Obsessional experiments for Linear Logic Proof-nets

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Abstract

We address the question of injectivity of coherent semantics of linear logic proof-nets. Starting from Girard's definition of experiment, we introduce the key-notion of "injective obsessional experiment", which allows to give a positive answer to our question for certain fragments of linear logic, and to build counter-examples to the injectivity of coherent semantics in the general case.
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Introduction

Denotational semantics associates with every proof in a given formal system a set in some space, which is usually called *the interpretation* or *the semantics* of the proof. This association can be seen as a way to define a (semantic) equivalence relation on proofs (of the same formula): roughly speaking, two proofs are equivalent when they have the same interpretation.
The cut-elimination procedure for the proofs of a given logical system can also be seen as a way to define a (syntactical) equivalence relation on proofs. If the cut-elimination procedure enjoys the confluence property, this relation can be (roughly) defined as: two proofs are equivalent when they have the same normal form.

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Now, a *very* natural question arises: do these two equivalence relations (sometimes/always) coincide?

Proofs of linear logic (LL) are represented as "proof-nets", a graph-theoretic presentation (introduced in [Gir87]) which gives a more geometric account of proofs. With several sequent calculus proofs is associated a unique proof-net: this reminds the case of both natural deduction proofs and λ-calculus terms, with respect to sequent calculus proofs (in the restricted framework of minimal logic). A net is both a canonical representative of a set of sequent calculus proofs and a computational object in itself (with a much better behaviour w.r.t. cut-elimination than sequential proofs).

In the present paper we ask the previous question for LL proof-nets. A first thing is clear (by definition of denotational semantics): two syntactically equivalent proof-nets are also semantically equivalent. If, for a given semantics s, the answer to our question is positive, we say that s is *injective*: two different proof-nets without cuts have different interpretations in the semantics s.

This kind of question has been studied for the pure and simply typed λ-calculus in several papers (see for example [Sta83]). It is somehow surprising that, while the question of "surjectivity" of the semantics (also known as "full completeness") has been extensively studied for LL, the question of injectivity was never studied before [TdF00b].

The technique we use to prove the injectivity property is "to rebuild" a cut-free proof-net from its semantic interpretation. Notice that this allows also to "semantically compute" the normal form of a proof-net. Indeed, one computes the semantics of a given proof-net (with cuts) and one rebuilds the unique cut-free proof-net with the given semantics: the thus obtained net can only be the normal form of the proof-net we started from. This approach to computation is known, among the researchers of the functional programming language community, as "normalization by evaluation" (see [BES98] and [DRR01]).

To ask the question of injectivity of some semantics of LL, is also a way to ask whether it is possible to make "more identifications" than proof-nets. In other terms, it is a way to "measure" the quality of the representation of proofs as proof-nets. In [OTdF01], we show that some fragment F of LL containing the additive connectives enjoys the injectivity property, and we use this high-quality representation of the proofs of F to define a (sliced) cut-elimination procedure enjoying outstanding properties: this gives (for the fragment F) a rather convincing solution to the problem of normalization in presence of the additives.

We restrict our analysis to the coherent multiset-based semantics of LL, and extend the positive results of the paper to the relational semantics in appendix B. Actually, the main tool of our approach (injective obsessional experiments) is fully relevant to the relational case (as shown in [TdF00b]). As the question had never been studied, there are a lot of (sometimes) simple properties to prove: we tried to be convincing without being tedious, and some (easy) proofs are left to the reader, who can also refer to [TdF00b].

The paper is composed of 4 chapters (with several sections) and 2 appendixes:

- chapter 1: The question of injectivity
- chapter 2: Injectivity and obsessionality
- chapter 3: Injective experiments for $(?\,\wp)$LL
- chapter 4: Positive and negative results
- appendix A: Proof-nets and coherent semantics
- appendix B: About injectivity for relational semantics.

In chapter 1, we address the question of injectivity in a precise way, and give a positive answer in the multiplicative case.
We start in section 1.1, by generalizing to multiplicative and exponential LL the notion of experiment of [Gir87]: an experiment associates with every edge a of a proof-net a multiset of elements of the web of the coherent space interpreting the "type" of a (the formula associated with the edge a). The "result" of an experiment of a proof-net R, is the set of labels associated by the experiment with the conclusions of R. The interpretation of a proof-net is the set of all the results of the experiments of the proof-net (definition 1.1.4). We end the section with some useful properties of experiments. In section 1.2, we state our problem in mathematical terms and introduce the canonical representatives of syntactical equivalence classes: standard proof-nets. We (easily) prove, in section 1.3, that coherent semantics is injective for the multiplicative fragment of LL, by means of the notion of "injective experiment".
In chapter 2, we introduce and study "obsessional experiments". Thanks to this new tool, we prove a sufficient condition of local injectivity: if, for a given (standard) proof-net, there exists a particular experiment, then this proof-net is the unique standard proof-net of its semantic equivalence class.

In section 2.1, we define obsessional experiments and state some of their properties. In section 2.2, we show the regularity of obsessional experiments: we prove (propositions 2.2.6 and 2.2.7) that the obsessional feature of an experiment can be read in its result, using in a crucial way the uniformity of coherent semantics (lemma 2.2.3). In section 2.3, we show the power of obsessional experiments: we prove (corollary 2.3.8) that they allow to partially reconstruct a (standard) proof-net. Finally, in section 2.4, we give the sufficient condition of local injectivity (theorem 2.4.8): if there exists, for a (standard) proof-net R, an "injective 1-experiment", then there are no other (standard) proof-nets with the same semantics as R.

In chapter 3, we define the subsystem $(?\wp)$LL of multiplicative and exponential proof-nets, and we prove that every (standard) proof-net of $(?\wp)$LL satisfies the sufficient condition of local injectivity of chapter 2.
In section 3.1, we define two operations on (standard) proof-nets: linearization and par-mutilation. We then show that by first linearizing and then par-mutilating a (standard) $(?\wp)$LL proof-net, one obtains a proof-net without boxes, without $\wp$ links and with only terminal contraction links (remark 3.1.5). Section 3.2 is devoted to prove that for such a proof-net there always exists an injective 1-experiment (proposition 3.2.4). We conclude the chapter by proving, in section 3.3, that the existence of such an experiment is preserved when one "comes back" from the linearized and par-mutilated proof-net to the original one (lemma 3.3.2 and proposition 3.3.4).

In chapter 4, we prove the positive and the negative results of the paper, summed up in the last section 4.3. Even though the existence of an injective 1-experiment of a (standard) proof-net *is not* necessary for the proof-net to be alone in its semantic equivalence class (as shown in remark 2.4.10), it certainly is a crucial property. Indeed, the existence of an injective 1-experiment for every (standard) $(?\wp)$LL proof-net allows to prove (in section 4.1) injectivity for $(?\wp)$LL, and the fact that such an experiment does not exist for every (standard) multiplicative and exponential proof-net allows to build a counterexample to the injectivity of coherent semantics in the general case (in section 4.2). Among the notable subsystems of $(?\wp)$LL (for which injectivity is proven), let us mention the "weakly polarized" fragment of LL, which contains the simply typed λ-calculus (theorem 4.1.5, corollary 4.1.6, theorem 4.1.7).

Appendix A is devoted to define the (well-)known notions of coherent space and proof-net. Contrary to the first one, the notion of proof-net is not canonical: in order to answer the question addressed in this paper, we need to refer to a precise definition (see A.2.5 and A.2.6).

Appendix B extends the positive results of the paper to relational semantics.

1 The question of injectivity

This chapter is mainly devoted to set the stage and formulate the question of injectivity in precise mathematical terms. We first introduce (section 1.1) the notion of experiment and mention some of its properties. We then turn to syntax (section 1.2), by introducing the canonical representatives of "syntactical equivalence classes": standard proof-nets. This allows to precisely state our problem (1.2.3). Finally, we solve (in section 1.3) the problem in the usual "perfect" fragment of LL: the multiplicative fragment. The (very easy) proof has the virtue to indicate the way to follow in order to attack the problem in the more difficult (and interesting) multiplicative and exponential fragment.
We use the (well-known) notions of proof-net and of coherence space. We refer to appendix A, where all the main definitions are given, so as several conventions and notations used in the paper.

1.1 Experiments: the semantics of Linear Logic Proof-nets

We come now to the crucial notion of experiment (introduced in [Gir87]). Experiments have almost never been used after Girard's first paper on LL. The only other works dealing with this notion are (at least as far as we know) [DdW94], [Ret97], and more recently [TdF00b] and [Bar01]. We give here a precise definition of experiment for the multiplicative and exponential fragment of LL, and recall the main results of [Gir87].

The following definition is the extension to multiplicative and exponential LL of the definition of [Gir87]: the main difference is due to the presence of exponential boxes. An experiment is not a simple labeling of the edges of a proof-net anymore. While in the absence of exponential boxes an experiment e associates with every edge a of type A of a proof-net a unique element of $|\mathcal{A}|$, this is not the case in general: it might associate with a an element of $|\mathcal{A}|$, several elements of $|\mathcal{A}|$, or the empty set. We will then say that e associates with a a *multiset* of elements of $|\mathcal{A}|$.

1.1.1. DEFINITION. **(experiment)** We define the notion of experiment of a proof-structure S by induction on the depth p of S.
Let e be an application which associates with every edge a of type A of S a multiset $e(a)$ of elements of $|\mathcal{A}|$, in such a way that when a has depth

0 the multiset $e(a)$ contains exactly one element. The application e is an **experiment** of S when the following conditions hold.
If $p = 0$, then:

- If $a = a_1$ is the conclusion of an axiom link with conclusions the edges a_1 and a_2 of type A and $A^\perp$ respectively, then $e(a_1) = e(a_2)$.
- If a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.
- If a is the conclusion of a $\wp$ (resp. $\otimes$) link with left premise a_1 and right premise a_2, then $e(a) = \{(x_1, x_2)\}$, where $e(a_1) = \{x_1\}$ and $e(a_2) = \{x_2\}$.
- If a is the conclusion of a dereliction link with premise a_1, then $e(a) = \{\{x_1\}\}$, where $e(a_1) = \{x_1\}$.
- If a is the conclusion of a weakening link, then $e(a) = \{\emptyset\}$.
- If a is the conclusion of a contraction link of arity k ($k \geq 2$), with premises $a_1, \ldots, a_k$, then $e(a) = \{x_1 \cup \ldots \cup x_k\}$, where $e(a_i) = \{x_i\}$ (for every $i \in \{1, \ldots, k\}$).

If $p > 0$, then e has to satisfy the same conditions as in case $p = 0$. Moreover, for every box $B^!_n$ with depth 0 in S and whose front door n has conclusion c of type $!C$ and whose auxiliary doors have conclusions $a_1, \ldots, a_m$ ($m \geq 0$) of type, respectively, $?A_1, \ldots, ?A_m$, let $S_n = S_{B^!_n}$ be the biggest subproof-structure of S contained in $B^!_n$. Let c' be the premise of the !-link n and (for every $i \in \{1, \ldots, m\}$) let a'_i be the premise of the pax link of $B^!_n$ having a_i as conclusion. Clearly, c' and $a'_1, \ldots, a'_m$ are the conclusions of the proof-structure S_n.
In order for the application e to be an experiment of S, for every such box $B^!_n$ there has to exist a *unique* multiset[1] $\{e_1, \ldots, e_{k_n}\}$ ($k_n \geq 0$) of experiments of S_n satisfying the following conditions:

- for every edge a of S_n, $e(a) = e_1(a) \cup \ldots \cup e_{k_n}(a)$,
- $e(c) = \{\{x_1, \ldots, x_{k_n}\}\}$, where $e_j(c') = \{x_j\}$ ($\forall j \in \{1, \ldots, k_n\}$),
- $\forall i \in \{1, \ldots, m\}$ one has $e(a_i) = \{x^i_1 \cup \ldots \cup x^i_{k_n}\}$, where $\forall j \in \{1, \ldots, k_n\}$ we have $e_j(a'_i) = \{x^i_j\}$.

[1]This simply means that for a given experiment e the multiset is unique, but -at least for the moment- it might be the case that there exists *another* experiment e', with the same result as e (see the end of the definition) and with a different multiset of experiments associated with the box $B^!_n$.

If the conclusions of S are the edges $a_1, \dots, a_l$ of type, respectively, $A_1, \dots, A_l$, and e is an experiment of S s.t. $\forall i \in \{1, \dots, l\}$ $e(a_i) = \{x_i\}$, then we shall say that $(x_1, \dots, x_l) \in |\mathcal{A}_1 \wp \dots \wp \mathcal{A}_l|$ is the *conclusion* or the *result* of the experiment e of S. We shall also denote it by $x_1, \dots, x_l$.

1.1.2. REMARK. Let a be an edge of the proof-structure S and e an experiment of S. We will often refer to the elements of $e(a)$ as to "the labels" associated by the experiment e with the edge a.
It is *crucial* to notice that the previous definition implies that the following conditions are fulfilled (inductively, with respect to the depth):

- the label $x_1 \cup \dots \cup x_k$ of the conclusion a of type $?C$ of a contraction link with depth 0 satisfies $x_1 \cup \dots \cup x_k \in |?\mathcal{C}|$
- the label $\{x_1, \dots, x_{k_n}\}$ of the conclusion a of type $!C$ of an of course link with depth 0 satisfies $\{x_1, \dots, x_{k_n}\} \in |!\mathcal{C}|$
- the label $x_1^i \cup \dots \cup x_{k_n}^i$ of the conclusion a_i of type $?A_i$ of a pax link with depth 0 satisfies $x_1^i \cup \dots \cup x_{k_n}^i \in |?\mathcal{A}_i|$.

