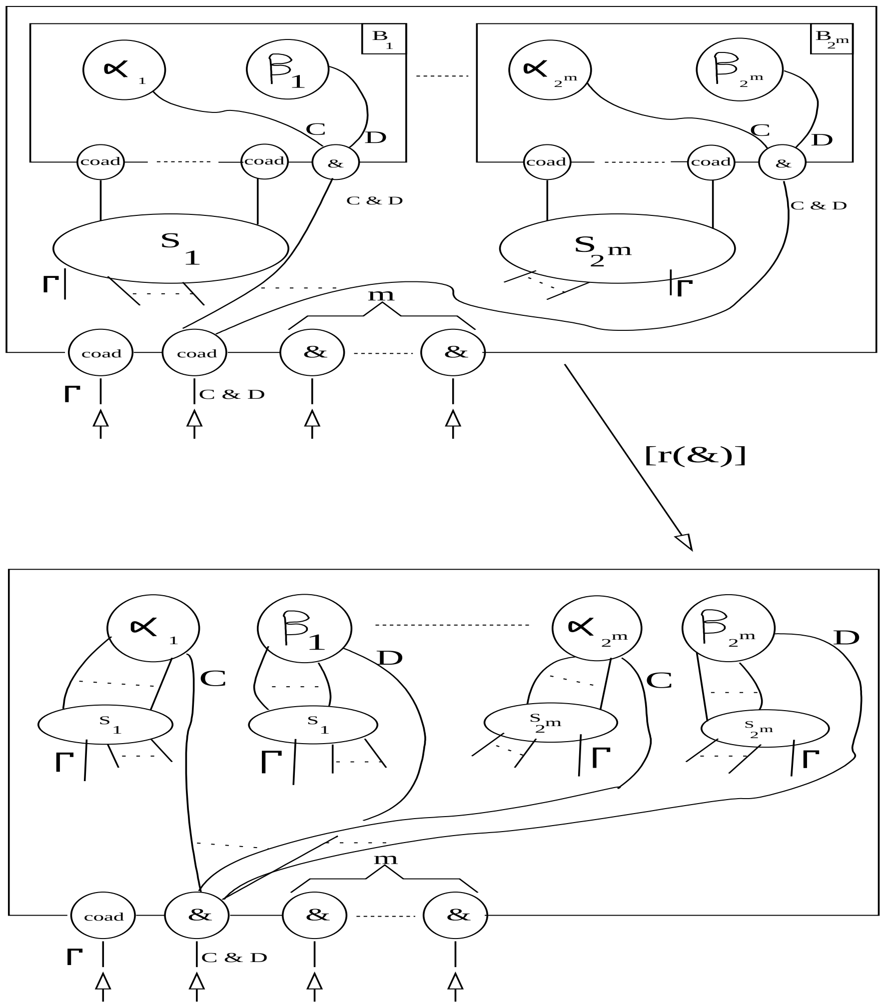


Figure 5. The special $[r(\&)]$ e.r.s.

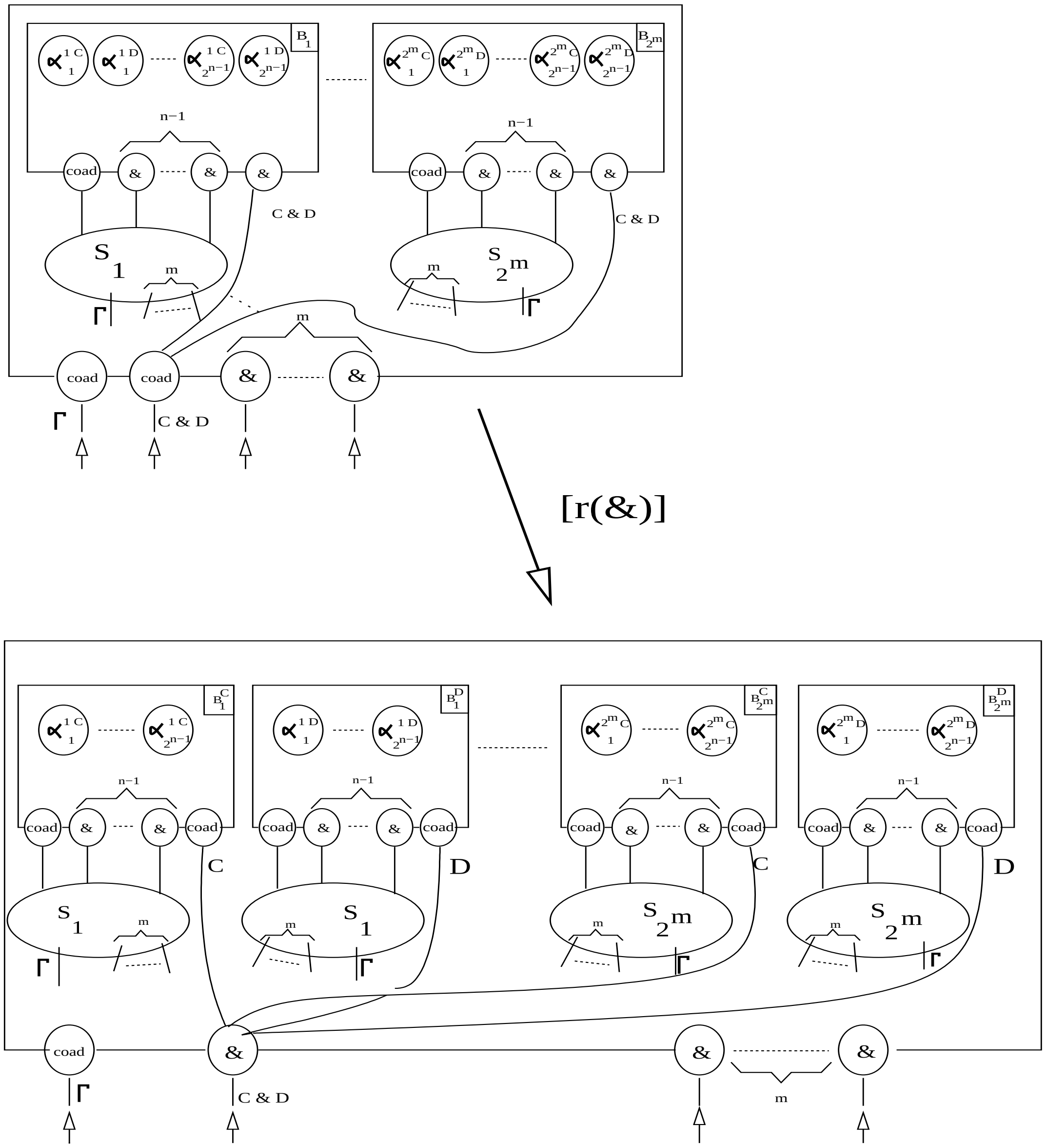


Figure 6. The non special $[r(\&)]$ e.r.s.

4.2. Remark. To be precise, one should show that the e.r.s. thus defined all preserve the (ACC) condition of definition 2.4, that is one should generalize to our framework theorem 2.12. But again, this is straightforward.

4.3. Remark. The e.r.s. $[r(\&)]$ is a way to superimpose several &-links: this makes the relation between a box and its &-door(s) much more flexible. A box is now nothing but a way to indicate the "superimposition" of a certain number of proofs.
The *cognoscenti* might also notice that the two normal proof-nets (of PN_2) of figure 3 are equivalent "up to reversion": by "reverting" one of them one obtains the other one. This operation of "reversion" has been recently exploited in several ways (see [Gir91a], [DJS97], [LQTdF01]). The proof-net of figure 3 containing a multibox can then also be though as a canonical representative of the equivalence class induced by reversion.

4.4. Remark. The step $[r(\&)]$ does not modify the notions of lift and residue of a box, defined in 2.13. Let us stress the fact that one should not think of a box as a subproof-net but rather as an edge (quiet a special one, of course) of the graph. By the way, this idea was already present in definition 2.13, because a box and its residues could also contain different proof-nets.
For all the logical links *which are not* &-links, definition 2.13 of lift and residue still holds.
This is not the case of the &-links: these are the unique (logical) links which may now have several lifts (after a step $[r(\&)]$).

5 Strong Normalization

In this section, we denote by $\to_u$ the "usual" rewriting rule on the set of PN_a proof-nets, i.e. if $R \to_u R'$, then there exists a *cut*-link x of R s.t. $R \overset{[x]}{\to} R'$. We denote by $\overset{[r(\&)]}{\longrightarrow}$ the e.r.s. $[r(\&)]$, and we will write $R \to R'$ to indicate that $R \to_u R'$ or $R \overset{[r(\&)]}{\longrightarrow} R'$. As usual, we will denote by $\twoheadrightarrow$ the reflexive and transitive closure of $\to$.
We prove that the separation property still holds for PN_a proof-nets (with the rewriting rule $\to$), and that $\twoheadrightarrow$ is strongly normalizing on PN_a: there is no infinite reduction sequence starting from a proof-net of PN_a.

The following definition is a variant of the definition of "size of a proof-net" given in [Gir87], that we adapted to our framework.

In [Gir87], this notion is used to give a combinatorial proof of the so-called "small" strong normalization theorem for PN_2: cut-elimination of PN_2 is strongly normalizing, provided one does not perform contraction e.r.s. It is for a similar purpose that we define our variant of "size of a proof-net".

5.1. DEFINITION. **(size of a proof-net).** Let R be a proof-net and let $B_1, \dots, B_n$ be the boxes with depth 0 in R. Suppose that there are k edges with depth 0 in R *which are not conclusions of a door of a box.* The size of R, which is denoted by $s(R)$, is defined by:
(1) if $n = 0$, then $s(R) = 2^k$
(2) if R is a box B with m &-doors and if $\alpha_1 \dots \alpha_{2^m}$ are the components of R, then $s(B) := (s(\alpha_1) + \dots + s(\alpha_{2^m})) \cdot 2^m$
(3) if $n \geq 1$, then $s(R) := s(B_1) \cdot \ldots \cdot s(B_n) \cdot 2^k$.

5.2. REMARK. (i) Our variant of size stresses the fact that the conclusions of the &-doors of a box are not linked to the box istelf (this was not the case in the original definition of [Gir87]): in the evaluation of the size, they are just treated like the other formulas, and the box is only used to superimpose (i.e. to make the sum of the sizes of) its components.
(ii) The conclusion of a *coad*-link of a proof-net R is not considered when computing $s(R)$, only the premises are.

The size thus defined does not depend on the complexity of formulas, so that the following proposition should extend without problems to second order quantifiers.

5.3. PROPOSITION. *Let* R *and* R' *be proof-nets s.t.* $R \to R'$.
(i) If $R \xrightarrow{[x]}_u R'$ and $[x] \neq [ccad]$, then $s(R') < s(R)$.
(ii) If $R \xrightarrow{[ccad]} R'$, then $s(R') = s(R)$.
(iii) If $R \xrightarrow{[r(\&)]} R'$, then $s(R') = s(R)$.

Proof: It is just a matter of checking it, step by step. We are going to describe the "changes of size" associated with every e.r.s. of definition 4.1, and this will end the proof.
We only consider the case of a *cut*-link with depth 0 (in case of $\longrightarrow_u$ steps) and of external boxes with depth 0 (in case of $[r(\&)]$ steps). The extension to the general case is straightforward.
If $[x] = [ax]$, then a size $r \cdot 2 \cdot 2$ is replaced by a size r.

If $[x] = [\wp/\otimes]$, then a size $s \cdot 2^6$ is replaced by a size $s \cdot 2^4$.

If $[x] = [\&/\oplus]$ is special, then a size $(\alpha_C + \alpha_D) \cdot 2 \cdot 2 \cdot t$ is replaced by a size $\alpha_C \cdot t$. We supposed that the types of the premises of the *cut*-link were $C\&D$ and $C^\perp \oplus D^\perp$, and we denoted the size of the component α_C (having a conclusion of type C which is one of the premises of the &-link erased by the e.r.s.) like the component itself. In the sequel of the proof, we will always use this kind of convention.

If $[x] = [\&/\oplus]$ is non special (see figure 4), then a size

$$s \cdot 2 \cdot 2 \cdot (\alpha_1^C + \alpha_1^D + \ldots + \alpha_{2^{n-1}}^C + \alpha_{2^{n-1}}^D) \cdot 2^{n-1} \cdot t$$

is replaced by a size

$$s \cdot (\alpha_1^C + \ldots + \alpha_{2^{n-1}}^C) \cdot 2^{n-1} \cdot t$$

If $[x] = [ccad]$, then a size

$$s \cdot (\alpha_1 + \ldots + \alpha_{2^n}) \cdot 2^n \cdot t$$

is replaced by a size

$$(s \cdot \alpha_1 + \ldots + s \cdot \alpha_{2^n}) \cdot 2^n \cdot t$$

In this case the size is the same before and after reduction (as stated by the proposition).

If the reduction step is a special $[r(\&)]$ step (see figure 5), then a size

$$((\alpha_1 + \beta_1) \cdot s_1 \cdot 2 + \ldots + (\alpha_{2^m} + \beta_{2^m}) \cdot s_{2^m} \cdot 2) \cdot 2^m \cdot t$$

is replaced by a size

$$(\alpha_1 \cdot s_1 + \beta_1 \cdot s_1 + \ldots + \alpha_{2^m} \cdot s_{2^m} + \beta_{2^m} \cdot s_{2^m}) \cdot 2^{m+1} \cdot t$$

In this case the size is the same before and after reduction.

If the reduction step is a non special $[r(\&)]$ step (see figure 6), then a size

$$(s_1 \cdot 2^{n-1} \cdot 2 \cdot (\alpha_1^{1C} + \alpha_1^{1D} + \ldots + \alpha_{2^{n-1}}^{1C} + \alpha_{2^{n-1}}^{1D}) + \ldots + s_{2^m} \cdot 2^{n-1} \cdot 2 \cdot (\alpha_1^{2^m C} + \alpha_1^{2^m D} + \ldots + \alpha_{2^{n-1}}^{2^m C} + \alpha_{2^{n-1}}^{2^m D})) \cdot 2^m \cdot t$$

is replaced by a size

$$(s_1 \cdot 2^{n-1} \cdot (\alpha_1^{1C} + \ldots + \alpha_{2^{n-1}}^{1C}) + s_1 \cdot 2^{n-1} \cdot (\alpha_1^{1D} + \ldots \alpha_{2^{n-1}}^{1D}) + \ldots$$

$$\ldots + s_{2^m} \cdot 2^{n-1} \cdot (\alpha_1^{2^m C} + \ldots + \alpha_{2^{n-1}}^{2^m C}) + s_{2^m} \cdot 2^{n-1} \cdot (\alpha_1^{2^m D} + \ldots \alpha_{2^{n-1}}^{2^m D})) \cdot 2^{m+1} \cdot t$$

In this case the size is the same before and after reduction. □

The following proposition is the generalization of proposition 2.14 to PN_a.

5.4. Proposition. **(separation property)** Let T, T' be two proof-nets s.t. $T \stackrel{\sigma}{\twoheadrightarrow} T'$, and let $\mathbb{B}$ be an additive box of T.
If $\mathbb{B}'$ and $\mathbb{B}''$ are two (different) residues of $\mathbb{B}$ in T', then $\mathbb{B}'$ and $\mathbb{B}''$ are separated.

proof: The main difference with proposition 2.14 lies in the presence in PN_a of $\stackrel{[r(\&)]}{\longrightarrow}$ e.r.s..
Basically, the result holds for the following reasons:
(i) Each time a box is duplicated, all its residues are separated
(ii) If B_1 and B_2 are two separated boxes, then any two residues $\vec{B_1}$ and $\vec{B_2}$ of B_1 and B_2 are still separated.
The detailed proof follows.
We have to show the existence of a box B' of T' which separates $\mathbb{B}'$ and $\mathbb{B}''$.
We proceed by induction on σ. If $T' = T$, then the result is obvious.
Otherwise $T \stackrel{\sigma_1}{\twoheadrightarrow} T_1 \stackrel{[x]}{\rightarrow} T'$, where either x is a *cut*-link of T_1 or $[x] = [r(\&)]$.
Let $\mathbb{B}_1'$ (resp. $\mathbb{B}_1''$) be the lift of $\mathbb{B}'$ (resp. $\mathbb{B}''$) in T_1.
In case $\mathbb{B}_1' = \mathbb{B}_1'' = \mathbb{B}_1$, then (because $\mathbb{B}'' \neq \mathbb{B}'$) $\mathbb{B}_1$ is duplicated during the e.r.s. $[x]$. This means that $[x] = [ccad]$ or $[x] = [r(\&)]$.
In case $[x] = [ccad]$, let $\mathcal{B}_1$ be the box of T_1 s.t. $[x]$ is $\mathcal{B}_1$-attractive, and let S be the subproof-net of T_1 swallowed by $\mathcal{B}_1$ when performing $[x]$. $\mathbb{B}_1$ is (obviously) a subproof-net of S. The box B' of T' that we are looking for is the unique residue of $\mathcal{B}_1$ in T'.
In case $[x] = [r(\&)]$, let $\mathcal{B}_1$ be the box of T_1 s.t. its unique residue in T' has one more front door than $\mathcal{B}_1$ (see figures 5 and 6): $\mathbb{B}_1$ is contained in $\mathcal{B}_1$, and the box B' of T' that we are looking for is the unique residue of $\mathcal{B}_1$ in T'.
Suppose now that $\mathbb{B}_1' \neq \mathbb{B}_1''$. By induction hypothesis, there exists a box $\mathcal{B}_1$ of T_1 which separates $\mathbb{B}_1'$ and $\mathbb{B}_1''$.
Let's then consider the three possibilities:

(i) $\mathcal{B}_1$ has no residue in T': then $[x] = [\&/\oplus]$ or $[x]$ is a special $[r(\&)]$ e.r.s..

We claim that the case $[x] = [\&/\oplus]$ has to be excluded. Indeed, if $[x] = [\&/\oplus]$ let B_1 be the box of T_1 having a front door whose conclusion is a premise of the *cut*-link x. If $B_1 \neq \mathcal{B}_1$, then $\mathcal{B}_1$ is contained in a component of B_1 erased by $[x]$: in this case neither $\mathbb{B}_1'$ nor $\mathbb{B}_1''$ would have a residue in T', which is a contradiction. If $B_1 = \mathcal{B}_1$ and this box has no residue in T', then it has a unique front door: in this case, because $\mathcal{B}_1$ separates $\mathbb{B}'_1$ and $\mathbb{B}''_1$, at least one among $\mathbb{B}'_1$ and $\mathbb{B}''_1$ would have no residue in T', which is again a contradiction.

This means that $[x]$ is a special $[r(\&)]$ e.r.s., and $\mathcal{B}_1$ has two components and no residue in T': it is one among the 2^m boxes erased by $[x]$ (see figure 5). Let $\mathcal{B}_2$ be the box of T_1 s.t. its unique residue in T' has one more front door than $\mathcal{B}_2$: the unique residue of $\mathcal{B}_2$ in T' is the box that we are looking for.