If R is a cut-free proof-net of multiplicative LL, a "correct" assignment of labels to the conclusions of the axiom links of R always induces an experiment of R. The fact that an experiment of a cut-free proof-net R' of multiplicative and exponential LL has to fulfill the previous conditions, implies that a "correct" assignment of labels to the conclusions of the axiom links of R' *does not* necessarily induce an experiment of R'.

1.1.3. REMARK. Let $\Gamma = A_1, \dots, A_l$ be the sequent conclusion of the two proof-nets R and R'. Let e (resp. e') be an experiment of R (resp. R') with result γ (resp. γ'). By definition, there exists a permutation σ (resp. σ') of $\{1, \dots, l\}$ such that $\gamma = (x_1, \dots, x_l) \in |\mathcal{A}_{\sigma(1)} \wp \dots \wp \mathcal{A}_{\sigma(l)}|$ (resp. $\gamma' = (x'_1, \dots, x'_l) \in |\mathcal{A}_{\sigma'(1)} \wp \dots \wp \mathcal{A}_{\sigma'(l)}|$).
We shall write $\gamma = \gamma'$ when for every $i \in \{1, \dots, l\}$ one has $x_i = x'_i$ and $A_{\sigma(i)} = A_{\sigma'(i)}$. This means that the equality $\gamma = \gamma'$ induces a bijection between the edges conclusions of R and the edges conclusions of R': the one which associates with the conclusion $a_{\sigma(i)}$ of type $A_{\sigma(i)}$ of R the conclusion $a_{\sigma'(i)}$ of type $A_{\sigma'(i)}$ of R'.
We shall say that a given conclusion a of R "corresponds" to a given conclusion a' of R' (or that a' is "the corresponding edge" of R'), when a' is the image of a through the bijection.

The following definition is the one of [Gir87], which is here extended to the case of multiset-based coherent semantics. The reader should notice that the semantics of a proof-net R depends on the choice of the coherent spaces associated with the atomic subformulas of R's conclusions (see definition A.1.1).

1.1.4. DEFINITION. Let R be a proof-net with conclusion Γ.
$[\![R]\!] := \{\gamma \in |\wp\,\Gamma| :$ there exists an experiment e of R with conclusion $\gamma\}$.
$[\![R]\!]$ is said to be *the interpretation* or *the semantics* of R.

1.1.5. THEOREM. *If R is a proof-net with conclusion Γ, then $[\![R]\!] \in \wp\Gamma$ (here $\wp\,\Gamma$ is the space interpreting the formula $\wp\,\Gamma$).*

Proof: For the coherent (set-based) semantics, it is proven in [Gir87]. In the multiset case, one has to extend the previous result. This is done in [Bar01] for multiplicative and exponential LL. □

The interpretation defined in 1.1.4 yields a denotational semantics of proof-nets for multiplicative and exponential LL, as stated by the following theorem.

1.1.6. THEOREM. *Let R be a proof-net and let R' be a proof-net obtained from R by applying one step of cut-elimination. Then $[\![R]\!] = [\![R']\!]$.*

Proof: The proof is given in [Gir87] for the coherent (set-based) semantics, and can be straightforwardly extended to the multiset case. □

Notation: The set of proof-nets (defined in appendix A) only makes use of the formulas of multiplicative and exponential LL, and will be denoted in this paper by $MELL$.

We now state some more properties of the notion of experiment, which will be used in the sequel.

1.1.7. DEFINITION. **(restriction of an experiment to a subproof-net)** Let R be a proof-net, let e be an experiment of R with conclusion γ and let R_1 be a subproof-net of R. We are going to define the multiset $e|_{R_1}$, whose elements are experiments of R_1, by induction on the depth p of the conclusions of R_1 in R.

If $p=0$, then $e|_{R_1} = \{e_1\}$, where e_1 is the experiment of R_1 defined by: for every edge a of R_1, $e_1(a) = e(a)$.
Let $p+1$ be the depth of the conclusions of R_1 in R, let B be the box of R with depth 0 containing R_1 and let R_B be the biggest subproof-net of R contained in B. Let n be the cardinality of the unique label associated by e with the conclusion of the pal door of B and let $e_1, \ldots, e_n$ be the n ($n \geq 0$) experiments of R_B from which the unique experiment of the multiset $e|_B$ is built (following definition 1.1.1)[2]. We define: $e|_{R_1} = e_1|_{R_1} \cup \ldots \cup e_n|_{R_1}$ (if $n=0$, we have $e|_{R_1} = \emptyset$).
If $e|_{R_1} = \{e_1, \ldots, e_l\}$, we'll denote by $\gamma|_{R_1}$ the multiset $\{\gamma_1, \ldots, \gamma_l\}$ where γ_i is the conclusion of the experiment e_i of R_1, $\forall i \in \{1, \ldots, l\}$.

We now state some useful lemmas, the proofs of which are left to the reader.

1.1.8. LEMMA. *Let e be an experiment of R, R_1 a subproof-net of R and $e|_{R_1} = \{e_1, \ldots, e_n\}$. For every edge a of R_1, $e(a) = e_1(a) \cup \ldots \cup e_n(a)$.*

1.1.9. LEMMA. *Let e be an experiment of the (non empty) proof-net R, and let a be an edge of R. We have that $e(a) = \emptyset$ iff there exists a box B of R containing a s.t. if we call c the conclusion of the pal door of B, then we have that $e(c) = \{n[\emptyset]\}$ for some integer n different from zero.*

1.1.10. LEMMA. *Let a be an edge of the proof-net R with depth p and let e be an experiment of R. Let $c_1, \ldots, c_p$ of type, respectively, $!C_1, \ldots, !C_p$ be the conclusions of the p pal doors of the boxes of R containing a.
If $p \geq 1$ and $\forall i \in \{1, \ldots, p\}$ the cardinality of all the elements of $e(c_i)$ is equal to n_i (resp. if $p=0$), then $e(a)$ is a multiset of cardinality $n_1 \cdot \ldots \cdot n_p$ (resp. of cardinality 1).*

1.1.11. DEFINITION. Let a be an edge of type A of the proof-net R and let e be an experiment of R. Let $x \in e(a)$ (i.e. let $x \in |\mathcal{A}|$ be *one* of the labels associated by e with the edge a). For every **occurrence** of subformula C of A, we define the multiset "multiset projection of x on C", denoted by $|x|_C$, by induction on the (logical) complexity of the formula $A \backslash C$:
- if $C = A$, then $|x|_C = \{x\}$

[2] We use here the unicity, for a given experiment e, of the multiset $\{e_1, \ldots, e_{k_n}\}$ of definition 1.1.1.

- if $E = C \otimes D$ or $E = C \wp D$ (resp. $E = D \otimes C$ or $E = D \wp C$) is an occurrence of subformula of A, then $|x|_C = \{y \in |\mathcal{C}| : \exists z \in |\mathcal{D}|$ s.t. $(y, z) \in |x|_E\}$ (resp. $|x|_C = \{y \in |\mathcal{C}| : \exists z \in |\mathcal{D}|$ s.t. $(z, y) \in |x|_E\}$)
- if $D = ?C$ or $D = !C$ where D is an occurrence of subformula of A, then $|x|_C = \{y \in |\mathcal{C}| : \exists z \in |\mathcal{D}|$ s.t. $y \in z \in |x|_D\}$.

1.1.12. Remark. (i) Let e (resp. e') be an experiment of the proof-net R, let a be an edge of type A of R, let C be an occurrence of subformula of D and D an occurrence of subformula of A. Let x (resp. x') be a label associated by e (resp. e') with the edge a. If $|x|_D = |x'|_D$, then $|x|_C = |x'|_C$.
(ii) Let a be an edge of type A of the proof-net R and e be an experiment of R. Let $x \in |\mathcal{A}|$ be *one of the* labels associated with a by e. One can extend the previous definition to the case where C is not an occurrence of subformula of A, by defining in that case $|x|_C = \emptyset$.
(iii) With the notations of (ii), if $|x|_C \neq \emptyset$, then C is an occurrence of subformula of A. Observe that the converse is wrong: one might have $|x|_C = \emptyset$ with C occurrence of subformula of A.

1.1.13. Remark. In [Gir87], there is a very nice proof of the following property for the multiplicative fragment of LL:
"an experiment of a given proof-net is uniquely determined by its result", otherwise stated:
let e (resp. e') be an experiment with result γ (resp. γ') of the proof-net R. If $\gamma = \gamma'$, then $e = e'$.
In its proof, Girard uses the longtrip correctness criterion. E. Duquesne and J. Van de Wiele extended this result to pure LL proof-nets (cf. [DdW94]), and in [Bar01] one can find a similar proof in the $MELL$ case for the set-based and multiset-based coherent semantics.
This last result will be used in the sequel but not in an essential way: it just simplifies our statements. All the results presented in the paper can be proven without using it.

Notice that this property entails the unicity of the multiset $\{e_1, \ldots, e_{k_n}\}$ of definition 1.1.1 once the *result* of e is known: in case we deal with a proof-net, the footnote of definition 1.1.1 cannot hold (this is not true for general proof-structures).

1.2 Standard proof-nets

To state our problem in a correct way, we need some preliminary remarks.

Let c be a binary connective of LL, and let $c^\perp$ be the dual connective. The semantics that we consider will identify the axiom with conclusions $AcB, A^\perp c^\perp B^\perp$ and the canonical proof of $AcB, A^\perp c^\perp B^\perp$ (obtained from the two axiom links with conclusions $A, A^\perp$ and $B, B^\perp$). We will often refer to this last proof(-net) as to the **η-expansion** of the axiom (terminology which obviously comes from the λ-calculus). A similar remark holds for unary connectives (the exponentials "!" and "?"). If R is obtained by applying to the proof-net R' some η-expansions, there is clearly no hope for our semantics to distinguish R from R'.
One can also easily see that the considered semantics cannot "see" whether the conclusion of a weakening or a contraction link is the premise of a pax link. And a contraction or a weakening link whose conclusion is the premise of a contraction link is also "semantically invisible".

In order to state the problem, we then have to define more precisely the two equivalence relations we want to compare.

1.2.1. DEFINITION. Let R be a proof-net of $MELL$. We shall say that R is **standard** when:
- R is cut-free
- every conclusion of an axiom link of R is of atomic type
- if a is the conclusion of a $?w$ or $?co$ link of R, then a is not a premise of a pax link nor of a $?co$ link.

It is fairly obvious that by performing the necessary η-expansions, by (possibly) erasing some $?w$ and $?co$ links, and by "pushing" (when this is possible) the links $?w$ and $?co$ out of the boxes, one can associate with every cut-free proof-net of $MELL$ a unique standard proof-net.
Notice, by the way, that (except for the η-expansions) a standard proof-net is nothing but a proof-net of the "nouvelle syntaxe" defined by V. Danos and L. Regnier (for example in [Reg92]).

We now have to be precise on what we mean by "R and R' are semantically equivalent" (in a given semantics), where R and R' are two proof-nets with the same conclusions. Indeed, we cannot simply say that this means $[\![R]\!] = [\![R']\!]$ (referring to definition 1.1.4), because this equality depends on the interpretation of atomic formulas. We will say that two proof-nets with the same conclusions are semantically equivalent when they have the same semantics *for every* possible interpretation of the atoms of the types of their conclusions.

We use in the following definition the confluence property of ($MELL$) proof-nets, proven in [Dan90].

1.2.2. DEFINITION. Let R and R' be two proof-nets with the same conclusions, let s be a semantics of $MELL$, and let's denote by $[\![T]\!]_s$ the element of $\wp\Gamma$ associated by the semantics s with the proof-net T with conclusion Γ. Let R_0 (resp. R'_0) be the unique standard proof-net associated with the unique normal form of R (resp. R').
We shall say that R and R' are **semantically equivalent** when $[\![R]\!]_s = [\![R']\!]_s$ (for every interpretation of the atoms of the types of the conclusions of R). We shall simply write $[\![R]\!]_s = [\![R']\!]_s$ (or $[\![R]\!] = [\![R']\!]$ when there will be no ambiguity).
We shall say that R and R' are **syntactically equivalent** or $\beta\eta$**-equivalent** when $R_0 = R'_0$. We will then write $R \simeq_{\beta\eta} R'$.

We can finally state our

1.2.3 Problem

Let R and R' be two proof-nets with the same conclusions, let s be the coherent multiset-based semantics of $MELL$. If $R \simeq_{\beta\eta} R'$, then $[\![R]\!]_s = [\![R']\!]_s$.
Do we have "if $[\![R]\!]_s = [\![R']\!]_s$, then $R \simeq_{\beta\eta} R'$"? Otherwise said: "is the semantics s injective?"

Convention: From now on, all the proof-nets considered in this paper will be standard $MELL$ proof-nets.

We can then re-state the previous question in the following way:
"Let R and R' be two proof-nets with the same conclusions. If R and R' are semantically equivalent, do we have $R = R'$?"

We now introduce some notions which will be used in the sequel (and mainly in the following chapter).

1.2.4. DEFINITION. With every proof-net R are naturally associated the following graphs:
- the open proof-structure, denoted by $OPS(R)$, obtained from R by erasing all the axiom links of R
- the linear proof-structure (resp. the open linear proof-structure), denoted by $LPS(R)$ (resp. denoted by $OLPS(R)$), obtained from R (resp. from

$OPS(R)$) by erasing the connexions between the different doors of a given box (the "rectangular frame" of definition A.2.1).

1.2.5. Remark. Let us be more precise on what we mean by "erasing the connexions between the different doors of a given box". In $LPS(R)$ and $OLPS(R)$, we keep the pax and of course links of R. This means (in particular) that the notion of depth is still meaningful in $LPS(R)$ and $OLPS(R)$: an edge/a link α has depth p when the path with starting edge/link α and terminal edge a conclusion of R crosses exactly p of course or pax links (different from α).
The only information which is lost in $LPS(R)$ (resp. $OLPS(R)$) is the following: we do not know anymore whether two given pax links (or a pax and an of course link) with the same depth are two doors of the same box.

1.2.6. Remark. (i) Let a be the conclusion of m, which is a pax or a dereliction link of the proof-net R. Because the conclusion of every contraction and weakening link is not a premise of a pax link (and because every conclusion of an axiom link has an atomic type), there exists a unique dereliction link n with conclusion the edge a' of R and s.t. the path with starting edge a' and terminal edge a conclusion of R crosses a after a certain number of pax doors. We will say that n is **the dereliction link above** a or **above** m. (Of course one might have $n = m$).
(ii) Let a (resp. $a_1, \ldots, a_k$) be the conclusion (resp. the premises) of a contraction link m with arity k. Because of the position of the structural rules in a standard proof-net, $a_1, \ldots, a_k$ are conclusions of pax or dereliction links. Thanks to (i), we can then associate with the link m k dereliction links $n_1, \ldots, n_k$ (where n_i is the dereliction link above a_i) which will be called **the** k **dereliction links above** m.

The following definition of "graph of an edge in a proof-net" can be given for every cut-free proof-net of $MELL$ (not necessarily standard).
Let R be a proof-net and a an edge of R. The graph of a in R is, essentially, the subgraph of R whose links are the elements of the following set of links: $\{n : n$ link of R s.t. there exists a path ϕ_a^n of R with starting link an axiom or a weakening and terminal edge a, which contains $n\}$.
However, this definition is not precise enough (specially w.r.t. the axiom links). This is why we give the inductive "construction" of this graph.

1.2.7. Definition. Let R be a proof-net and let a be an edge of R. Let's fix a "graphical representation" $R^\star$ of R: we mean that we fix the order of

the contraction links of R (see remark A.2.2). Let k_a be the cardinality of the set $\{n : n \text{ link of } R^\star \text{ s.t. there exists a path } \phi_a^n \text{ of } R^\star \text{ with starting link an axiom or a weakening and terminal edge } a, \text{ which contains } n\}$.
The tree of a in $R^\star$, denoted by $G_a^{R^\star}$ is an oriented tree, which we define by induction on k_a:
- if $k_a = 1$, then a is a conclusion of an axiom link or the conclusion of a weakening link (these are the only links with no premises) of $R^\star$. If a is a conclusion of an axiom link, then $G_a^{R^\star}$ is the edge a, and we will write $G_a^{R^\star} = \{a\}$. If a is the conclusion of a weakening link, then $G_a^{R^\star}$ is the weakening link and its conclusion, and we will write $G_a^{R^\star} = \{n\} \cup \{a\}$.
Otherwise, let n be the link of $R^\star$ with conclusion a:
- if n is a logical link with two premises and a conclusion (i.e. if $n = \otimes, \wp$), then let a_1 be the left premise and a_2 the right premise of n in $R^\star$. One has $k_{a_1} < k_a$ and $k_{a_2} < k_a$. We can then define $G_a^{R^\star}$ as the graph obtained by connecting the two graphs $G_{a_1}^{R^\star}$ and $G_{a_2}^{R^\star}$ by means of the link n with left premise a_1 and right premise a_2, and with conclusion a. We will write $G_a^{R^\star} = G_{a_1}^{R^\star} \cup G_{a_2}^{R^\star} \cup \{n\} \cup \{a\}$.
- if n is a link with one premise and one conclusion (i.e. if $n = !, ?de, pax$), then let a' be the premise of n. One has $k_{a'} < k_a$. We can then define $G_a^{R^\star}$ as the graph obtained from $G_{a'}^{R^\star}$ by adding the link n with premise a' and with conclusion a. We will write $G_a^{R^\star} = G_{a'}^{R^\star} \cup \{n\} \cup \{a\}$.
- if n is a $?co$ link with $h \geq 2$ premises, then let (from left to right in the given representation $R^\star$ of R) $a_1, \dots, a_h$ be the premises of n in $R^\star$. One has $k_{a_i} < k_a$ (for $i \in \{1, \dots, h\}$). We can then define $G_a^{R^\star}$ as the graph obtained by connecting the h graphs $G_{a_1}^{R^\star}, \dots, G_{a_h}^{R^\star}$ by means of the link n with premises (from left to right) $a_1, \dots, a_h$, and with conclusion a.

It is rather obvious that the relation $\sim$ defined by $G_a^{R^\star} \sim G_a^{R^\bullet}$ iff $R^\star$ and $R^\bullet$ are two graphical representation of the proof-net R, is an equivalence relation.
The **graph of** a **in** R, denoted by G_a^R or by G_a if there is no ambiguity, is the equivalence class of $G_a^{R^\star}$ w.r.t. $\sim$.

1.2.8. Remark. **(and Definition)** (i) The graph G_a^R of the previous definition is an equivalence class of trees which can only differ in the order of the premises of the contraction links.

(ii) If a is an edge of the proof-net R and if e is an experiment of R, we will denote by $e|_{G_a^R}$ the restriction of e to the edges of G_a^R. **Warning!** If a' is an edge of the proof-net R' and if e' is an experiment of R', we will write $e|_{G_a^R} = e'|_{G_{a'}^{R'}}$ when *there exists* a tree $\mathcal{T}$ (resp. $\mathcal{T}'$) of the equivalence class

G_a^R (resp. $G_{a'}^{R'}$) such that ($\mathcal{T} = \mathcal{T}'$ and) $e|_{\mathcal{T}} = e'|_{\mathcal{T}'}$.

1.2.9. REMARK. (i) Let R be a proof-net, a an edge of type A of R and c an edge of type C of G_a^R. Then C is an occurrence of subformula of A.
(ii) Let R be a proof-net, a and c two edges of R. If $a \in G_c^R$, then G_a^R is a subgraph of G_c^R.
(iii) Let R be a proof-net, $a_1, \ldots, a_n$ be the conclusions of R. The graph $OLPS(R)$ is nothing but the juxtaposition of the graphs $G_{a_1}^R, \ldots, G_{a_n}^R$.
(iv) Let a and a' be two edges of the proof-net R. $G_a \cap G_{a'} = \emptyset$ if and only if $a \notin G_{a'}$ and $a' \notin G_a$.

1.3 The case of MLL

The first idea is to try to prove that the syntactical and the semantic relations mentioned in the introduction (and precisely defined in the previous section) do coincide. We are going to see that it is indeed the case for the multiplicative fragment of LL. This is easy to show, and the reason why we call theorem this result (1.3.4) is that its proof suggests the approach (which will be developed in the sequel) to the much more complex $MELL$ case.
We denote by MLL the subset of $MELL$ proof-nets containing only axiom, cut, $\otimes$ and $\wp$ links.
Observe that every experiment of an MLL proof-net R associates with every edge a unique label: in the multiplicative case, an experiment of R is a labeling of the edges of R.
It is also immediate that, if R is an MLL proof-net, then $OPS(R) = OLPS(R)$.

The following lemma is an obvious remark: if one knows the type of all the conclusions of the MLL proof-net R, then one knows the proof-net "up to the axiom links".

1.3.1. LEMMA. *If R and R' are two MLL proof-nets with the same conclusions, then $OPS(R) = OPS(R')$.*

The previous lemma stresses the fact that the unique information that we have to extract from the interpretation of an MLL proof-net R to be able to say that the semantics is injective, is the set of the pairs of edges of atomic type which are conclusions of the same axiom link of R.

1.3.2. LEMMA. *Let R (resp. R') be an MLL proof-net with conclusion Γ, and let e (resp. e') be an experiment of R (resp. R') with result γ (resp. γ'). If $\gamma = \gamma'$, then $(OPS(R) = OPS(R')$ and) $e|_{OPS(R)} = e'|_{OPS(R')}$.*

Here is the only point which requires a little bit of attention. We introduce the notion of "injective experiment", which will be extended to $MELL$ in the sequel.

1.3.3. DEFINITION. Let R be an MLL proof-net and let e be an experiment of R. We will say that e is an **injective experiment** when $\forall a, a'$ edges of the same atomic type of R such that $a \neq a'$, one has $e(a) \neq e(a')$.

We can now prove the injectivity "theorem" for MLL.

1.3.4. THEOREM. **(Injectivity for MLL).** *Let R and R' be two proof-nets with the same conclusions. If $[\![R]\!] = [\![R']\!]$, then $R = R'$.*

Proof: Let e be an injective experiment of R with result γ.
Notice that such an experiment always exits, because there is no condition on the labels of the edges of a standard MLL proof-net: it contains no of course, *cut*, ?*co*, nor pax link (remember remark 1.1.2). We only have to choose an interpretation of R's atomic formulas containing a sufficient number of elements.
Because $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ. From lemma 1.3.2, $(OPS(R) = OPS(R')$ and) $e|_{OPS(R)} = e'|_{OPS(R')}$. Because e is injective, for every atomic edge a_1 of type X of $OPS(R) = OPS(R')$, there exists a unique edge a_2 of $OPS(R) = OPS(R')$ of type $X^\perp$ such that $e(a_1) = e'(a_1) = e(a_2) = e'(a_2)$. Then $R = R'$. □

1.3.5. REMARK. The previous proof points out the following: every injective experiment of an MLL proof-net R contains *all the informations* contained in R.
One might then wonder: "what are the other experiments for?"
We should not forget that denotational semantics does not yield a *static* representation of a given proof π: it mainly describes the possible interactions of π with the other proofs. Notice, by the way, that the interpretation of a proof-net by means of the results of its injective experiments is not correct: it does not yield an invariant of cut-elimination (just imagine two axiom links connected by a cut link).

2 Injectivity and obsessionality

The first idea is to apply the method used to prove theorem 1.3.4 to $MELL$, i.e. to prove the analogue of lemma 1.3.2 and to extend the notion of injective experiment to $MELL$ proof-nets.
But we immediately stumble on several problems. The first one is that the type A of a conclusion a of an $MELL$ proof-net R is not enough to know G_a^R (due to the presence of pax, $?w$ and $?co$ links). Worse, suppose you know all the labels that the experiments of R associate with an edge a which is the conclusion of a link n, suppose this is enough to "guess" which kind of link n is, and suppose that n is a $?co$ link with arity k. How can you "guess", for a given label of a, the way to split it into the k labels of the premises of a?
We discovered that the interpretation of a proof-net always contains the results of a very specific kind of experiment, *obsessional experiments*, which turned out to be "at the heart of semantics".

We introduce obsessional experiments in section 2.1. These experiments have a particular nature: they are both regular and powerful. Their regularity is explored in sections 2.1 and 2.2: we prove that coherent semantics "can read the obsessional feature of an experiment in its result" (propositions 2.2.6 and 2.2.7). In section 2.3 we show the power of our new tool by proving the analogue of lemma 1.3.2 (corollary 2.3.8): as a consequence, we show that an obsessional experiment of a proof-net R allows to determine R "up to the axiom links and the boxes" (theorem 2.3.12). Finally, we give in section 2.4 a sufficient condition of "local injectivity" (theorem 2.4.8). This last result is the starting point of next chapters: henceforth to answer our original question (problem 1.2.3), we try to fulfill the sufficient condition of theorem 2.4.8.

2.1 Obsessional experiments

We introduce now the main tool of our analysis: obsessional experiments. These experiments are very regular, as witnessed by the properties stated in the present section.

2.1.1. DEFINITION. **(n-obsessional experiment)** Let R be a proof-net, let e be an experiment of R, and let $n \geq 1$ be an integer.
We will say that e is an n-obsessional experiment of R iff
(1) for every edge a of R of atomic type X, for every $x, y \in |\mathcal{X}|$, if $x \in e(a)$ and $y \in e(a)$, then $x = y$

(2) for every edge c of type $!C$ of R, $e(c) \neq \emptyset$ and for every $y \in e(c)$, $card(y) = n$.
When $n = 1$, we will also say that e is a 1-experiment.

2.1.2. REMARK. (i) Let e be an n-obsessional experiment of the proof-net R and let a be an edge of R of atomic type with depth p_a in R. The experiment e satisfies the hypothesis of lemma 1.1.10 where $\forall i \in \{1, \ldots, p_a\}$ $n_i = n$. Then $e(a)$ is a multiset of cardinality n^{p_a}, containing n^{p_a} occurrences of the same element of $|\mathcal{X}|$.
(ii) For 1-experiments, conditions (1) and (2) of the definition above are equivalent to condition (2).
Notice also that every experiment of a proof-net without boxes is always a n-obsessional experiment (for every $n \geq 1$): for these proof-nets, the notion of n-obsessional experiment does not make much sense.
(iii) To understand what an n-obsessional experiment is, the reader might visualize it in the following way: let's fix a proof-net R and start from the subproof-nets of R which contain no box. Consider, for each one of these subproof-nets, an experiment. Until we do not meet a door of some box, the top-down propagation of the labels of the edges of R is completely deterministic. Either it fails or it succeeds (remember remark 1.1.2), but we have no choice. When we meet a door of a box, we stop and wait for friends (i.e. we wait until every premise of every door of the box has a label). When everybody has arrived, we have an experiment e of the content R_B of every box B of R which contains no other box. We then take n copies of that same experiment e: we get an experiment of B (the condition on the labels of the conclusions of the pax and pal doors are necessarily satisfied). We start again the game, following the same rules. This construction (when it succeeds) yields an n-obsessional experiment of R; and the results of the present section entail that every n-obsessional experiment of R can be built in such a way.
The reader certainly agrees, by now, with the terminology chosen for this kind of experiment.
Notice also, that the notion of obsessional experiment has a true meaning only in a multiset framework.