(ii) $\mathcal{B}_1$ has one residue in T': in this case the box B' of T' that we are looking for is the unique residue of $\mathcal{B}_1$ in T'.

(iii) $\mathcal{B}_1$ has several residues in T': let $B'_{11}, \ldots, B'_{1k}$ be these $k \geq 2$ boxes of T'. Either $[x] = [ccad]$ or $[x] = [r(\&)]$.

(iii.1): If $[x] = [ccad]$, then $\mathcal{B}_1$ is contained in the subproof-net S of T_1 copied k times by the e.r.s. $[x]$. Let's call B_2 the box of T_1 s.t. $[x]$ is B_2-attractive, and let's call B'_2 the (unique) residue of B_2 in T'.

As subproof-nets of $\mathcal{B}_1$ (then of S), $\mathbb{B}'_1$ and $\mathbb{B}''_1$ have both k residues in T'. More precisely, each box $B'_{11}, \ldots, B'_{1k}$ contains a residue of $\mathbb{B}'_1$ and a residue of $\mathbb{B}''_1$. Let's call $\mathbb{B}'_{11}, \ldots, \mathbb{B}'_{1k}$ (resp. $\mathbb{B}''_{11}, \ldots, \mathbb{B}''_{1k}$) the residues of $\mathbb{B}'_1$ (resp. of $\mathbb{B}''_1$) contained in $B'_{11}, \ldots, B'_{1k}$ respectively. We have that $\mathbb{B}' = \mathbb{B}'_{1i}$ and $\mathbb{B}'' = \mathbb{B}''_{1j}$, where $i, j \in \{1, \ldots, k\}$.

If $i = j$, then the box B'_{1i} is the box of T' that we are looking for. If $i \neq j$, then the box B'_2 is the box of T' that we are looking for.

(iii.2): If $[x] = [r(\&)]$, then we use the notations of figures 5 and 6, and call $\mathcal{B}_2$ the box of T_1 s.t. its unique residue in T' has one more front door than $\mathcal{B}_2$.

In case $[x]$ is a special $[r(\&)]$ e.r.s. (figure 5), $\mathcal{B}_1$ is necessarily a box of one of the subgraphs S_i ($i \in \{1, \ldots, 2^m\}$), and the box B' of T' that we are looking for is either one of the residues of $\mathcal{B}_1$ in T' or the unique residue of $\mathcal{B}_2$ in T'.

In case $[x]$ is a non special $[r(\&)]$ e.r.s. (figure 6), $\mathcal{B}_1$ is either a box of one of the subgraphs S_i ($i \in \{1, \ldots, 2^m\}$) and in that case we argue like before, or $\mathcal{B}_1$ is one among the boxes $B_1, \ldots, B_{2^m}$ of figure 6. (Notice that $\mathcal{B}_1$ cannot be contained in one of the components of one among $B_1, \ldots, B_{2^m}$, because it would have a unique residue in T'). If $\mathcal{B}_1$ is the box B_i (for some $i \in \{1, \ldots, 2^m\}$), then $\mathbb{B}'_1$ (resp. $\mathbb{B}''_1$) is a box (with depth zero) of the component $\alpha_j^{iE'}$ (resp. $\alpha_k^{iE''}$) of B_i, for some $j, k \in \{1, \ldots, 2^{n-1}\}$, $j \neq k$, and $E', E'' \in \{C, D\}$. If $E' = E'' = C$

(resp. $E' = E'' = D$), then the residue B_i^C (resp. B_i^D) of B_i in T' is the box that we are looking for. If $E' \neq E''$, then the unique residue of $\mathcal{B}_2$ in T' is the box that we are looking for. $\square$

5.5. REMARK. The separation property allows to bound "a priori" the maximal depth of the reducts of a proof-net R: this is the basis of the following definition. Indeed, because a residue of a box B cannot be contained in another residue of B (thanks to the separation property), we know that the maximal depth of every reduct of R will never be greater than the number of boxes of R.
The reader should notice that this is not at all the case of exponential boxes (see [TdF00b] for a counterexample).

We now define a measure on proof-nets, which will allow to prove the strong normalization theorem (5.8). This measure is a t-uple, and we will then refer to the lexicographical order.

5.6. DEFINITION. Let T be a proof-net containing exactly n boxes, and R be any reduct of T. Let B be a box of R. We define:
$compl_{\&}(B)$:=number of *coad*-links of B whose conclusion is labeled by a formula having & as main connective.
For every $i \leq n$, $\mathcal{D}_i := \{B : B \text{ is a box with depth } i \text{ in } R\}$, and $compl_{\&}^i(R) := \Sigma_{B \in \mathcal{D}_i} compl_{\&}(B)$.
$\|R\|_{\&}$:= (number of links with depth 0,$compl_{\&}^0(R)$,..., number of links with depth $n-1$,$compl_{\&}^{n-1}(R)$).

5.7. PROPOSITION. *Let T be a proof-net containing exactly n boxes, and let R and R' be two reducts of T. One has:*
(i) If $R \stackrel{[ccad]}{\longrightarrow} R'$, then $\|R'\|_{\&} < \|R\|_{\&}$
(ii) If $R \stackrel{[r(\&)]}{\longrightarrow} R'$, then $\|R'\|_{\&} < \|R\|_{\&}$.

Proof: (i) Let $i \geq 0$ be the depth of the box B of R attractive for the $[ccad]$ step. Let $B_i \dots, B_1$ (with B_k of depth $k-1$) be the boxes of R containing B.
The number of links of R with depth j is the same before and after the reduction step, $\forall j < i$.
Moreover (if it exists) B_i, so as every box of R which is not contained in B_i, has a unique residue in R', and this residue has the same *coad* and &-doors,

and its conclusions the same types. In particular, this is the case for every box of R with depth $j < i$, and then $compl^j_{\&}(R') = compl^j_{\&}(R)\ \forall j < i$.
The number of links with depth i in R' is strictly less than the number of links with depth i in R, so that $\|R'\|_{\&} < \|R\|_{\&}$.

(ii) Let B be the box of R whose number of &-doors increases, and let $\vec{B}$ be its residue in R'. Suppose that B has depth i ($i \leq n-1$) and is contained in the boxes $B_i, \ldots, B_1$.
The number of links of R' with depth j, for every $j \leq i$, is the same as the number of links with depth j of R.
Moreover, like in case (i), (if it exists) B_i, so as every box of R which is not contained in B_i, has a unique residue in R', and this residue has the same *coad* and &-doors, and its conclusions the same types. In particular, this is the case for every box of R with depth $j < i$, and then $compl^j_{\&}(R') = compl^j_{\&}(R)\ \forall j < i$.
By definition of $\xrightarrow{[r(\&)]}$, we have that $compl_{\&}(\vec{B}) < compl_{\&}(B)$, from which we deduce $compl^i_{\&}(R') < compl^i_{\&}(R)$. Then $\|R'\|_{\&} < \|R\|_{\&}$. □

5.8. Theorem. *The rewriting rule $\longrightarrow$ is strongly normalising for PN_a proof-nets.*

Proof: We prove that if $T \twoheadrightarrow R \longrightarrow R'$, then $(s(R'), \|R'\|_{\&}) < (s(R), \|R\|_{\&})$, following the lexicographical order.
This is an immediate consequence of propositions 5.3, 5.7, 5.4. □

5.9. Remark. After theorem 5.8, one could (naïvely) dream of proving local confluence of the rewriting rule $\longrightarrow$, and then the confluence property for $\longrightarrow$.
This is clearly impossible. To convince herself/himself, the reader can consider the proof-net obtained from a box B by cutting a proof-net S out of a *coad*-door of B. There are in general several [*ccad*] associated with such a cut: one can copy S itself but also a (strict) subproof-net of S. Obviously, the two choices can lead to different normal forms.
One cannot exclude that some “ad hoc” restrictions might solve this kind of conflict. But our personal feeling is that something is still missing, and a deeper understanding of the structure of proof-nets should be reached first (see also the paragraph 8.2).
In the next sections, we will then rather try to exploit our notion of “multi-box” (which is not ad hoc!) to define a normalisation procedure using the

e.r.s. defined in section 4, that associates with every proof-net of PN_a a unique cut-free one.

6 On the structure of PN_a proof-nets

As stated in the introduction, we now want to use our new tool (multiboxes) to define a new way of computing the additive commutative steps. Coming back to figure 3, one is tempted to define as the [*ccad*]-reduct of the proof-net with cut, the proof-net with the multibox. However, figure 3 deals with a very particular case of additive commutative cut, and we already mentioned that the general case is much more complicated.
The cut-elimination procedure we want to define proceeds by first eliminating all the *cut*-links different from the *ccad* ones. Once these steps are performed (in any order), we want to use the recovered symmetry of the nets with multiboxes, in order to eliminate all the *ccad cut*-links at once. We start the process at depth 0, and then continue, inside the multiboxes with depth 0.
To compute these new (and "big") commutative steps in an (as much as possible) "reasonable" way, we'd better know the structure of our nets. Indeed, suppose a net R is composed of two nets R_1 and R_2, connected by a $\otimes$-link, and suppose that R contains only *ccad cut*-links (at depth 0). It is clearly useless to duplicate this $\otimes$-link (this is the typical case of a so-called "cut splitting" $\otimes$-link, see definition 6.8): one will rather handle R_1 and R_2 separately, performing a "big" commutative step in R_1, another "big" step in R_2, and finally connecting the two thus obtained nets by means of a $\otimes$-link. This is why we introduce in this section the notion of "minimal" proof-net (definition 6.9): a proof-net is minimal when it contains no cut splitting $\otimes$-link. We decompose (in definition 6.13) every proof-net into minimal ones (with cuts), and then apply (in section 7) the cut-elimination procedure to every minimal proof-net containing only *ccad cut*-links.