Lemmas 2.1.3, 2.1.4 and proposition 2.1.6 are rather intuitive, and their proofs are left to the reader.

2.1.3. LEMMA. *Let e be an n-obsessional experiment of the proof-net R and let R_1 be a subproof-net of R. If $e|_{R_1} = \{e_1, \ldots, e_l\}$, then $\forall i \in \{1, \ldots, l\}$*

e_i is an n-obsessional experiment of R_1.

2.1.4. LEMMA. *Let e and e' be two n-obsessional experiments of the proof-net R. If for every edge a of atomic type of R $e(a) = e'(a')$, then $e = e'$.*

2.1.5. REMARK. The previous lemma is wrong if e and e' are not (both) obsessional: consider the proof-net obtained from an axiom link with conclusions a_1 of type X and a_2 of type $X^{\perp}$, by adding what is needed (i.e. two ! links, a $?de$ link and two pax links) to obtain the proof-net with conclusions $!!X, ?X^{\perp}$. The edges a_1 and a_2 are contained in two boxes: let's call B_1 the smaller one and B_2 the bigger one. Let e be the experiment of the subproof-net of R containing only the axiom link (and its conclusions) s.t. $e(a_1) = e(a_2) = \{x\}$, where $x \in |\mathcal{X}|$. Let e_1 (resp. e_2) be the experiment of R obtained from e by taking 2 (resp. 4) copies of e to exit from B_1 and 8 (resp. 4) copies of the experiment thus obtained to exit from B_2. As $4 \times 4 = 2 \times 8$, we will indeed have that $e_1(a_1) = e_2(a_1)$ and $e_1(a_2) = e_2(a_2)$, despite the fact that $e_1 \neq e_2$.

2.1.6. PROPOSITION. *Let e be an n-obsessional experiment of the proof-net R' and let R be a subproof-net of R'. If the depth of the conclusions of R in R' is p, then $e|_R = \{e_1, \ldots, e_{n^p}\}$, where $e_1 = \ldots = e_{n^p}$ is an n-obsessional experiment of R.*
In particular, for every edge a of type A of R' with depth p, $e(a)$ is a multiset of cardinality n^p containing n^p occurrences of the same element of $|\mathcal{A}|$.

We will constantly use, in the sequel, the following remark (and specially (ii)).

2.1.7. REMARK. (i) Let e be an n-obsessional experiment of the proof-net R, c an edge of type $!C$ of R and c' the premise of the pal door whose conclusion is c. Let y be the unique element of $e(c)$. By definition of experiment, there exist $z_1, \ldots, z_n \in e(c')$ s.t. $y = \{z_1, \ldots, z_n\}$. We know from the previous proposition that $e(c')$ contains a unique element $z \in |\mathcal{C}|$: then $z_1 = \ldots = z_n = z$.
(ii) Let e (resp. e') be an n-obsessional experiment of the proof-net R (resp. of the proof-net R') and let a (resp. a') be an edge of depth p in R (resp. of depth p' in R'). If the unique element of $e(a)$ is equal to the unique element of $e'(a')$ and if $p = p'$, then $e(a) = e'(a')$.

2.1.8. REMARK. Let R be a proof-net without $?co$ links. If one associates ("correctly") with every edge of atomic type α of R an element x_α of the web of the coherent space $\mathcal{A}$, then there exists a (unique) n-obsessional experiment e_n of R s.t. the unique element of $e_n(\alpha)$ is x_α (for every edge α of atomic type). Indeed, the fact that e_n is obsessional guarantees that the compatibility conditions of remark 1.1.2 (required by definition 1.1.1) when passing through the pax and pal doors are always satisfied. Notice that this is not the case for the $?co$ links, and this will be discussed in all details in the sequel of the paper.

2.2 Obsessional results

In the present section, we answer the following question: is it possible, from the result of an experiment, to deduce whether or not it is an obsessional experiment?
We first prove some general lemmas, and then show that the answer to our question is positive (proposition 2.2.6), thanks to the uniformity property of coherent semantics (lemma 2.2.3).

The following lemma is a consequence of the definitions. Its proof is rather long and not very interesting: we therefore omit it, and leave it as an exercise to the reader.

2.2.1. LEMMA. *Let R be a proof-net, let c be an edge of R of type C and let e be an experiment of R. For every x, the two following statements are equivalent:*
(1) There exists an edge d of type A of G_c^R, with $A \neq ?F$ (for every formula F), such that $x \in e(d)$
(2) There exists $y \in e(c)$ such that $x \in |y|_A$, with $A \neq ?F$ (for every formula F).

2.2.2. PROPOSITION. *Let R be a proof-net, let c be a conclusion of R of type C, and let e be an experiment of R. Let A be an occurrence of subformula of C s.t. $A \neq ?F$ (for every formula F), and let $d_1, \ldots, d_k$ be the k edges ($k \geq 0$) of type A s.t. $\forall i \in \{1, \ldots, k\}$ $d_i \in G_c^R$.*
If $e(c) = \{y\}$, then $|y|_A = e(d_1) \cup \ldots \cup e(d_k)$.

Proof: Straightforward consequence of the previous lemma. □

Up to now, we did never use the fact that the labels of the edges of type $!A$ (resp. $?A$) are *cliques* of the coherent space $\mathcal{A}$ (resp. $\mathcal{A}^\perp$). (By the way, this means that everything what we have written up to now, is true also for the relational semantics of appendix B.) This fact will now be used to prove proposition 2.2.6, which states that coherent semantics "can read the obsessional feature of an experiment in its result". In the proof, we will make an essential use of the following lemma (which can be seen as a way to express the "uniformity" of coherent semantics).

2.2.3. LEMMA. **(Uniformity property).** Suppose that the proof-net R is a box B having as conclusions the edge d of type $!D$ and the edges $d_1, \ldots, d_k$ of types, respectively, $?D_1, \ldots, ?D_k$. Let's call d' (resp. $d'_1, \ldots, d'_k$) the premise of the pal door of B (resp. the premises of the pax doors with conclusions $d_1, \ldots, d_k$ of B): $d', d'_1, \ldots, d'_k$ are the conclusions of R_B. Let e be an experiment of R s.t. $e(d) = \{\{y_1, \ldots, y_n\}\}$, and let $e_1, \ldots, e_n$ be the n experiments of R_B from which e is built: one has $\forall i \in \{1, \ldots, n\}$ $e_i(d') = \{y_i\}$. Finally, let $y_i, t^i_1, \ldots, t^i_k$ be the result of the experiment e_i of R_B ($\forall i \in \{1, \ldots, n\}$).
If, for some $i_1, i_2 \in \{1, \ldots, n\}$ one has $y_{i_1} = y_{i_2}$, then $\forall j \in \{1, \ldots, k\}$ one has $t^{i_1}_j = t^{i_2}_j$.

Proof: The results of the experiments $e_1, \ldots, e_n$ of R_B all belong to the same clique of the coherent space $\mathcal{A} = \mathcal{D} \wp ?\mathcal{D}_1 \wp \ldots \wp ?\mathcal{D}_k$ (from theorem 1.1.5), then $(y_{i_1}, t^{i_1}_1, \ldots, t^{i_1}_k) \frown\!\!\!\smile (y_{i_2}, t^{i_2}_1, \ldots, t^{i_2}_k)(\mathcal{A})$. Moreover, $\forall j \in \{1, \ldots, k\}$ the label associated by e with the conclusion d_j of type $?D_j$ of R is $t^1_j \cup \ldots \cup t^n_j$, which is an element of $|?\mathcal{D}_j| = \mathcal{D}^\perp_j$. Then, in particular, $t^{i_1}_j \cup t^{i_2}_j$ is a clique of $\mathcal{D}^\perp_j$, i.e. $t^{i_1}_j \smile\!\!\!\frown t^{i_2}_j(?\mathcal{D}_j)$. This means that, for $j \in \{1, \ldots, k\}$ one never has $t^{i_1}_j \frown t^{i_2}_j(?\mathcal{D}_j)$. Because we also have $y_{i_1} = y_{i_2}$ we cannot have $y_{i_1} \frown y_{i_2}$. The only remaining possibility to have $(y_{i_1}, t^{i_1}_1, \ldots, t^{i_1}_k) \frown\!\!\!\smile (y_{i_2}, t^{i_2}_1, \ldots, t^{i_2}_k)(\mathcal{A})$ is then that $\forall j \in \{1, \ldots, k\}$ $t^{i_1}_j = t^{i_2}_j$. □

We use in the proof of the following lemma the result of [Bar01] mentioned in remark 1.1.13.

2.2.4. LEMMA. *Let n be a strictly positive integer. Let e be an experiment of the proof-net R s.t. for every edge c of type $!C$ (for some formula C) of R, if we call c' (of type C) the premise of the of course link having c as conclusion, we have that:*
(1) $e(c) \neq \emptyset$
(2) for every $y \in e(c)$, there exists $z \in e(c')$ s.t. $y = \{n[z]\}$.
Then e is an n-obsessional experiment of R.

Proof: By induction on a sequentialization of R.
If R is an axiom link, then the lemma holds.
Otherwise, let n be a terminal link of R, i.e. such that there exists a sequentialization of R whose last rule is the rule corresponding to n. If R is not a box, then $n = \otimes, \wp, ?de, ?co, ?w$, and the result is a straightforward application of the induction hypothesis.
Let's consider the case of a box B with conclusions the edge d of type $!D$ and the edges $d_1, \ldots, d_k$ of types, respectively, $?D_1, \ldots, ?D_k$. Let's call d' (resp. $d'_1, \ldots, d'_k$) the premise of the pal door of B (resp. the premises of the pax doors with conclusions $d_1, \ldots, d_k$ of B): $d', d'_1, \ldots, d'_k$ are the conclusions of R_B. By hypothesis $e(d) = \{\{n[y]\}\}$. Then, there exist n experiments $e_1, \ldots, e_n$ of R_B s.t. $\forall i \in \{1, \ldots, n\}$ $e_i(d') = \{y\}$. Let then $y, t_1^i, \ldots, t_k^i$ be the result of the experiment e_i of R_B ($\forall i \in \{1, \ldots, n\}$). We are going to prove that:
(a) $\forall i_1, i_2 \in \{1, \ldots, n\}$ $\forall j \in \{1, \ldots, k\}$ we have that $t_j^{i_1} = t_j^{i_2}$
(b) $\forall i \in \{1, \ldots, n\}$, e_i is an n-obsessional experiment of R_B.
Property (a) is a consequence of lemma 2.2.3. To prove property (b), by induction hypothesis it will be enough to prove that $\forall i \in \{1, \ldots, n\}$ the experiment e_i of R_B satisfies the hypothesis (1) and (2) of the lemma. If for some edge c of R_B $e_i(c) = \emptyset$, then (by lemma 1.1.9) there necessarily exists an edge g of type $!G$ of R_B such that $\emptyset \in e_i(g)$. But by lemma 1.1.8, one would then have $\emptyset \in e(g)$, which contradicts (2). Similarly, suppose there exists an edge c of type $!C$ (for some formula C) and $y \in e_i(c)$ s.t. $\forall z \in e_i(c')$ $y \neq \{n[z]\}$, where c' is the edge of R_B premise of the of course link having c as conclusion. In this case, either the cardinality of y (as a multiset) is different from n or there exist $z_1, z_2 \in e_i(c')$ s.t. $z_1 \neq z_2$ and $z_1, z_2 \in y$. By lemma 1.1.8, in both cases there exists $y \in e(c)$ s.t. $\forall z \in e(c')$ $y \neq \{n[z]\}$, thus contradicting the hypothesis of the lemma.
This entails that e_i is an n-obsessional experiment of R_B. Moreover, thanks to the property (a) above (and to remark 1.1.13), $e_1 = \ldots = e_n$. The experiment e of R is then n-obsessional (remember remark 2.1.2). □

2.2.5. Remark. Let e (resp. e') be an n-obsessional experiment of the proof-net R (resp. of the proof-net R') and let a (resp. a') be an edge of R (resp. of R') conclusion of the link m (resp. m') whose premises are $a_1, \ldots, a_k$ (resp. $a'_1, \ldots, a'_k$), with $k \geq 0$.
If m and m' are two links of the same kind, and if $\forall i \in \{1, \ldots, k\}$ $e(a_i) = e'(a_i)$, then $e(a) = e'(a')$.
Notice that this is wrong (in general) if e and e' are not n-obsessional.

We show why the property is wrong if we omit the hypothesis of n-obsessionality. Take for example $k = 2$, m and m' of type $\otimes$, $e(a_1) = e'(a'_1) = \{x_1, x_2\}$ and $e(a_2) = e'(a'_2) = \{y_1, y_2\}$, with $x_1 \neq x_2$ and $y_1 \neq y_2$. One might very well have $e(a) = \{(x_1, y_1), (x_2, y_2)\}$ and $e'(a') = \{(x_1, y_2), (x_2, y_1)\}$.

2.2.6. PROPOSITION. *Let* R be a proof-net, $\gamma \in [\![R]\!]$ and n a strictly positive integer. The two following statements are equivalent:

(i) for every edge a of type A conclusion of R and for every (occurrence of) subformula $!C$ of A, if x is the label of γ associated with a, we have:

if $z \in |x|_{!C}$, then there exists $t \in |\mathcal{C}|$ s.t. $z = \{n[t]\}$.

(ii) the experiment e of R with result γ is an n-obsessional experiment.

Proof: Let e be an n-obsessional experiment of R with result γ, and let a be an edge of type A conclusion of R. Let $!C$ be an (occurrence of) subformula of A and let x be the label that e associates with a. Let k be the number of edges of G^R_a of type $!C$. If $k = 0$, then $|x|_{!C} = \emptyset$ and we are done. Otherwise, $k \geq 1$, and by proposition 2.2.2, $|x|_{!C} = e(c_1) \cup \ldots \cup e(c_k)$, where $c_1, \ldots, c_k$ are the k edges of type $!C$ s.t. $\forall i \in \{1, \ldots, k\}$ one has $c_i \in G^R_a$. If $z \in |x|_{!C}$, then there exists $i \in \{1, \ldots, k\}$ s.t. $z \in e(c_i)$. By remark 2.1.7 (i), we then have indeed $z = \{n[t]\}$, for some $t \in |\mathcal{C}|$.
Conversely, let e be the experiment of R with result γ. We show that e satisfies the hypothesis of lemma 2.2.4, by applying lemma 2.2.1.
By contradiction, suppose that e does not satisfy the hypothesis of lemma 2.2.4. There exists an edge c of type $!C$ conclusion of an of course link with premise c' and satisfying one of the two following conditions:
(1) $e(c) = \emptyset$
(2) $\exists y \in e(c)$ s.t. $\forall z \in e(c')$ $y \neq \{n[z]\}$.
In case (1), by lemma 1.1.9, there exists a box B of R s.t. if we call d the conclusion of type $!D$ of the pal door of B, then $e(d) = \{m[\emptyset]\}$, for some integer m different from zero. Let's call a of type A the conclusion of R such that $d \in G^R_a$ and let $x \in |\mathcal{A}|$ be such that $e(a) = \{x\}$. Lemma 2.2.1 applied to e gives $\emptyset \in |x|_{!D}$, which contradicts (i).
In case (2), one has two possibilities:
(2.1) $\exists y \in e(c)$ s.t. $card(y) \neq n$
(2.2) $\exists y \in e(c)$ s.t. $card(y) = n$ and $\exists z_1, z_2 \in e(c')$, $z_1 \neq z_2$ s.t. $z_1, z_2 \in y$.
In both cases, let a of type A be the conclusion of R such that $c \in G^R_a$ and let $x \in |\mathcal{A}|$ be such that $e(a) = \{x\}$. Lemma 2.2.1 applied to e gives $y \in |x|_{!C}$, which contradicts again (i). □

The following proposition is the consequence of the previous one which will be used in the sequel to answer the question of injectivity.

2.2.7. PROPOSITION. *Let R and R' be two proof-nets with the same conclusions. Let e (resp. e') be an experiment of R (resp. of R') with result γ (resp. γ').*
If e' is an n-obsessional experiment of R' and if $\gamma = \gamma'$, then e is an n-obsessional experiment of R.

Proof: We just have to show that the result γ of e satisfies (i) of proposition 2.2.6. This is immediate, because e' is n-obsessional and its result $\gamma' = \gamma$ satisfies (i) of the proposition 2.2.6. □

2.3 Recovering OLPS

We show, in this section, that the coherent semantics of a proof-net R determines R "up to the axiom links and the boxes" (theorem 2.3.12).
The notion of n-obsessional experiment allows to prove the analogue for $MELL$ of lemma 1.3.2: corollary 2.3.8. This is the second step allowing to argue like for 1.3.4 (the first step being proposition 2.2.7): suppose there exists an n-obsessional experiment e of R with result γ; because $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ, and we know by proposition 2.2.7 that e' is itself n-obsessional, which allows to apply corollary 2.3.8. To conclude, we prove that for every proof-net R and for every integer $n \geq 1$, there exists an n-obsessional experiment of R.
We use here in a crucial way the obsessional feature of our obsessional experiments.

We first have to introduce one more notion: the structural graph of a link.

2.3.1. DEFINITION. Let R be a proof-net, let n be a contraction link of arity k, a pax link, or a dereliction link. Let a be the conclusion of n. Let $a_1, \ldots, a_k$ (resp. $a'_1, \ldots, a'_k$) be the conclusions (resp. the premises) of the k dereliction links above a ($k \geq 1$).
The **structural graph** of n in R, **denoted by** SG^R_a is the subgraph of G^R_a obtained by erasing $G^R_{a'_i}$ for every $i \in \{1, \ldots, k\}$.
We will say that the edge a (resp. the edges $a_1, \ldots, a_k$) is (resp. are) the conclusion (resp. the hypothesis) of SG^R_a. For every edge c of SG^R_a we will say that the number p of the pax links crossed by the path of SG^R_a with starting edge c and terminal edge a is the depth of c in SG^R_a.

The previous definition will mainly (but not only) be used when a is the conclusion of a contraction link.

2.3.2. REMARK. Let a be an edge of the proof-net R which is the conclusion of a contraction link n of arity k, and let $a_1, \ldots, a_k$ be the conclusions of the k dereliction links $m_1, \ldots, m_k$ above a. For every edge c (resp. for every link m) of SG_a^R s.t. $c \notin \{a, a_1, \ldots, a_k\}$ (resp. $m \notin \{n, m_1, \ldots, m_k\}$), c is the conclusion of a pax link (resp. m is a pax link).

The following lemma is indeed the main ingredient of the main result of this section (proposition 2.3.7). Every letter stands for a non negative integer, and we deal (as usual) with *multisets*.

2.3.3. LEMMA. *Let $1 \leq l, m < n$. We have $\forall p_1, \ldots, p_l, p'_1, \ldots, p'_m$: the equality $n^{p_1} + \ldots + n^{p_l} = n^{p'_1} + \ldots + n^{p'_m}$ implies the equality $\{p_1, \ldots, p_l\} = \{p'_1, \ldots, p'_m\}$ (in particular $l = m$).*

Proof: This result is a straightforward consequence of the unicity of the decomposition of every integer in base n. □

The following definition associates with every proof-net an integer (its ?*co*-size), which depends on the arity of the contraction links of R. This "size" is often used in the sequel, and allows to simplify several statements.

2.3.4. DEFINITION. Let k be the maximal arity of the contraction links of the proof-net R. The ?*co*-**size** of R, denoted by $h(R)$, is the following integer:

- $h(R) = max(1, k)$, if there exists at least one box in R
- $h(R) = 0$ otherwise.

We shall often consider integers $n > h(R)$: in case $h(R) = 0$, the reader can actually read $n = 1$.

We come now to a first application of the notion of obsessional experiment.

2.3.5. LEMMA. *Let $h(R)$ be the ?co-size of the proof-net R, and let $n > h(R)$. Let e be an n-obsessional experiment of R. Let a be an edge of R of type $?A$ and let x be the unique element of $e(a)$ (following proposition 2.1.6). Then:*
(i) a is the conclusion of a weakening link iff $x = \emptyset$
(ii) a is the conclusion of a dereliction link iff x is a singleton
(iii) a is the conclusion of a pax link iff there exists an integer $p \geq 1$ s.t. $card(x) = n^p$
(iv) a is the conclusion of a contraction link with arity k iff there exist $p_1, \dots, p_k$ non negative integers s.t. $k \geq 2$ and $card(x) = n^{p_1} + \dots + n^{p_k}$.

Proof: As an example, we prove (iv).

(iv.1) If a is the conclusion of a contraction link with arity k ($k \geq 2$), then let $a_1, \dots, a_k$ be the edges conclusion of the k dereliction links above a and let $p_1, \dots, p_k$ be, respectively, their depths in SG_a^R. Let $\{y_i\}$, $\forall i \in \{1, \dots, k\}$, be the unique element of $e(a_i)$. By definition of n-obsessional experiment, the unique element of $e(a)$ is the multiset of cardinality $n^{p_1} + \dots + n^{p_k}$ containing n^{p_i} occurrences of y_i $\forall i \in \{1, \dots, k\}$.

(iv.2) Because $card(x) > 1$, by (*i*) and (*ii*), the edge a is the conclusion of a pax link or of a contraction link. If a were the conclusion of a pax link, then by (*iii*) there would exist $p \geq 1$ s.t. $card(x) = n^p$. But then one would have $n^p = n^{p_1} + \dots + n^{p_k}$, thus contradicting lemma 2.3.3 (because $k \geq 2$). This means that a is indeed the conclusion of a contraction link. Let l be the arity of this link. By (*iv*.1), there exist $p'_1, \dots, p'_l$ non negative integers s.t. $card(x) = n^{p'_1} + \dots + n^{p'_l}$. Then $n^{p_1} + \dots + n^{p_k} = n^{p'_1} + \dots + n^{p'_l}$. Because (when $h(R) \neq 0$) we have $n > h(R) \geq k, l \geq 1$, we can apply lemma 2.3.3: $\{p'_1, \dots, p'_l\} = \{p_1, \dots, p_k\}$. In particular $l = k$, which means that a is the conclusion of a contraction link with arity k. $\square$

Let e (resp. e') be an n-obsessional experiment of the proof-net R (resp. R'), with $n > 1$. Let a (resp. a') be an edge of R (resp. of R') of type $?C$ conclusion of a contraction link with arity $k \geq 2$. Let $a_1, \dots, a_k$ (resp. $a'_1, \dots, a'_k$) be the hypothesis of SG_a^R (resp. of $SG_{a'}^{R'}$) and $p_1, \dots, p_k$ (resp. $p'_1, \dots, p'_k$), respectively, their depths in SG_a^R (resp. in $SG_{a'}^{R'}$).
Suppose that $e(a) = e'(a')$ and let $t \in |?\mathcal{C}|$ be the unique element of $e(a) = e'(a')$ (following proposition 2.1.6). We define, $\forall x \in t$:

- $k_x := card(\{i \in \{1,\ldots,k\} : \{x\}$ is the unique element of $e(a_i)\})$ (resp. $k'_x := card(\{i \in \{1,\ldots,k\} : \{x\}$ is the unique element of $e'(a'_i)\}))$
- the multiset $\{p^x_1,\ldots,p^x_{k_x}\} \subseteq \{p_1,\ldots,p_k\}$ (resp. the multiset $\{p'^x_1,\ldots,p'^x_{k'_x}\} \subseteq \{p'_1,\ldots,p'_k\}$) of the depths in SG^R_a (resp. in $SG^{R'}_{a'}$) of the edges $c \in \{a_1,\ldots,a_k\}$ (resp. of the edges $c' \in \{a'_1,\ldots,a'_k\}$) such that $\{x\}$ is the unique element of $e(c)$ (resp. of $e'(c')$).

2.3.6. Lemma. *If* $\forall x \in t$, $k_x = k'_x$ and $\{p^x_1,\ldots,p^x_{k_x}\} = \{p'^x_1,\ldots,p'^x_{k'_x}\}$, then $SG^R_a = SG^{R'}_{a'}$ and $e|_{SG^R_a} = e'|_{SG^{R'}_{a'}}$.

Proof: We leave it to the reader to convince herself/himself that the following holds and is enough to conclude:
$\forall c_1, c_2 \in \{a_1,\ldots,a_k\}$, $c_1 \neq c_2$, c_1 (resp. c_2) of depth h_1 (resp. h_2) in SG^R_a and s.t. $\{y_1\}$ (resp. $\{y_2\}$) is the unique element of $e(c_1)$ (resp. of $e(c_2)$), there exists an edge $c'_1 \in \{a'_1,\ldots,a'_k\}$ (resp. an edge $c'_2 \in \{a'_1,\ldots,a'_k\}$) of depth h_1 (resp. h_2) in $SG^{R'}_{a'}$ s.t. $c'_1 \neq c'_2$ and $\{y_1\}$ (resp. $\{y_2\}$) is the unique element of $e'(c'_1)$ (resp. of $e'(c'_2)$). □

2.3.7. Proposition. *Let a (resp. a') be an edge of type A of the proof-net R (resp. of the proof-net R'). Let $h(R)$ (resp. $h(R')$) be the ?co-size of R (resp. R'), and let $n > max(h(R), h(R'))$.*
Let e (resp. e') be an n-obsessional experiment of R (resp. of R'). If $e(a) = e'(a')$, then:
(i) $G^R_a = G^{R'}_{a'}$
(ii) $e|_{G^R_a} = e'|_{G^{R'}_{a'}}$.

Proof: By induction on $s(G^R_a)$, the number of links of G^R_a.
If $s(G^R_a) = 0$, then a is an edge of R which is the conclusion of an axiom link, and then A is an atomic formula. This means that a' is also the conclusion of an axiom and $G^R_a = G^{R'}_{a'}$. (ii) is a straightforward consequence of the hypothesis $e(a) = e'(a')$.
Otherwise, let m be the link of R having a as conclusion. The non trivial cases are the ones in which $A =?D$ for some formula D, specially when a is the conclusion of a contraction link. We nevertheless give a complete proof, precisely in order to let the reader feel the difference between the cases.
- If $m = \otimes$ or $m = \wp$, then the edge a' of R' is also conclusion of a link $m' = \otimes$ or $m' = \wp$. Let a_1 and a_2 (resp. a'_1 and a'_2) be the premises of m in R (resp. of m' in R'). The edges a_i and a'_i $(i = 1, 2)$ have the same type, and by definition $G^R_a = G^R_{a_1} \cup G^R_{a_2} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup G^{R'}_{a'_2} \cup \{m'\} \cup \{a'\}$.