6.1 Conventions

We will use in the sequel the notion of **empire** of a formula, defined in [Gir87], and used in several works (like for example [Gir91b]).
A definition can be found in the paper just mentioned, but another one, more intuitive, is the following: "the empire of a formula A of a PN_a proof-net R, denoted by $e(A)$, is the biggest subproof-net of R having A among its conclusions".

The reader can refer in the sequel to any of these (of course equivalent) definitions. We will not recall in all details the properties of empires (which can be found in the works previously mentioned).

We will use the expressions "terminal link" and "last link".

A link n is a **last link** of a proof-net R, when the conclusion(s) of n is(are) a conclusion(s) of R (a *cut*-link is never a last link). A link n is a **terminal link** of a proof-net R, when there exists a sequentialization of R whose last rule is the one corresponding to the link n.

A terminal link, different from a *cut*-link, is always a last link. Conversely, every $\wp$ or $\oplus$-link which is a last link of a proof-net R, is also a terminal link of R. This is of course *not* the case of the $\otimes$-links: a $\otimes$-link might very well be a last link of R without being a terminal link.

The notion of "path" used in the sequel is the one of "straight path" introduced in [DR95], which can be generalized to our framework: a straight path is an oriented path changing direction only when crossing a *cut*-link or an axiom link.

In the sequel, we will simply write "path" always meaning "straight path".

Definition. ([**DR95**]) Let R be a proof-net. A path of R is a sequence of edges or reverted edges (i.e. a path may take an edge from its goal to its source). We denote by $\alpha, \beta, \ldots$ the edges of a proof-net (oriented as usual, following definition 2.1) and by $\alpha^*, \beta^*, \ldots$ the previous edges "reverted" (i.e. oriented now in the opposite direction). Let now a (resp. b) be an edge or a reverted edge whose goal (resp. source) is the link n; we denote by ab the path consisting of the edge a followed by the node n, itself followed by the edge b.

We will say that the path Φ of R is a **straight path** if:

(i) Φ does not contain any $\alpha^*\alpha$ nor any $\alpha\alpha^*$

(ii) if α and β are two distinct premises (and then, according to our notations, two edges) of the same link n, and if $\alpha\beta^*$ is a subpath of Φ, then n is a *cut*-link.

6.2. Definition. We will say that a path with first link n (resp. edge a) of a proof-net R is **negligible**, when it only crosses $\wp$ or $\oplus$-links (except perhaps its last link).

A negligible path with (all its links and edges at) depth zero is **maximal**, when its last link is a $\otimes$ or a *cut*-link.

Observe that if A is an edge of a proof-net R, there always exists a negligible path with starting edge A. If the edge A has depth 0, it is not always the case that there exists a maximal negligible path with starting edge A. When such a path doesn't exist, the last edge of the longest negligible path with starting edge A is a conclusion of R.

Notations:

Let R' be a subproof-net with conclusions Γ' of the proof-net R. We denote by $R \backslash R'$ the proof-net with hypothesis obtained by replacing in R the subproof-net R' by a link H with conclusions Γ'.
Let S be a subgraph of the proof-net R. If the graph obtained by erasing in R the links, the edges and the boxes of S is still a proof-net, then it will be denoted by $R \backslash S$.
If A and C are the premises of the $\otimes$-link n of the proof-net R, we will write $e(A) \cup e(C) \cup \{n\}$ meaning the subproof-net of R obtained by connecting $e(A)$ and $e(C)$ by means of the $\otimes$-link n.

6.3. Remark. (i) Let R be a proof-net containing a *cut*-link with premises A and $A^{\perp}$. Then $e(A) \cap e(A^{\perp}) = \emptyset$.
(ii) Let R' be a subproof-net with conclusions Γ of the proof-net R, and suppose that one (then every) formula of Γ has depth 0 in R. If every path of R with starting edge a formula of Γ is a negligible path which is not maximal, then every link of $R \backslash R'$ (different from the Hypothesis link) is a $\wp$ or a $\oplus$-link, crossed by at least one among these negligible paths.
(iii) Let R' be a subproof-net with conclusions Γ' of the proof-net R. If there exists a path with starting edge $A \in \Gamma'$, which crosses no *cut*-link, and with last edge D, then there exists a subproof-net of R having D among its conclusions and containing all the formulas of Γ' (in particular $\Gamma' \subseteq e(D)$ and $R' \subseteq e(D)$). One has, actually, $A \in e(D)$ (and $D \notin e(A)$) and then $R' \subseteq e(A) \subseteq e(D)$.

Little digression:

In order to decompose a proof-net into "minimal" ones, we first have to decompose a proof-net containing some terminal $\otimes$-links into a set of proof-nets. More precisely, if the proof-net R has exactly n terminal $\otimes$-links, one can associate with R a set E_R of $n+1$ proof-nets, such that R is obtained by connecting the elements of E_R by means of the n $\otimes$-links.

To be very precise, we show that E_R is well defined (it depends only on the proof-net R). The reader who is not interested in this little exercice can directly jump to the definition 6.9 of minimal proof-net.
What we are actually going to see, is that R can be though as a tree whose nodes are the $n+1$ proof-nets of E_R, and the leaves of the tree are the elements of E_R of the form $e(A)$, where A is a formula of R and $e(A)$ does not contain any of the n $\otimes$-links.

6.4. LEMMA. *Let R be a proof-net and R_1 a subproof-net of R. Let n be a $\otimes$-link which is a last link of R and a last link of R_1.
If n is a terminal link of R, then it is also a terminal link of R_1.*

Proof: We will denote in the sequel by $e_1(A)$ (resp. $e(A)$) the empire in R_1 (resp. R) of the formula A.
Let $C \otimes D$ be the conclusion of the link n. If n is not a terminal link of R_1, there exists a $\wp$-link of R_1 (then of R) with conclusion $A \wp B$ and such that $A \in e_1(C) \subseteq e(C)$, $B \in e_1(D) \subseteq e(D)$ and $A \wp B \notin e_1(C) \cup e_1(D)$.
If $A \wp B \in e(C)$, then $B \in e(C)$, which contradicts $B \in e(D)$ (remember that n is a terminal link of R). Similarly, one cannot have $A \wp B \in e(D)$. But this means that $A \wp B$ is not an edge of R: again, a contradiction. $\square$

6.5. LEMMA. *Let R be a proof-net with conclusion Γ, let $\otimes_1, \ldots, \otimes_n$ be $n \geq 1$ terminal links of R, and let $C_i \otimes D_i$ be the conclusion of the link $\otimes_i$. There exist $k \geq 1$ formulas $A_1, \ldots, A_k$, where $A_i \in \{C_i, D_i\}$, such that R is the following disjoint union:*

$$R = e(A_1) \cup \ldots \cup e(A_k) \cup \{\otimes_1, \ldots, \otimes_k\} \cup R_1,$$

where R_1 is (either the empty set or) a subproof-net of R and $e(A_j)$ does not contain any link of the set $\{\otimes_1, \ldots, \otimes_n\}$ (for every $j \in \{1, \ldots, k\}$).

Proof: In this proof, the union of two sets of links will always be a *disjoint* union.
Because $\otimes_1$ is a terminal link of R, we have $R = e(C_1) \cup e(D_1) \cup \{\otimes_1\}$. If $n = 1$, we are done. Otherwise, $e(C_1)$ or $e(D_1)$ contains one among the links $\otimes_j$ ($j \neq 1$). Let's suppose (for example) it is the case for $e(D_1)$. We can also suppose (maybe after a renaming of the links) that $\otimes_2 \in e(D_1)$.
Because $\otimes_2$ is terminal in R, $R = e(C_2) \cup e(D_2) \cup \{\otimes_2\}$. Suppose for example that $\otimes_1 \in e(C_2)$. In this case $\otimes_2 \notin e(C_1)$, and the fact that $\otimes_1 \in e(C_2)$ (then $C_1 \in e(C_2)$) implies that $e(C_1) \subseteq e(C_2)$ (see for example lemma 5 of [Gir96]).

Let $R_1 = e(D_1)$; we will denote in the sequel by $e_1(A)$ the empire in R_1 of the formula A. We want to show that $e(D_2) = e_1(D_2)$. Observe that the inclusion $e_1(D_2) \subseteq e(D_2)$ is obvious. Conversely, were A a formula of $e(D_2)$ which is not a formula of $e_1(D_2)$, then $A \notin e(C_2) \supseteq e_1(C_2)$. By applying now lemma 6.4, we obtain that $\otimes_2$ is a terminal link of the proof-net R_1: $R_1 = e_1(C_2) \cup e_1(D_2) \cup \{\otimes_2\}$. Then $A \notin R_1$, which would imply $A \in e(C_1)$ or $A = C_1 \otimes D_1$. Remember now that $e(C_1) \subseteq e(C_2)$ and $C_1 \otimes D_1 \in e(C_2)$: we would get $A \in e(C_2)$, which is a contradiction.
Observe that $e(D_2) = e_1(D_2)$ ($\subsetneq e(D_1)$) contains strictly less links of $\{\otimes_1, \dots, \otimes_n\}$ than $e(D_1)$. This means that by applying the previous process to $e(D_2)$ (and then recursively) instead of $e(D_1)$, one will eventually get for some $i \in \{1, \dots, n\}$ a subproof-net $e(A_i)$ of R (where $A_i \in \{C_i, D_i\}$), which does not contain any link of the set $\{\otimes_1, \dots, \otimes_n\}$. $\square$

6.6. DEFINITION. **(decomposition 1)** Let R be a proof-net with conclusion Γ, and (at least) n terminal $\otimes$-links, $n \geq 1$.
We are going to associate with R its first *decomposition*. We define a (multi)set of proof-nets $E_R = \{T_1, \dots, T_{n+1}\}$ such that R can be obtained by connecting the elements of E_R by means of these n $\otimes$-links.
To obtain E_R, apply the previous lemma, taking the maximal integer $k \geq 1$ such that $R = e(A_1) \cup \ldots \cup e(A_k) \cup \{\otimes_1, \dots, \otimes_k\} \cup R_1$. Then apply the same lemma (still with the maximal k) to the proof-net R_1, recursively (we need for this lemma 6.4).