We have that $s(G^R_{a_1}) < s(G^R_a)$, $s(G^R_{a_2}) < s(G^R_a)$. Let x be the unique element of $e(a) = e'(a')$ (following proposition 2.1.6). By definition of experiment, there exists $x_1 \in e(a_1)$ and $x_2 \in e(a_2)$ (resp. $x'_1 \in e'(a'_1)$ and $x'_2 \in e'(a'_2)$) s.t. $x = (x_1, x_2) = (x'_1, x'_2)$. Then $x_1 = x'_1$ is the unique element of $e(a_1)$ and of $e'(a'_1)$, and $x_2 = x'_2$ is the unique element of $e(a_2)$ and of $e'(a'_2)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, then a_1, a_2, a'_1 and a'_2 all have the same depth. This implies that $e(a_1) = e'(a'_1)$ and $e(a_2) = e'(a'_2)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$, $G^R_{a_2} = G^{R'}_{a'_2}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, $e|_{G^R_{a_2}} = e'|_{G^{R'}_{a'_2}}$. The conclusion follows then immediately.

- If m is an of course link, then the edge a' of R' is also the conclusion of an of course link m'. Let a_1 (resp. a'_1) be the premise of m in R (resp. of m' in R'). The edges a_1 and a'_1 have the same type, and by definition $G^R_a = G^R_{a_1} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$. Let x be the unique element of $e(a) = e'(a')$ (following proposition 2.1.6). By definition of n-obsessional experiment, there exist $x_1, \ldots, x_n \in e(a_1)$ (resp. $x'_1, \ldots, x'_n \in e'(a'_1)$) s.t. $x = \{x_1, \ldots, x_n\} = \{x'_1, \ldots, x'_n\}$. Proposition 2.1.6 gives then $x_1 = \ldots = x_n = x'_1 = \ldots = x'_n$ i.e. the unique element of $e(a_1)$ is also the unique element of $e'(a'_1)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, then a_1 and a'_1 also have the same depth. Then $e(a_1) = e'(a'_1)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, which allows to conclude.

- If m is a dereliction link of R, then $A = ?C$ and by lemma 2.3.5, the unique element of $e(a) = e'(a')$ is a singleton. By that same lemma, a' is also the conclusion of a dereliction link m'. Let a_1 (resp. a'_1) be the premise of m in R (resp. of m' in R'). The edges a_1 and a'_1 have the same type, and by definition $G^R_a = G^R_{a_1} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$. Let x be the unique element of $e(a) = e'(a')$ (following proposition 2.1.6). By definition of experiment, there exists $y_1 \in e(a_1)$ (resp. $y'_1 \in e'(a'_1)$) s.t. $x = \{y_1\} = \{y'_1\}$. Then $y_1 = y'_1$, i.e. the unique element of $e(a_1)$ is also the unique element of $e'(a'_1)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, then a_1 and a'_1 also have the same depth. Then $e(a_1) = e'(a'_1)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, which allows to conclude.

- If m is a pax link of R, then let x be the unique element of $e(a)$ (following proposition 2.1.6). By lemma 2.3.5, there exists $p \geq 1$ s.t. $card(x) = n^p$. By that same lemma, because $e(a) = e'(a')$, a' is necessarily the conclusion of a pax link m' of R'. Let a_1 (resp. a'_1) be the premise of m in R (resp. of m'

in R'). The edges a_1 and a'_1 have the same type, and by definition $G^R_a = G^R_{a_1} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$. By definition of n-obsessional experiment, there exists $x_1, \ldots, x_n \in e(a_1)$ (resp. $x'_1, \ldots, x'_n \in e'(a'_1)$) s.t. $x = x_1 \cup \ldots \cup x_n = x'_1 \cup \ldots \cup x'_n$. Proposition 2.1.6 gives then $x_1 = \ldots = x_n = x'_1 = \ldots = x'_n$, i.e. the unique element of $e(a_1)$ is also the unique element of $e'(a'_1)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, then a_1 and a'_1 also have the same depth. Then $e(a_1) = e'(a'_1)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, which allows to conclude.

- If m is a weakening link, then by lemma 2.3.5, the unique element of $e(a) = e'(a')$ is $\emptyset$. By that same lemma, because $e(a) = e'(a')$, a' is necessarily the conclusion of a weakening link, and we are done.

- If m is a contraction link with arity $k \geq 2$, then let t be the unique element of $e(a) = e'(a')$. By lemma 2.3.5, there exist $p_1, \ldots, p_k$ non negative integers s.t. $card(t) = n^{p_1} + \ldots + n^{p_k}$. By that same lemma, because $e(a) = e'(a')$, a' is necessarily the conclusion of a contraction link m' with arity k.

Let $a_1, \ldots, a_k$ (resp. $a'_1, \ldots, a'_k$) be the hypothesis of SG^R_a (resp. of $SG^{R'}_{a'}$) and let $p_1, \ldots, p_k$ (resp. $p'_1, \ldots, p'_k$) be, respectively, their depths in SG^R_a (resp. in $SG^{R'}_{a'}$). With these notations (which are the ones of lemma 2.3.6), we are going to prove that the hypotheses of lemma 2.3.6 are satisfied. Let $x \in t$ and $\{p^x_1, \ldots, p^x_{k_x}\} \subseteq \{p_1, \ldots, p_k\}$ (resp. $\{p'^x_1, \ldots, p'^x_{k'_x}\} \subseteq \{p'_1, \ldots, p'_k\}$) be the multiset of the depths in SG^R_a (resp. in $SG^{R'}_{a'}$) of the edges $c \in \{a_1, \ldots, a_k\}$ (resp. of the edges $c' \in \{a'_1, \ldots, a'_k\}$) such that $\{x\}$ is the unique element of $e(c)$ (resp. of $e'(c')$). Let α be the cardinality of t and let α_x be the number of occurrences of x in t. Because e and e' are two n-obsessional experiments, we have that $\alpha_x = n^{p^x_1} + \ldots + n^{p^x_{k_x}} = n^{p'^x_1} + \ldots + n^{p'^x_{k'_x}}$. Because (when $h(R) \neq 0$ or $h(R') \neq 0$) we have $n > h(R), h(R') \geq k_x, k'_x \geq 1$, we can apply lemma 2.3.3, and obtain $\{p^x_1, \ldots, p^x_{k_x}\} = \{p'^x_1, \ldots, p'^x_{k'_x}\}$. The hypothesis of lemma 2.3.6 are then satisfied, and we have that $SG^R_a = SG^{R'}_{a'}$ and $e|_{SG^R_a} = e'|_{SG^{R'}_{a'}}$. In particular, *possibly after a renaming of the hypothesis of* SG^R_a *and of* $SG^{R'}_{a'}$ (remember that G^R_a and $G^{R'}_{a'}$ are equivalence classes of trees), we have that $\forall i \in \{1, \ldots, k\}$ $e(a_i) = e'(a'_i)$, and $s(G^R_{a_i}) < s(G^R_a)$. The induction hypothesis gives then $G^R_{a_i} = G^{R'}_{a'_i}$ and $e|_{G^R_{a_i}} = e'|_{G^{R'}_{a'_i}}$, $\forall i \in \{1, \ldots, k\}$. And we are done. □

2.3.8. COROLLARY. *Let* R *and* R' *be two proof-nets with conclusion* Γ, *let* $h(R)$ *(resp.* $h(R')$*) be the* ?*co*-*size of* R *(resp.* R'*), and let* $n >$

$max(h(R), h(R'))$. Let e (resp. e') be an n-obsessional experiment of R (resp. of R') with conclusion γ (resp. γ'). If $\gamma = \gamma'$, then $OLPS(R) = OLPS(R')$ and $e|_{OLPS(R)} = e'|_{OLPS(R')}$.

We just proved the analogue (for $MELL$) of lemma 1.3.2. Let's then try to conclude that when $[\![R]\!] = [\![R']\!]$, one has $OLPS(R) = OLPS(R')$: let e be an n-obsessional experiment of R (**Question: does it exist?**) with result γ. Because $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ, which by proposition 2.2.7 is n-obsessional. If the answer to the previous question is positive, we can apply corollary 2.3.8: $OLPS(R) = OLPS(R')$. In order to answer this question, we introduce the notion of "simple experiment". We also generalize the notion of injective experiment to $MELL$ (it will be used in the sequel).

2.3.9. DEFINITION. Let R be a proof-net and let e be an n-obsessional experiment of R. We will say that e is **injective** (resp. **simple**) when $\forall a, a'$ edges of the same atomic type of R such that $a \neq a'$, the unique element of $e(a)$ is different from (resp. equal to) the unique element of $e(a')$.

The existence of an n-obsessional experiment has to be proven. Indeed, even for n-obsessional experiments, even for standard proof-nets, we cannot say "a priori" that the labels of the premises of a contraction link satisfy the condition of remark 1.1.2 (required by definition 1.1.1), as already mentioned in remark 2.1.8.
We are going to prove that a simple n-obsessional experiment does always exist, for every (standard) $MELL$ proof-net (proposition 2.3.11).

Notation: If e is an n-obsessional experiment of the proof-net R and a is an edge of R, we will denote, from now on, by $|e(a)|$ the unique element of $e(a)$. We did not introduce the notation before, due to the possible confusion with the notion of projection (definition 1.1.11), which will rarely be used in the sequel of the paper (and it will be clear from the context which of the two notions the notation refers to).

2.3.10. LEMMA. *Let R be a proof-net, let a and a' be two edges of R of type A. Let e be an n-obsessional experiment of R ($n \geq 1$).
If $\forall \alpha \in G_a$, and $\forall \alpha' \in G_{a'}$, with α and α' of the same atomic type, one has $|e(\alpha)| = |e(\alpha')|$, then $|e(a)| \smile\!\!\!\!\frown |e(a')|(\mathcal{A})$.*

Proof: By induction on $p = s(G_a) + s(G_{a'})$, where for every edge b of R, $s(G_b) =$ number of links of G_b. The details are left to the reader. □

2.3.11. PROPOSITION. *Let R be a proof-net. If n is a strictly positive integer, then there exists a simple n-obsessional experiment of R.*

Proof: Let $\mathcal{X}$ be a coherent space and $x \in |X|$. If we interpret every atomic formula of R by the coherent space $\mathcal{X}$ and if we associate with every axiom link of R the label x, then the previous lemma shows that we can always perform a contraction between two different edges of the same type.
More formally, one can proceed by induction on a sequentialization of R, the only significant case being the one of the terminal contraction link, for which one makes use of the previous lemma. □

2.3.12. THEOREM. *Let R and R' be two proof-nets with the same conclusions. If $[\![R]\!] = [\![R']\!]$, then $OLPS(R) = OLPS(R')$.*

Proof: Let $h(R)$ (resp. $h(R')$) be the ?*co*-size of R (resp. R'), and let $n > max(h(R), h(R'))$. Let e_n be a simple n-obsessional experiment of R (which exists by proposition 2.3.11). Because $[\![R]\!] = [\![R']\!]$, there exists an experiment e'_n of R' with the same result as e_n. By proposition 2.2.7, e'_n is an n-obsessional experiment of R'. From corollary 2.3.8, $OLPS(R) = OLPS(R')$.
□

2.4 Local injectivity

We give, in this section, a sufficient condition of "local injectivity" (theorem 2.4.8): if there exists an injective 1-experiment of a given proof-net, then it is "alone" in its (semantic) equivalence class.

Let's begin by trying to push further the similarity between the $MELL$ case and the MLL one, and let's try to argue like for 1.3.4: let $h(R)$ (resp. $h(R')$) be the ?*co*-size of R (resp. R'), and let $n > max(h(R), h(R'))$; let e be an injective n-obsessional experiment of R (**Question: does it exist?**) with result γ. Because $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ, which by proposition 2.2.7 is n-obsessional.
Provided the answer to the previous question is positive, we can apply corollary 2.3.8: $OLPS(R) = OLPS(R')$ and $e|_{OLPS(R)} = e'|_{OLPS(R')}$. The injectivity of e (then of e') allows to conclude in that case $LPS(R) = LPS(R')$, and we'll see that *in the absence of weakenings* this implies that $R = R'$.

Let R be a proof-net. If one associates with every axiom link l with conclusions α_l and $\alpha_l^\perp$ of R, an element x_l of the web of the coherent space $\mathcal{A}$, in

such a way that if $l \neq l'$ then $x_l \neq x_{l'}$, does there exist an n-obsessional experiment e_n of R s.t. for every axiom link l one has $|e_n(\alpha_l)| = |e_n(\alpha_l^\perp)| = x_l$? Definition 1.1.1 of experiment (see also remark 1.1.2) clearly shows that in the absence of cut links, the constrained labels are the ones of the conclusions of the of course, pax, and $?co$ links.
We show that if e_n exists for $n = 1$ (i.e. if there exists an injective 1-experiment of the proof-net R), then e_n exists for every $n > h(R)$ (i.e. there exists also an injective n-obsessional experiment of R, for every "big enough" n): e_n is the n-obsessional experiment "induced" by the 1-experiment e_1 (proposition 2.4.6). The existence of such an experiment e_1 of R is then proven to be enough to conclude that R is "alone" in its (semantic) equivalence class (theorem 2.4.8).
In the rest of the paper, our key-question will be whether or not, for a given proof-net (or a given set of proof-nets), there exists an injective 1-experiment. The following chapters give the answer for some particular fragments of $MELL$: when it is positive, we obtain a positive answer to our original problem 1.2.3, when it is negative, we can build counterexamples, thus answering negatively to problem 1.2.3.

Convention: Let e be a 1-experiment of the proof-net R. For every edge a of type A of R, $e(a) = \{x\}$ for some $x \in |\mathcal{A}|$ (from proposition 2.1.6).
In the rest of the paper, we will identify the multiset (of cardinality 1) $e(a)$ with its unique element x.

2.4.1. DEFINITION. Let R be a proof-net and let e_1 be an injective 1-experiment of R. We say that e_n is an n-obsessional experiment induced by e_1, if e_n is an n-obsessional experiment of R such that: for every atomic edge a of R $|e_n(a)| = e_1(a)$.

2.4.2. REMARK. (i) The existence of an n-obsessional experiment induced by a given injective 1-experiment is not obvious a priori. But if it exists, it is unique (by lemma 2.1.4 and proposition 2.1.6) and obviously injective.
(ii) The reader might be tempted to think that if e_n is the n-obsessional experiment induced by the injective 1-experiment e_1 of R, then for every edge a, $|e_n(a)| = e_1(a)$. She/he can, but she/he shouldn't, because this is wrong (in general): suppose that $|e_n(c')| = e_1(c')$, where c' is the premise of an of course link of R with conclusion c; one has $|e_n(c)| = \{n[|e_n(c')|]\}$, and $e_1(c) = \{e_1(c')\}$.

The proofs of the following lemmas are simple applications of the definition of coherence in the spaces interpreting LL formulas (see A.1.1). We give an example by proving the first one, and leave the following ones to the reader.

2.4.3. LEMMA. *Let R be a proof-net containing at least one box, let $h(R)$ be the ?co-size of R, and let $n > h(R)$. Let a and a' be two different edges of R of the same type A. Let e_1 be an injective 1-experiment of R and let e_n be the n-obsessional experiment of R induced par e_1. We have that:*
(1) If there exists an edge α of atomic type s.t. $\alpha \in G_a$ or $\alpha \in G_{a'}$, then $|e_n(a)| \neq |e_n(a')|$
(2) Otherwise, either $G_a = G_{a'}$ and then $|e_n(a)| = |e_n(a')|$, or $G_a \neq G_{a'}$ and then $|e_n(a)| \smile |e_n(a')|$.

Proof: (1) Let $x = |e_n(a)|$ (resp. $x' = |e_n(a')|$), and suppose for example that $\alpha \in G_a$ is of type X.
If $a' \in G_a$ or if $a \in G_{a'}$, then necessarily $A = ?B$ (remember that a and a' have the same type), and because $n > 1$, one has $x \neq x'$.
Otherwise $G_a \cap G_{a'} = \emptyset$. And then $\alpha \notin G_{a'}$. For every atomic edge $\beta \in G_{a'}$ of type X $|e_n(\beta)| \neq |e_n(\alpha)|$ (by definition of injective experiment). By lemma 2.2.1, we then have $|e_n(\alpha)| \in |x|_X$ and $|e_n(\alpha)| \notin |x'|_X$. Then $x \neq x'$.
(2) If such an edge α does not exist, then none of the edges of G_a and of $G_{a'}$ is an edge of atomic type. In G_a so as in $G_{a'}$ there is then at least one weakening link. And every "leaf" of G_a and $G_{a'}$ is the conclusion of a weakening link.
If $G_a = G_{a'}$, then (because e_n is n-obsessional) $e_n(a) = e_n(a')$ (and in particular $|e_n(a)| = |e_n(a')|$).
If $G_a \neq G_{a'}$, we proceed by induction on $k = s(G_a) + s(G_{a'})$, the sum of the number of links of G_a and of $G_{a'}$.
If $k = 2$, then a and a' are conclusions of two weakening links and $G_a = G_{a'}$.
Otherwise, if $A = B \otimes C$ (resp. $A = B \wp C$), then let b and b' be the premises of type B and let c and c' be the premises of type C of the $\otimes$ (resp. $\wp$) links having a and a' as conclusions. By definition, $|e_n(a)| = (|e_n(b)|, |e_n(c)|)$ and $|e_n(a')| = (|e_n(b')|, |e_n(c')|)$. We have $G_b \neq G_{b'}$ or/and $G_c \neq G_{c'}$. Suppose for example that $G_b \neq G_{b'}$: by induction hypothesis, we then have $|e_n(b)| \smile |e_n(b')|$. In the $\otimes$ case, this is enough to conclude that $|e_n(a)| \smile |e_n(a')|$. In the $\wp$ case, observe that if $G_c = G_{c'}$ then we already proved that $|e_n(c)| = |e_n(c')|$, while if $G_c \neq G_{c'}$ then by induction hypothesis $|e_n(c)| \smile |e_n(c')|$: whatever happens we indeed have that $|e_n(a)| \smile |e_n(a')|$.
If $A = !B$, then the conclusion is a straightforward application of the induction hypothesis.

If $A = ?B$, let m (resp. m') be the link having a (resp. a') as conclusion. Suppose for example that $s(G_a) \geq s(G_{a'})$, and let's first consider the case where $a' \in G_a$. One has $|e_n(a')| \subseteq |e_n(a)|$, and then $|e_n(a')| \cup |e_n(a)|$ is a clique of $|?\mathcal{B}| = \mathcal{B}^{\perp}$: then $|e_n(a')| \stackrel{\smile}{\frown} |e_n(a)|(?\mathcal{B})$. Because $a \neq a'$ and $n > 1$, $|e_n(a)| \neq |e_n(a')|$. Then $|e_n(a')| \smile |e_n(a)|$.
We can now turn our attention to the case $a' \notin G_a$, i.e. $G_a \cap G_{a'} = \emptyset$. If m or m' (but not both!) is a weakening link, then $|e_n(a')| \smile \emptyset$ or $|e_n(a)| \smile \emptyset$: we can then suppose that m and m' are $?de$, pax or $?co$ links.
Let then $a_1, \dots, a_l$ (resp. $a'_1, \dots, a'_q$) be the premises of the l (resp. q) dereliction links above a (resp. a'). Observe that because $G_a \cap G_{a'} = \emptyset$ one has $a_i \neq a'_j$ $\forall i \in \{1, \dots, l\}$ and $\forall j \in \{1, \dots, q\}$.
There is no edge of atomic type in G_{a_i} nor in $G_{a'_j}$, $\forall i \in \{1, \dots, l\}$ and $\forall j \in \{1, \dots, q\}$ (because there is no such edge in G_a nor in $G_{a'}$). $\forall i \in \{1, \dots, l\}$ and $\forall j \in \{1, \dots, q\}$ one has then two possibilities for G_{a_i} and $G_{a'_j}$: either $G_{a_i} = G_{a'_j}$ and then $|e_n(a_i)| = |e_n(a'_j)|$, or $G_{a_i} \neq G_{a'_j}$ and then (by induction hypothesis) $|e_n(a_i)| \smile |e_n(a'_j)|$. In any case, we will then have $|e_n(a')| \stackrel{\smile}{\frown} |e_n(a)|$.
If $SG_a = SG_{a'}$ (see definition 2.3.1 for the notation), then (because $G_a \neq G_{a'}$) there necessarily exists $i \in \{1, \dots, l\}$ such that $\forall j \in \{1, \dots, q\}$ $G_{a_i} \neq G_{a'_j}$, or there exists $j \in \{1, \dots, q\}$ such that $\forall i \in \{1, \dots, l\}$ $G_{a_i} \neq G_{a'_j}$: in both cases the induction hypothesis gives $|e_n(a)| \neq |e_n(a')|$.
If $SG_a \neq SG_{a'}$, then by lemmas 2.3.5 and 2.3.3 $card(|e_n(a)|) \neq card(|e_n(a')|)$ and then $|e_n(a)| \neq |e_n(a')|$. (Notice that we use here the hypothesis $n > h(R)$, in a crucial way: if we omit it the lemma is wrong). □

2.4.4. LEMMA. *Let a and a' be two different edges of the same type A of the proof-net R, and let e_1 be an injective 1-experiment of R.*

(1) If $a \in G_{a'}$ or $a' \in G_a$, either one among a and a' is the conclusion of a $?co$ link and then $e_1(a) \neq e_1(a')$, or it is not the case and then $e(a) = e(a')$.
(2) If $G_a \cap G_{a'} = \emptyset$ and there exists an edge α of atomic type s.t. $\alpha \in G_a$ or $\alpha \in G_{a'}$, then $e_1(a) \neq e_1(a')$.
(3) If $G_a \cap G_{a'} = \emptyset$ and there exists no edge of atomic type α s.t. $\alpha \in G_a$ or $\alpha \in G_{a'}$, then $e_1(a) \stackrel{\smile}{\frown} e_1(a')$.

2.4.5. LEMMA. *Let R be a proof-net containing at least one box, let $h(R)$ be the $?co$-size of R, and let $n > h(R)$. Let a and a' be two different edges of R of the same type A. Let e_1 be an injective 1-experiment of R and let e_n be the n-obsessional experiment of R induced par e_1.*

If $e_1(a) \smile e_1(a')$, then $|e_n(a)| \smile |e_n(a')|$.

Proof: By induction on $k = s(G_a) + s(G_{a'})$, the sum of the number of links of G_a and of $G_{a'}$. $\square$

2.4.6. PROPOSITION. *Let R be a proof-net, let $h(R)$ be the ?co-size of R, and let $n > h(R)$.
If there exists an injective 1-experiment of R, then there exists also an injective n-obsessional experiment of R: it is the (unique) n-obsessional experiment of R induced by e_1.*

Proof: If R contains no boxes, then $h(R) = 0$ and (remember the convention of definition 2.3.4) $n = 1$: in this case the result is obvious with $e_n = e_1$.
Otherwise, we proceed by induction on a sequentialization π of R. Of course, we consider a slightly modified sequent calculus, the contraction rule of which is a k-ary rule (with $k \geq 2$ active formulas in the sequent premise of the rule).
The more significant case is when the last rule r_m of π is a contraction with arity k. Let then π_1 be the subproof of π obtained by erasing r_m so as its conclusion and let R_1 be the subproof-net of R obtained by erasing the last contraction link m of arity k corresponding to r_m. The proof π_1 is a sequentialization of R_1, and the restriction e_1^1 of the injective 1-experiment e_1 of R to R_1 is an injective 1-experiment of R_1. Let $a_1, \dots, a_k$ be the premises of m and a its conclusion. One has $e_1^1(a_i) \stackrel{\smile}{\frown} e_1^1(a_j)$ $\forall i, j \in \{1, \dots, k\}$. Let e_n^1 be the n-obsessional experiment of R_1 given by the induction hypothesis. Notice that because $\forall i \in \{1, \dots, k\}$ a_i has depth 0 in R_1, $e_n^1(a_i)$ is a singleton.
If for some $i, j \in \{1, \dots, k\}$ $e_1^1(a_i) = e_1^1(a_j)$, then (because a_i and a_j are premises of the same ?co link of R we have $G_{a_i} \cap G_{a_j} = \emptyset$) by lemma 2.4.4 there is no atomic edge in G_{a_i} nor in G_{a_j}: in this case lemma 2.4.3 allows to conclude that $|e_n^1(a_i)| \stackrel{\smile}{\frown} |e_n^1(a_j)|$.
$\forall i, j \in \{1, \dots, k\}$ we have two possibilities: either $e_1^1(a_i) \smile e_1^1(a_j)$ and then by lemma 2.4.5 $|e_n^1(a_i)| \smile |e_n^1(a_j)|$, either $e_1^1(a_i) = e_1^1(a_j)$ and then we just proved that $|e_n^1(a_i)| \stackrel{\smile}{\frown} |e_n^1(a_j)|$. In any case we have $|e_n^1(a_i)| \stackrel{\smile}{\frown} |e_n^1(a_j)|$ $\forall i, j \in \{1, \dots, k\}$. We can then define the experiment e_n of R s.t. $\forall a'$ edge of R, if $a' \neq a$ then $e_n(a') = e_n^1(a')$, and $e_n(a) = \{|e_n^1(a_1)| \cup \ldots \cup |e_n^1(a_k)|\}$.
Let us also mention the case of a last promotion rule for π. We argue like before: the experiment e_n is obtained from the experiment e_n^1 given by the induction hypothesis by "repeating n times" e_n^1, following the definition of n-obsessional experiment (see remark 2.1.2). $\square$

Let us now prove the following (very simple) characterization of exponential boxes, in the absence of weakenings.

2.4.7. PROPOSITION. *Let R be a proof-net which contains no weakening links, and let R' be a proof-net with the same conclusions as R.*
If $LPS(R) = LPS(R')$, then $R = R'$.

Proof: We will use the following characterization of boxes:

Let l (resp. m) be an of course or a pax link with depth p in the proof-net T (T contains no weakening links). The links l and m are two doors of the same box if and only if there exists an oriented path Φ *not necessarily straight* (see definition A.3.2), having l as starting link and m as terminal link, and s.t. *every* link of Φ different from l and m has depth (strictly) greater than p.

The conclusion follows then from the remark that the paths of $LPS(T)$ are exactly the paths of T, for every proof-net T. □

The following theorem states that a given proof-net (without weakenings) is "alone" in its (semantic) equivalence class, provided there exists for him an injective 1-obsessional experiment. This can be seen as a condition of "local injectivity".

2.4.8. THEOREM. *Let R be a proof-net, and suppose there exists an injective 1-experiment of R.*
Then, for every proof-net R' with the same conclusions as R, from $[\![R]\!] = [\![R']\!]$ it follows that $LPS(R) = LPS(R')$. Moreover, if R contains no weakening links, then $R = R'$.