6.7. REMARK. The procedure described in the previous definition shows that the set $E_R = \{T_1, \dots, T_{n+1}\}$ of proof-nets is uniquely determined by the proof-net R.

We introduce now the notion of "minimal proof-net", and show how every proof-net of PN_a can be decomposed into minimal ones (definition 6.13).

6.8. DEFINITION. Let R be a proof-net, and $n = cut, \otimes$ a link of R with depth 0 and with premises A_1 and A_2.
We will say that n is **cut splitting** for R, or that n **splits the cuts** of R, when every *cut*-link of R different from n is a link of $e(A_1) \cup e(A_2)$.

6.9. DEFINITION. **(minimal proof-net)** The proof-net R is said to be **minimal**, when there is no cut splitting $\otimes$-link for R.

6.10. Lemma. **(splitting the cuts)** Let R be a proof-net. If there exists a *cut*-link with depth 0 in R, then one has two possibilities:
(i) there exists a cut splitting $\otimes$-link for R
(ii) (i) is not satisfied, and there exists a cut splitting *cut*-link for R.

Proof: Let $\sharp R$:=number of links of R. We proceed by induction on $\sharp R$, using the sequentialization theorem.
If R has a terminal axiom link, then this link is also the unique link of R and there are no *cut*-links with depth 0 in R.
Otherwise, suppose that (i) is not satisfied. Because there is no $\otimes$-link which is cut splitting for R, R has no terminal $\otimes$-link.

- If R is a box, then there are no *cut*-links with depth 0 in R.
- If among the terminal links of R there is a *cut*-link c, then c is cut splitting for R.
- If among the terminal links of R there is a $\oplus$ (resp. $\wp$) link, then R can be obtained from a proof-net R_1 by adding this $\oplus$ (resp. $\wp$) link and its conclusion. If there exists a *cut*-link with depth 0 in R, then it is also a *cut*-link of R_1. Moreover, every link n which is cut splitting for R_1 is also cut splitting for R. Such a link n does necessarily exist (by induction hypothesis), and it then has to be a *cut*-link (because R does not satisfy (i)). □

6.11. Lemma. *Let R be a proof-net whose last links are all axiom links, doors of some boxes or $\otimes$-links, and suppose there exists a cut-link with depth 0 in R.*
If there exists a cut splitting $\otimes$-link for R, then there exists a terminal $\otimes$-link of R.

Proof: Let $\otimes_0$ with conclusion $A_0 \otimes D_0$ be a cut splitting $\otimes$-link for R. By definition 6.8, $\otimes_0$ has depth 0 in R.
Let then Φ_0 be the path with starting link $\otimes_0$ and last edge a conclusion of R. Because $\otimes_0$ is cut splitting, Φ_0 cannot cross any *cut*-link. The last link crossed by Φ_0 is then a $\otimes$-link: let $\otimes_1$ be this link (one might have $\otimes_0 = \otimes_1$). Let A_1 be the last edge crossed by Φ_0 before $\otimes_1$, and suppose that $A_1 \otimes D_1$ is the conclusion of $\otimes_1$ (if the conclusion is $D_1 \otimes A_1$ we argue, obviously, in an analog way). If $\otimes_0 = \otimes_1$, we define $A_1 = A_0$ and $D_1 = D_0$. Observe that if $A_1 \neq A_0$, then $e(A_1)$ contains all the *cut*-links of R.

If $R = e(A_1) \cup e(D_1) \cup \{\otimes_1\}$, then $\otimes_1$ is the terminal link that we are looking for.
Otherwise, there exists a $\wp$-link having as premises a conclusion of $e(A_1)$ and a conclusion of $e(D_1)$. Like before, let Φ_1 be the path having as first link this $\wp$-link and having as last link a $\otimes$-link and as last edge a conclusion of R. Let's call $\otimes_2$ this $\otimes$-link, A_2 the last edge crossed by Φ_1 before $\otimes_2$, and suppose that $A_2 \otimes D_2$ is the conclusion of $\otimes_2$. In *any* case, $e(A_2)$ contains all the *cut*-links of R, and $e(A_2) \supsetneq e(A_1)$. If $R = e(A_2) \cup e(D_2) \cup \{\otimes_2\}$, then we are done. Otherwise, we start the process again.
To conclude, observe that $\forall i \geq 1$, one has $e(A_{i+1}) \supsetneq e(A_i)$, which entails that for some $k \geq 1$, we will necessarily have $R = e(A_k) \cup e(D_k) \cup \{\otimes_k\}$. $\square$

6.12. Corollary. *If* there is no terminal $\otimes$, $\wp$, nor $\oplus$-link in the proof-net R, then R is minimal.

Proof: The absence of terminal $\wp$ and $\oplus$-links, implies the absence of last $\wp$ and $\oplus$-links. This means that the last links of R are all axiom links, doors of some boxes or $\otimes$-links.
If there is no *cut*-link with depth 0 in R, then every sequentialization of R must end with a &-rule or with an axiom. In the first case R is a box, in the second one it is an axiom link. And in both cases we are done.
If there exists a *cut*-link with depth 0 in R, then we can apply the previous lemma: if there were a cut splitting $\otimes$-link for R, then there would be a terminal $\otimes$-link too. $\square$

If among the last links of a proof-net R there are some $\wp$ or/and $\oplus$-links, then one can erase these links and their conclusions, until one obtains a subproof-net of R whose last links are all $\otimes$, ax, or doors of some boxes. Such a subproof-net will be denoted by $R^{\wp\oplus}$ in the following definition.

6.13. Definition. **(Decomposing R by means of minimal proof-nets)** Let R be a proof-net. We define by induction on the number of links of R, the set $D(R)$ of the minimal proof-nets associated with R:
- if the unique link of R is an axiom link, then $D(R) := \emptyset$
- if there exists a terminal $\wp$ or $\oplus$-link, then $D(R) := D(R^{\wp\oplus})$
- if there exists no terminal $\wp$ nor $\oplus$-link, and there exist exactly $n \geq 1$ terminal $\otimes$-links of R, then let $\{R_1, \dots, R_{n+1}\}$ be the set associated with R by definition 6.6. We define $D(R) := D(R_1) \cup \ldots \cup D(R_{n+1})$

- if there is no terminal $\otimes$,$\wp$, nor $\oplus$-link of R, and if there exists a terminal *cut*-link, then $D(R) := \{R\}$ (we know that R is minimal thanks to the corollary 6.12)
- otherwise, R is necessarily a box, and $D(R) := \emptyset$.

6.14. Remark. Notice that the previous definition decomposes proof-nets into minimal ones *with cuts*. Suppose for example that R is obtained by connecting the two minimal proof-nets (without terminal $\wp$-links, nor terminal $\oplus$-links nor terminal $\otimes$-links) R_1 and R_2 by means of a $\otimes$-link. If R_1 is cut-free and R_2 contains a *cut*-link with depth 0, then the previous definition gives: $D(R) = D(R_1) \cup D(R_2) = \{R_2\}$.
This is simply because we will use the decomposition into minimal proof-nets only to perform cut-elimination.

7 The Normalization procedure of PN_a

The normalization procedure we are going to define (definition 7.13) is the usual one *except* for the commutative steps. We start by eliminating the *cut*-links at depth 0 which are not of type *ccad* (using the e.r.s. of section 4), and this is easily done. The difficult point is to handle the *cut*-links of type *ccad* (with depth 0). To eliminate all of them at once, we can suppose (thanks to the previous section) that the proof-nets we deal with are minimal. We will then focus on minimal proof-nets, containing only *cut*-links of type *ccad* (at depth 0), and with at least such a *cut*-link.
We first give a "canonical" representation of these nets (definition 7.2), and show they always contain a particular box, which will be called "extremist" (lemma 7.4). With respect to these extremist boxes, the notion of "slice" of a proof-net makes sense (definition 7.6), and allows to define the "complete [*ccad*]-reduct" of a minimal proof-net containing only *cut*-links of type *ccad* (definition 7.10). We then show (proposition 7.12) that the complete [*ccad*]-reduct can also be reached by performing some [*ccad*] and some [$r(\&)$] e.r.s. (defined in section 4). The strong normalization theorem (theorem 5.8) of section 5 allows then to conclude that the thus defined cut-elimination procedure associates with every proof-net of PN_a a unique cut-free one (theorem 7.14).

We start with a preliminary result, which we use in the sequel. In the proof of the following lemma, the minimality of R is essential.

Let R be a minimal proof-net containing a $\otimes$-link with depth 0 and conclusion $A \otimes D$. In the following definition, R_{AD} is the subproof-net of R obtained by connecting the subproof-nets $e(A)$ and $e(D)$ (with conclusions A, Γ_A and D, Γ_D respectively), by means of a $\otimes$-link with conclusion $A \otimes D$.

7.1. LEMMA. **(of the non splitting $\otimes$-links)** The subproof-net R_{AD} of R satisfies one of the two following conditions:
(i) there exists a maximal negligible path Φ with starting edge $A \otimes D$
(ii) there exist $G \in \Gamma_A$ and $H \in \Gamma_D$ s.t. G and H are premises of a $\wp$-link, and there exists a maximal negligible path Φ with starting link precisely this $\wp$-link.
In both cases, if Z is the last edge of Φ, the subproof-net $e(Z)$ of R contains the edge $A \otimes D$ (and then also the subproof-net R_{AD}).

Proof: First observe that every conclusion of R_{AD} has depth 0, which implies that whenever a path with starting edge one of these conclusions crosses a *coad*-link or a &-link, this path has already crossed a *cut*-link.
Then observe that (by definition of empire) every formula of $\Gamma_A \cup \Gamma_D$ is either a premise of a $\wp$-link or a conclusion of R.
Let's suppose that (i) is not satisfied and prove (ii). Every path with starting edge $A \otimes D$ is then negligible and not maximal: this means that every such path contains only $\wp$-links and $\oplus$-links. Let Ψ be the longest path with starting edge $A \otimes D$. The last edge of Ψ is a conclusion of R.
We have two possibilities:

case 1: Every $\wp$-link of R has at most one premise among the edges $\Gamma_A \cup \Gamma_D$.

If Ψ crosses no $\wp$-links, then (because the correctness graphs of R are connected) every formula of $\Gamma_A \cup \Gamma_D$ is a conclusion of R, and the starting $\otimes$-link would then be cut splitting for R, thus contradicting its minimality.

If Ψ crosses a $\wp$-link, then (again, because the correctness graphs of R are connected) the premise of the $\wp$-links crossed by Ψ which is not an edge of Ψ has to be among $\Gamma_A \cup \Gamma_D$. Conversely, the conclusion of every $\wp$-link having a premise among $\Gamma_A \cup \Gamma_D$ is an edge of Ψ. This is enough to say that the proof-net with hypothesis $R \backslash R_{AD}$ contains only $\wp$-links or $\oplus$-links (see (ii) of remark 6.3). The starting $\otimes$-link would then be cut splitting for R, thus contradicting its minimality.

case 2: Suppose now that there exist n $\wp$-links ($n \geq 1$) with premises $G_i \in \Gamma_A$ and $H_i \in \Gamma_D$ ($i \in \{1, \ldots, n\}$). In this case, there exists a subproof-net R' of R with conclusions $A \otimes D$, $G_1 \wp H_1, \ldots, G_n \wp H_n$, $\Gamma_A \cup \Gamma_D \backslash \{G_1, H_1, \ldots, G_n, H_n\}$.

By conctradiction, let us suppose that all the paths with starting edge among $G_1 \wp H_1, \ldots, G_n \wp H_n$ are negligible and not maximal. For the same reasons as in case 1 (and mainly because the correctness graphs of R are connected), every path with starting edge a conclusion of R_{AD} is negligible and not maximal. From (ii) of remark 6.3, $R \backslash R_{AD}$ contains then only $\wp$-links or $\oplus$-links. This implies that the $\otimes$ link with conclusion $A \otimes D$ is cut splitting for R, thus contradicting its minimality.

Observe that were (i) not satisfied, the path Φ satisfying (ii) would have a starting link whose premises would be conclusions of R_{AD}, which implies that even in case (ii) one has $A \otimes D \in e(Z)$. □

The following definition is just a way to represent the proof-nets we want to study. The notations we are going to introduce will be constantly used in the sequel.

7.2. DEFINITION. **(canonical form)** Let R be a minimal proof-net and B a box with depth 0 in R s.t. there exists a conclusion of a *coad*-door of B which is a premise of a *cut*-link. The proof-net of figure 7 is called the canonical form of R w.r.t. B, where:
- the conclusions of the &-doors of B are $L_{s+1}, \ldots, L_k$, and the conclusions of the *coad*-doors of B are $A_1 \ldots A_n, L_1 \ldots L_s$, where $n \geq 1, k \geq 1, s \geq 0$
- none among $L_1, \ldots, L_s$ is the premise of a *cut*-link
- for every $i \in \{1, \ldots, n\}$ the graph $e(A_i^{\perp})$ is the empire of $A_i^{\perp}$ in R, and its conclusions are $A_i^{\perp}, \Gamma_i$ (Γ_i is a -possibly empty- multiset of formulas)
- S is a proof-net with hypothesis, and the conclusions of the unique Hypothesis link of S are $L_1 \ldots L_k, \Gamma_1 \ldots \Gamma_n$, while the conclusions of S are $M_1 \ldots M_p$ ($p \geq 1$), which are also the conclusions of R. We denote by S both the proof-net with hypothesis and the subgraph of the proof-net R.

The next lemma will be used in the proof of the following one.

Let R be a minimal proof-net s.t. every *cut*-link with depth 0 in R is a *ccad*. Let B be a box with depth 0 and s.t. one of the conclusions of the *coad*-doors of B is the premise of a *cut*-link. Let us consider the canonical representation of R w.r.t. B (figure 7).

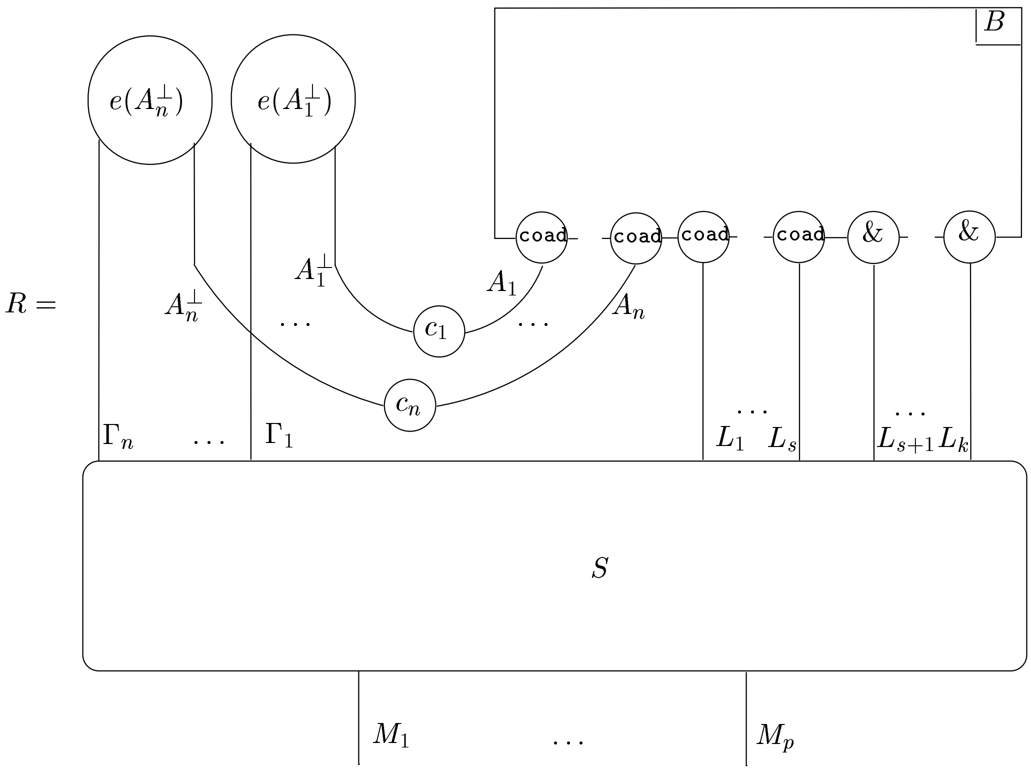


Figure 7. Canonical form of R with respect to B

7.3. LEMMA. *If there exists a cut-link in S, then there exists a cut-link c of S and a premise Q of c which is not the conclusion of a coad-link, and s.t. $e(Q) \supseteq R \backslash S$.*

Proof: We use the notations of figure 7 and definition 7.2.
First observe that (by *(ii)* of remark 6.3), because S contains a *cut*-link, there exists a maximal negligible path Φ_L with starting edge a conclusion L of $R \backslash S$. We then have two possibilities for S:
(i) there exists a maximal negligible path Φ_L with starting edge the conclusion L of $R \backslash S$, and whose last link is a *cut*-link
(ii) there exists a maximal negligible path Φ_L with starting edge the conclusion L of $R \backslash S$, and whose last link is a $\otimes$-link.

case (i): Let c be the *cut*-link which is the last link of Φ_L. Let G be the premise of c contained in Φ_L. If $G = L$, then (by definition of empire) $G \notin \Gamma_i$, for every $i \in \{1, \ldots, n\}$. We then have (by definition 7.2) $G \in \{L_{s+1}, \ldots, L_k\}$, i.e. G is the conclusion of a &-door of B: then $e(G) \supseteq R \backslash S$ and G is not a conclusion of a *coad*-link.

If $G \neq L$, then by definition of Φ_L, G is the conclusion of a $\wp$-link or of a $\oplus$-link and (applying *(iii)* of remark 6.3) $e(G) \supseteq R \backslash S$.

case (ii): Let $A_0 \otimes_0 D_0$ be the conclusion of the last link of Φ_L. One has $e(C_0) \supseteq R \backslash S$, for some $C_0 \in \{A_0, D_0\}$. From lemma 7.1 of the non splitting $\otimes$-links, there exists a maximal negligible path Φ_0 whose last edge C_1 is the premise of a *cut*-link or of the link $\otimes_1$ with conclusion $A_1 \otimes D_1$. One has $e(C_1) \supsetneq e(C_0)$.

More generally, lemma 7.1 allows to produce a sequence of formulas $C_0, \ldots, C_n$ (where $C_i \in \{A_i, D_i\}$), and a sequence of maximal negligible paths $\Phi_0, \ldots, \Phi_n$ such that:

- $e(C_n) \supsetneq \ldots \supsetneq e(C_0) \supseteq R \backslash S$

- for every $i \in \{1, \ldots, n\}$ the formula C_i is not the conclusion of a *coad*-link

- the last edge of Φ_i is C_{i+1} for $i \in \{0, \ldots, n\}$.

The number of links of the empires of the C_i strictly increases at each step of our construction: this implies that there exists an integer n s.t. the last edge C_{n+1} of Φ_n is the premise of a *cut*-link. The edge C_{n+1} is not the conclusion of a *coad*-link, and $e(C_{n+1}) \supseteq R \backslash S$. □

We now show the existence of "extremist" boxes in minimal proof-nets.

7.4. Lemma. **(of the extremist boxes)** Let R be a minimal proof-net s.t. every *cut*-link with depth 0 in R is a *ccad*, and suppose there exists such a *cut*-link.
There exists a box B of R (with depth 0) such that R has a canonical form w.r.t. B, and the proof-net with hypothesis S (of this canonical form) contains no *cut*-link.
We will say that such a box B is **extremist**.

Proof: Again, we use the notations of definition 7.2 and of figure 7.
For every box B of R with depth 0 s.t. there exists a conclusion of a *coad*-door of B which is premise of a *cut*-link (i.e. a box w.r.t. which one can represent R in canonical form), let us define:
$c_B := \{c_1, \ldots, c_n\} \cup \{c' : c' \text{ is a } cut\text{-link of } e(A_i^\perp), \text{ for } i \in \{1, \ldots, n\}\}$.
Let c_0 be a *cut*-link with depth 0 in R. Let D_0 and $D_0^\perp$ be the two premises of c_0. Because a *cut*-link with depth 0 is a *ccad*, we can suppose that D_0 is the conclusion of a *coad*-link of B_0. Let's then consider the canonical form of R w.r.t. B_0. If B_0 is extremist, then we are done.
Otherwise, from lemma 7.3, S_0 contains a *cut*-link c_1 s.t. one of the premises of c_1 (let's call it D_1) is not the conclusion of a *coad*-link, and $e(D_1) \supseteq R \backslash S_0$. The edge $D_1^\perp$ is necessarily the conclusion of a *coad* door of the box B_1 of S_0. The proof-net R has then a canonical form w.r.t. B_1, and $e(D_1)$ is among the subproof-nets of R which are cut out of the *coad*-doors of B_1 (remember figure 7). Because $e(D_1) \supseteq R \backslash S_0$, we have $card(c_{B_1}) > card(c_{B_0})$ ($c_1 \notin c_{B_0}$).
Like in the proof of the previous lemma, in this way one can build a sequence of boxes of R, which has to be finite, and whose last element is the extremist box that we are looking for. $\Box$

7.5. Lemma. *In the subgraph S of the canonical form of the minimal proof-net R w.r.t. the extremist box B, there are only $\wp$-links and $\oplus$-links.*

Proof: If a $\otimes$-link were contained in S, then it would be cut splitting for R. $\Box$

We now have to define the elimination of the *cut*-links of type *ccad*. We will try to give a (reasonably) precise description of such a procedure. This is not that simple, and to understand in details the content of theorem 7.14 requires from the reader a certain amount of patience.

If the conclusion of every &-door of the multibox B is also a conclusion of the proof-net R containing B, then one can define the proof-net obtained from R by substituting for B one of its components. By generalizing this substitution to the case of several multiboxes, we obtain the following notion of "slice" of a proof-net.

7.6. DEFINITION. **(slice of a proof-net)** Let R be a proof-net with conclusion
$\Gamma, D^1_1\&E^1_1, \ldots, D^1_{k_1}\&E^1_{k_1}, \ldots, D^n_1\&E^n_1, \ldots, D^n_{k_n}\&E^n_{k_n}$. For every $j \in \{1, \ldots, n\}$, let B^j be a box of R s.t. the conclusions of the &-doors of B^j are $D^j_1\&E^j_1, \ldots, D^j_{k_j}\&E^j_{k_j}$.
For every $j \in \{1, \ldots, n\}$ and for every $l \in \{1, \ldots, k_j\}$, we associate the integer $i^j_l = 1$ (resp. $i^j_l = 2$) with the 2^{k_j-1} components of B^j with conclusion D^j_l (resp. E^j_l). We then have a bijection between the components of B^j and the sequences $i^j_1 \ldots i^j_{k_j}$. We will denote by $B^j_{i^j_1 \ldots i^j_{k_j}}$ the component of B^j with conclusions $C^j_1 \ldots C^j_{k_j}$ (where $C^j_l \in \{D^j_l, E^j_l\}$), associated with the sequence $i^j_1 \ldots i^j_{k_j}$.
Let $k = k_1 + \ldots + k_n$.
We then call **slice** of R w.r.t. $B^1, \ldots, B^n$, any of the 2^k following proof-nets:

$$R_{i^1_1 \ldots i^1_{k_1} \ldots i^n_1 \ldots i^n_{k_n}} := R[B^1_{i^1_1 \ldots i^1_{k_1}}/B^1, \ldots, B^n_{i^n_1 \ldots i^n_{k_n}}/B^n]$$

with conclusions $\Gamma, C^1_1, \ldots, C^1_{k_1}, \ldots, C^n_1, \ldots, C^n_{k_n}$, where $C^j_l \in \{D^j_l, E^j_l\}$.

7.7. DEFINITION. Let R be a minimal proof-net s.t. every *cut*-link with depth 0 in R is a *ccad*, and there exists such a *cut*-link.
An extremist box B of R will be said to be **clean** (in R), when the graph S of the canonical form of R w.r.t. B is empty. In particular, when B is clean, the conclusion of every &-door of B is a conclusion of R.

7.8. DEFINITION. Let R be a minimal proof-net s.t. every *cut*-link with depth 0 in R is a *ccad*.
Let us remove (recursively) every terminal $\wp$ and $\oplus$-link of R (and its conclusion) until we obtain a subproof-net of R whose last links are $\otimes$, *ax* or doors of some boxes. This subproof-net of R, denoted by $R^{\wp\oplus}$, is called the **clean subproof-net** of R (see also definition 6.13).

7.9. REMARK. Every extremist box of $R^{\wp\oplus}$ is clean.

7.10. DEFINITION. Let R be a minimal proof-net s.t. every *cut*-link with depth 0 in R is a *ccad*, and suppose there exists such a *cut*-link. Let $B^1, \ldots, B^n$ be the n extremist boxes of R. Let $D_1^j \& E_1^j, \ldots, D_{k_j}^j \& E_{k_j}^j$ be the $k_j \geq 1$ conclusions of the &-doors of B^j. Let $R^{\wp\oplus}$ be the clean subproof-net of R. The boxes $B^1, \ldots, B^n$ are clean in $R^{\wp\oplus}$.
We call **complete** $[ccad]$**-reduct** of R the proof-net obtained from R by replacing $R^{\wp\oplus}$ by the multibox B s.t. the conclusions of the &-doors of B are the formulas $D_1^1 \& E_1^1, \ldots, D_{k_1}^1 \& E_{k_1}^1, \ldots, D_1^n \& E_1^n, \ldots, D_{k_n}^n \& E_{k_n}^n$, and the components of B are the slices of R w.r.t. the n extremist boxes $B^1, \ldots, B^n$. More precisely, the components of B are the proof-nets $R_{i_1^1 \ldots i_{k_1}^1 \ldots i_1^n \ldots i_{k_n}^n}$ (defined in 7.6) with conclusions $C_1^1, \ldots, C_{k_1}^1, \ldots, C_1^n, \ldots, C_{k_n}^n$, where $C_l^j \in \{D_l^j, E_l^j\}$.

Let B be a box of the proof-net R, $F_1 \& G_1, \ldots, F_l \& G_l$ be the conclusions of the &-doors of B, and A be the conclusion of a *coad*-door of B. Suppose that $A, F_1 \& G_1, \ldots, F_l \& G_l$ are conclusions of R.
Let R' be a proof-net with conclusions $A^\perp, \Gamma$, let $B^1, \ldots, B^n$ be boxes of R' s.t. for every $j \in \{1, \ldots, n\}$ the conclusions of the &-doors of B^j are $D_1^j \& E_1^j, \ldots, D_{k_j}^j \& E_{k_j}^j$. Suppose that all these formulas are conclusions of R'.
Let's call T the proof-net obtained by cutting R and R' by means of a *cut*-link with premises A and $A^\perp$. (Notice that the conclusions of all the &-doors of $B, B^1, \ldots, B^n$ are conclusions of T).

7.11. LEMMA. *The proof-net obtained from T by first copying 2^l times the proof-net R' (i.e. by performing a $[ccad]$ e.r.s. of definition 4.1), and then by applying $k = k_1 + \ldots + k_n$ e.r.s. $\xrightarrow{[r(\&)]}$, is a multibox with $k+l$ &-doors with conclusions $F_1 \& G_1, \ldots, F_l \& G_l, D_1^1 \& E_1^1, \ldots, D_{k_1}^1 \& E_{k_1}^1, \ldots, D_1^n \& E_1^n, \ldots, D_{k_n}^n \& E_{k_n}^n$, and whose components are the 2^{k+l} slices of T w.r.t. $B, B^1, \ldots, B^n$.*

Proof: It is a consequence of the previous definitions. □

7.12. PROPOSITION. *Let R be a minimal proof-net s.t. every cut-link with depth 0 in R is a ccad, and there exists such a cut-link. Let $B^1, \ldots, B^n$ be the $n \geq 1$ extremist boxes of R, and $R^{\wp\oplus}$ the clean subproof-net of R.
The complete $[ccad]$-reduct of R can be obtained using some $[ccad]$ e.r.s. and some steps $[r(\&)]$, in the following way:*

- choose an extremist box B and the canonical form of R associated with B (figure 7)
- apply (in any order) the $[ccad]$ e.r.s. B-attractive consiting in copying the required number of times the empires $e(A_1^\perp), \ldots, e(A_n^\perp)$ (these steps can in fact be performed in $R^{\wp\oplus}$)
- apply the necessary $[r(\&)]$ steps to the *coad*-doors of the box thus obtained.

Proof: Just apply to any extremist box the previous lemma. □

In the following definition, we will refer to the "$\neg[ccad]$ rewriting rule" meaning the rewriting rule obtained from the "usual" rewriting rule (see the beginning of section 5) by forbidding $[ccad]$ e.r.s. In other terms, we forbid the steps $[ccad]$ and $[r(\&)]$ of definition 4.1.

7.13. DEFINITION. **(the normalization procedure for PN_a)** Let R be a proof-net. We apply to R the following normalization procedure:

1. Apply the rewriting rule $\neg[ccad]$ at depth 0, until you obtain a proof-net R_0, s.t. every *cut*-link of R_0 with depth 0 is (only) of type *ccad*
2. Decompose R_0 by means of minimal proof-nets (with cuts), following definition 6.13
3. Associate with every minimal proof-net of $D(R_0)$ its clean subproof-net (following definition 7.8)
4. Associate with each of the thus obtained subproof-nets its complete $[ccad]$-reduct.

In such a way, we obtain a proof-net with no *cut*-links at depth 0. We then apply that same procedure to each of the components of the boxes with depth 0, and go on until we reach a cut-free proof-net.

7.14. THEOREM. **(confluence of the procedure)** Let R be a proof-net. The normalization procedure just described uses only the e.r.s. of definition 4.1, and it allows to associate with R a unique cut-free proof-net.

Proof: Let us use the notations of the definition. Observe that the rewriting rule $\neg[ccad]$ is confluent: it is obviously strongly normalizing (for example as a consequence of theorem 5.8), and locally confluent (this is just an easy

checking of all possible cases, and it is also true in PN_2). The proof-net R_0 is then unique.
The unicity of the decomposition by means of minimal proof-nets is a consequence of definition 6.13. The unicity of the clean subproof-net is an immediate consequence of the definition. So as the unicity of the complete [*ccad*]-reduct, a consequence of definition 7.10.
The procedure will necessarily terminate, because only e.r.s. defined in 4.1 are used (see proposition 7.12), and we know that these steps yield a strongly normalizing rewriting rule (theorem 5.8). □

8 Concluding Remarks

We would like to conclude by pointing out to the reader two aspects of the syntax of LL (which we did not discuss), relating them to the additive multiboxes.

8.1 About the extension to full LL

The extension of PN_a to full LL doesn't seem to be problematic at all. One should take proof-nets as defined in [TdF01] (with jumps for weakenings), and replace the notion of "additive box" by the one of "additive multibox". The normalization procedure defined in section 7 should extend to full LL. The only technical problem might arise from the strong normalization theorem (theorem 5.8): one should first fill the last hole in the proof of the strong normalization theorem for PN_2 (see [TdF00a]), and extend it to the calculus with multiboxes, which might be non trivial.

8.2 Polarization, focalization, and hypersequentialization

The two crucial properties of focalization and reversion have been widely studied and exploited ([AP91], [Gir91a], [DJS97], [LQTdF01]).
Thanks to the notion of "positive tree" (coming from [Lau99]), the problems of confluence and strong normalization have been entirely solved in the polarized and focalized fragment of LL (see [LQTdF01]).
But polarized and focalized proof-nets have a very particular shape, and despite the fact that classical logic can be translated by means of these proof-nets, the feeling is that "something is missing". This is not only because (at least for the moment), Light Linear Logic cannot be expressed by means of these proof-nets (as we wrote in the introduction), but also because of the notion of proof-net coming from "ludics". The "ludics" program, launched

in [Gir99], heavily relies on the two properties of focalization and reversion. Its first accomplishment (the paper [Gir00]) introduces a notion of proof-net (hypersequentialized proofs) which uses a variant of our notion of multibox (see [TdF00b] for more details): it is a box with 2^n components but only one front door. The unicity of the front door (due to a strict alternation of polarities) makes the nets of [Gir00] much more rigid that ours (in particular, the -so perfect- system of MLL proof-nets is not a subsystem of these nets). Maybe, a new challenge on proof-nets lies in the development of a "parallel" ludics (thus keeping the properties of polarization and focalization), using our notion of multibox in a more flexible way.

A The linear sequent calculus for $MALL$

Axiom and cut:

$$\frac{}{\vdash A, A^{\perp}}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ cut$$

The Hypothesis rule (Hyp):

$$\frac{}{\vdash \Gamma}\ (Hyp)$$

Multiplicative logical rules:

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}\ \wp$$

Additive logical rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2 \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \&$$

B The denotational semantics of $MALL$

We consider here two standard denotational semantics of LL: coherent and relational semantics (see for example [TdF00b] for more details). A formula

A of LL is interpreted by a coherent space or a set, denoted by $\mathcal{A}$. These semantics are *concrete*: it is possible to *compute* the element of $\mathcal{A}$ associated with a given proof-net with conclusion A. This is done by using the notion of **experiment** (introduced in [Gir87]) that we now define.

The following definition is the extension to $MALL$ of the definition of [Gir87]. It is not exactly the restriction to $MALL$ of the one given in [TdF00b], but an equivalent one. The idea is to define an experiment of an $MALL$ proof-structure S as an experiment of a "slice" of S. The notion of slice is intuitively clear: we substitute (recursively), in S, for every additive box one of its components (the notion of component obviously makes sense for proof-structures too).
But let us be more precise. Let $C_1\&C_2$ be the conclusion of the front door of the additive box B, and let Γ be the multiset of the conclusions of the *coad*-doors of B. Suppose that there are no boxes inside B. The proof-structure $T = B$ has two slices. The first one is obtained by substituting for B its first component: we thus obtain a proof-structure with conclusion Γ, C_1, to which we add a $\&_1$-link: it is a *unary* &-link, with premise C_1 and conclusion $C_1\&C_2$. The second slice is obtained in the obvious way (we will add in that case a $\&_2$-link: a *unary* &-link, with premise C_2 and conclusion $C_1\&C_2$). Of course, these two slices are not (strictly speaking) proof-structures: they cannot be obtained applying definition 2.1. We will nevertheless still speak of proof-structures referring to the slices of T. Notice that in these "new" proof-structures there are no *coad*-links.
The reader will certainly agree that the formal definition of slice of an $MALL$ proof-structure is straightforward from the previous example. If T is a proof-structure with conclusion Γ, then every slice of T is a proof-structure with conclusion Γ.

B.1. DEFINITION. **(experiment)** An experiment e of a proof-structure T is an experiment of one of the slices of T. In case T has no slices, we consider T as its unique slice.
An experiment e of a slice S of a proof-structure T is an application which associates with every edge a of S of type A an element $e(a)$ of $|\mathcal{A}|$ (where $|\mathcal{A}|$ is the web of the space $\mathcal{A}$), satisfying the following conditions:

- If $a = a_1$ is the conclusion of an axiom link with conclusions a_1 of type A and a_2 of type $A^\perp$, then $e(a_1) = e(a_2)$
- If a is the conclusion of type $C_1 \wp C_2$ (resp. $C_1 \otimes C_2$) of a $\wp$ (resp.

$\otimes$) link with premises c_1 of type C_1 and c_2 of type C_2, then $e(a) = (e(c_1), e(c_2))$

- If a is the conclusion of type $C_1 \oplus C_2$ of a $\oplus$-link, with premise c_1 of type C_1 (resp. c_2 of type C_2), then $e(a) = (e(c_1), 0)$ (resp. $e(a) = (e(c_2), 1)$).
- If a is the conclusion of type $C_1 \& C_2$ of a $\&_1$-link (resp. $\&_2$-link) with premise c_1 of type C_1 (resp. c_2 of type C_2), then $e(a) = (e(c_1), 0)$ (resp. $e(a) = (e(c_2), 1)$).
- If a is the premise of a *cut*-link with premises a and b, then $e(a) = e(b)$.

Let T be a proof-structure whose conclusions are $a_1, \ldots, a_n$ of type, respectively, $A_1, \ldots, A_n$, and let e be an experiment of T. Let $\forall i \in \{1, \ldots, n\}$ $e(a_i) = x_i$. We shall say that $(x_1, \ldots, x_n) \in |\mathcal{A}_1 \wp \ldots \wp \mathcal{A}_n|$ is the conclusion (or the result) of the experiment e of R. We shall also denote it by $x_1, \ldots, x_n$.

The following definition is the one of [Gir87]. We denote by $\wp\, \Gamma$ the formula obtained by performing the $\wp$ between the formulas of Γ.

B.2. Definition. Let R be a proof-net with conclusion Γ. $[R] := \{\gamma \in |\wp\, \Gamma| :$ there exists an experiment e of R with conclusion $\gamma\}$.
$[R]$ is said to be *the intepretation* or *the semantics* of R.

The following theorems are the obvious extensions to our framework of the similar results proven in [Gir87].

B.3. Theorem. *If R is a proof-net with conclusion Γ, then $[R] \in \wp\Gamma$ (here $\wp\, \Gamma$ is the space interpreting the formula $\wp\, \Gamma$).*

B.4. Theorem. *Let R be a proof-net and x a cut-link of R. If $R \overset{[x]}{\to} R'$, then $[R] = [R']$.*

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