Proof: Let R' be a proof-net with the same conclusions as R, let $h(R)$ (resp. $h(R')$) be the ?*co*-size of R (resp. R'), and let $n > max(h(R), h(R'))$.
By proposition 2.4.6, there exists an injective n-obsessional experiment of R. Let e_n be this experiment and γ its result. Because $[\![R]\!] = [\![R']\!]$, by proposition 2.2.7 there exists an n-obsessional experiment e'_n of R' with result γ. Corollary 2.3.8 gives then $OLPS(R) = OLPS(R')$ and $e_n|_{OLPS(R)} = e'_n|_{OLPS(R')}$. The injectivity of e_n allows then to conclude that $LPS(R) = LPS(R')$. When R contains no weakening links, by proposition 2.4.7 $R = R'$.
□

2.4.9. Remark. The reader might be interested in the following (delicate) point. One could think that if for a given fragment F of $MELL$ (without weakenings) the answer to our new question (does there exist an injective 1-experiment of every proof-net of F?) were positive, the analogue of remark 1.3.5 would apply (which, in presence of exponentials, would be more striking). But this is not precisely the case: indeed, suppose the answer is positive for F, let R be a proof-net of F, e_1 an injective 1-experiment of R, $n > h(R)$, and e_n the injective n-obsessional experiment of R induced by e_1. If R' is a proof-net, different from R, with the same conclusions as R and such that $h(R') > n$, the experiment e_n might be not enough to distinguish R from R': we only know the existence of $m > h(R')$ and of an injective m-obsessional experiment e_m such that the result of e_m is not an element of $[\![R']\!]$ (which means that e_m allows to distinguish R and R').
Contrary to the MLL case, we cannot conclude that for every proof-net of F, there exists an experiment containing *all the informations* contained in R.

2.4.10. Remark. It is rather natural to wonder whether the converse of theorem 2.4.8 holds. One cannot exclude that "some kind of converse" does, but for sure, strictly speaking, the answer is negative, as we now show.
Consider the (standard) proof-net R associated with the following sequent calculus proof:

$$
\begin{prooftree}
\AxiomC{$\vdash X, X^{\perp}$}
\UnaryInfC{$\vdash ?X, X^{\perp}$}
\UnaryInfC{$\vdash ?X, ?X^{\perp}$}
\RightLabel{$?W$}
\UnaryInfC{$\vdash ?X, ?X^{\perp}, ?A$}
\AxiomC{$\vdash X, X^{\perp}$}
\UnaryInfC{$\vdash ?X, X^{\perp}$}
\UnaryInfC{$\vdash ?X, ?X^{\perp}$}
\RightLabel{$?W$}
\UnaryInfC{$\vdash ?X, ?X^{\perp}, ?B$}
\BinaryInfC{$\vdash ?X, ?X, ?A \otimes ?B, ?X^{\perp}, ?X^{\perp}$}
\RightLabel{$?co$}
\UnaryInfC{$\vdash ?X, ?A \otimes ?B, ?X^{\perp}, ?X^{\perp}$}
\RightLabel{$?co$}
\UnaryInfC{$\vdash ?X, ?A \otimes ?B, ?X^{\perp}$}
\end{prooftree}
$$

There cannot exist any injective experiment of R: if x and y are the two elements of $|\mathcal{X}|$ that the experiment e of R associates with the two axiom links of R, one has $x \mathrel{\substack{\smile\\ \frown}} y(\mathcal{X})$ and $x \mathrel{\substack{\smile\\ \frown}} y(\mathcal{X}^{\perp})$, that is $x = y$. This is due to the presence of the two contraction links of R. The same phenomenon will be used (in a more subtle way) in section 4.2 to prove that coherent semantics is not injective for $MELL$.
However, R is alone in its semantic equivalence class: if R' is a proof-net with the same conclusions as R and such that $[\![R]\!] = [\![R']\!]$, then by theorem

2.3.12 one has $OLPS(R) = OLPS(R')$. Well, in this very special case this implies $R = R'$.
This example is clearly related to the fact, already mentioned several times, that a proof-net is not a graph but an equivalence class of graphs (remark A.2.2).

3 Injective experiments for $(?\wp)$LL

A $(?\wp)$LL formula A is any formula built in the following way:

$$A ::= \quad X \quad | \quad ?A \wp A \quad | \quad A \wp ?A \quad | \quad A \wp A \quad | \quad A \otimes A \quad | \quad !A \quad .$$

An $MELL$ proof-net is a $(?\wp)$LL proof-net when the types of its conclusions are all subformulas of $(?\wp)$LL formulas. We use for $(?\wp)$LL proof-nets the same conventions as for $MELL$ ones: in particular, every $(?\wp)$LL proof-net will be standard (unless explicitly mentioned).
The chapter is devoted to prove that for every proof-net of $(?\wp)$LL there exists an injective 1-experiment (proposition 3.1.4). This result will be used in chapter 4: it allows to prove injectivity of coherent semantics for $(?\wp)$LL and (then) for some remarkable fragments of $MELL$.

Here is the outline of the proof. With every $(?\wp)$LL proof-net R one can associate its "linearized" $L(R)$ (which contains no boxes), and then the set of proof-nets $L(R)^{\wp}$ (which in general might be empty) obtained from the proof-net without boxes $L(R)$ by "removing" the $\wp$ links of $L(R)$.
For every proof-net R of $(?\wp)$LL, there exists a proof-net of $L(R)^{\wp}$ containing only terminal contraction links. We prove that for such a proof-net of $L(R)^{\wp}$ there exists an injective experiment (proposition 3.2.4). We then prove that this implies the existence of an injective experiment of $L(R)$ (lemma 3.3.2), and that every injective experiment of $L(R)$ yields an injective 1-experiment of R (proposition 3.3.4).

3.1 Definitions and result

We define two operations on proof-nets: linearization and par-mutilation. The first one associates with every proof-net a proof-net without boxes, and the second one associates with some proof-nets without boxes a set of proof-nets without boxes and without $\wp$ links.
We show, at the end of the section, how these two operations are used to prove the main result of the chapter (proposition 3.1.4).

3.1.1 Linearization

With every proof-net R one can associate in a canonical way a (unique) proof-net without boxes which will be called **the linearized** of R and will be denoted by $L(R)$: it is the proof-net obtained by erasing all the connexions between the doors of the boxes of R and by erasing every of course and pax link of R.
The fact that $L(R)$ is a standard proof-net is obvious. We denote by L the application which associates with every formula A the formula $L(A)$, obtained by erasing every occurrence of the connective "!" in A. Let us point out that the types of the conclusions of R are not (in general) the types of the conclusions of $L(R)$: if the conclusions of R are of type Γ, then the ones of $L(R)$ will be of type $L(\Gamma)$ (with the usual convention for this kind of notation).

REMARK. (i) Let R be a proof-net. There exists a canonical application L which associates with every edge a of type A of R the edge $L(a)$ of type $L(A)$ of $L(R)$. Notice that L is far from being injective, if there is at least one box in R.
(ii) Let R be a proof-net. If e_L is an injective experiment of $L(R)$, then there exists at most one 1-experiment e of R, such that for every atomic edge α of R (then of $L(R)$) $e(\alpha) = e_L(\alpha)$. We will say that e is the **delinearized** of e_L. Obviously, if e exists it is injective.

We now turn our attention to the presence of $\wp$ links. We define a procedure of "$\wp$-mutilation", allowing to associate with some proof-nets R, a set $L(R)^{\wp}$ (defined in (ii) of the remark of subsection 3.1.2) of proof-nets, obtained from the proof-net without boxes $L(R)$ by "removing" the $\wp$ links of $L(R)$.

3.1.2 The procedure of par-mutilation

Recall that a $?co$ link of a proof-net is terminal when its conclusion is also a conclusion of the proof-net.

Let R be a proof-net without boxes, and let Γ be the sequent conclusion of R. Let $A \in \Gamma$ be s.t. $C \wp D$ is an *occurrence* of subformula of A, and let a of type A be a conclusion of R.
In order to be able to apply our procedure, we need that for every edge h of type H of G^R_a such that $C \wp D$ is a subformula of H, there exists an edge ξ_h of type $C \wp D$ of G^R_h. Intuitively, we are requiring that there is no weakening

above a "introducing" an occurrence of $C \wp D$: in the particular case of a proof-net R without weakenings, the procedure can always be applied.

Let then $a_1, \ldots, a_k$ be the edges of G_a^R of type $C \wp D$ (we are always speaking of the *occurrence* of subformula $C \wp D$ of A). Let $n_1, \ldots, n_k$ be the k $\wp$ links with conclusions, respectively, $a_1, \ldots, a_k$, let $c_1, \ldots, c_k$ (resp. $d_1, \ldots, d_k$) be the premises of type C (resp. D) of $n_1, \ldots, n_k$.

Let's call R' the graph obtained from R in the following way.

For every $i \in \{1, \ldots, k\}$, we perform the first three operations, and we then apply the fourth one to the graph thus obtained, in order to get the graph R':

1. we erase the link n_i and its conclusion a_i
2. if a_i is a premise of a link m_i, then we replace the premise a_i of m_i by d_i
3. if the formula C is different from $?F$ (for every F), then we add a dereliction link with premise c_i and conclusion an edge g'_i of type $?C$ (otherwise $g'_i = c_i$ will be of type C)
4. If $k \geq 2$, then we add to the graph obtained after the application of the operations 1-3 to R, a contraction link with premises $g'_1, \ldots, g'_k$ and conclusion the edge g (of type $?C$ or C).

The reader will easily convince herself/himself that the graph thus obtained is a standard proof-net whose sequent conclusion is $\Gamma \backslash A, A[D/(C \wp D)], ?C$ (or $\Gamma \backslash A, A[D/(C \wp D)], C$). One simply has to notice that, if some edges among the c_i are conclusions of a $?co$ link, then the $?co$ link of R' with conclusion g will have more than k premises[3].

By permuting the role played by the edges c_i and d_i, we clearly obtain a proof-net whose sequent conclusion is $\Gamma \backslash A, A[C/(C \wp D)], ?D$ (or $\Gamma \backslash A, A[C/(C \wp D)], D$).

We will say that the proof-net R' is obtained from R by mutilation of a $\wp$ formula.

Remark. (i) The procedure of $\wp$-mutilation just described might not make sense in presence of weakening links. Indeed, if $C \wp D$ is an occurrence of subformula of the type A of the conclusion a of R, if there is an edge $b \in G_a^R$ of type $?B$, s.t. $C \wp D$ is an occurrence of subformula of $?B$, and if b is the

[3]To be precise, one should also mention that some edges among the c_i might be concclusions of $?w$ links. In case $k \geq 2$, by definition of standard proof-net, these links and edges disappear.

conclusion of a weakening link, then the procedure of $\wp$-mutilation cannot be applied. This is precisely what is avoided by the hypotheses "for every edge h of type H of G_a^R such that $C \wp D$ is a subformula of H, there exists an edge ξ_h of type $C \wp D$ of G_h^R".

(ii) By applying several times (provided it is possible) the procedure of $\wp$-mutilation to a proof-net $L(R)$, we might obtain a proof-net R^- containing no $\wp$ links: R^- will only contain atomic axiom links, $?de$, $?co$, $?w$, $\otimes$ links. Of course (when it exists) R^- is not necessarily unique: we will denote in the sequel by $L(R)^{\wp}$ the set of proof-nets without $\wp$ links obtained from $L(R)$ by a sequence of $\wp$-mutilations, following the procedure previously described. If for a proof-net R, there exists no sequence of $\wp$-mutilations starting from $L(R)$ and leading to a proof-net without $\wp$ links, then we will have $L(R)^{\wp} = \emptyset$.

It is important to notice that every $?co$ link of a proof-net of $L(R)^{\wp}$ which is not a link of R, is a terminal link.

(iii) The reader might wonder why we distinguished (in the procedure of $\wp$-mutilation) the mutilation of a formula $C \wp D$ from the one of a formula $?C \wp D$ (where $C \neq ?F$ for every formula F). This is in order to remain in the subsystem $(? \wp)$LL of $MELL$ (more precisely to still satisfy the property (P) of lemma 3.1.3): the formula $??C$ is not contained in this subsystem.

Convention:

We will say in the sequel that an edge a is a premise of a link n "up to a link m", when a is a premise of n or when a is a premise of m and the conclusion of m is a premise of n. We will use a similar convention for the terminal edges of a proof-net.

We now show how linearization and $\wp$-mutilation are used to prove the main result of the chapter (proposition 3.1.4).

3.1.3. Lemma. *Let R be a proof-net without boxes. If R satisfies the following property:*

(P) every edge of type $?F$ (for some formula F of LL) is (up to a $?co$ link) premise of a $\wp$ link or a terminal edge, and every $\wp$ link has at least a premise which is not of type $?F$ (for every formula F of LL)

then there exists an injective experiment of R.

Proof: By induction on the number k of $\wp$ links of R.

If $k = 0$, then every edge of type $?F$ of R is terminal (up to a contraction link). Let R' be the proof-net obtained from R by removing all the (necessarily terminal) weakening links of R. Every contraction link of R' (if any) is terminal, and there are no $\wp$ links in R': by proposition 3.2.4 (which will be proven in section 3.2), there exists an injective experiment of R'. Then (obviously) there exists an injective experiment of R.
If $k > 0$, then let a be a conclusion of R such that there exists a $\wp$ link in G_a^R. And let n be a $\wp$ link of G_a^R which is (one among) the "closest" to a. This clearly makes sense (it is easy to measure the distance from a in the tree G_a^R). We claim that for the $\wp$ link n, we can apply the procedure of $\wp$-mutilation described in 3.1.2. Indeed, (i) of the remark of subsection 3.1.2 does not apply in this case, because every non terminal weakening link of R is the premise of a $\wp$ link, and n is among the closest to a. We now have to apply the procedure carefully: if (using the notations of 3.1.2), one among the two premises of the a_i (for example the c_i) is of type $C = ?E$, then we apply the procedure in such a way to obtain a proof-net with conclusion $\Gamma\backslash A, A[D/C \wp D], C$ (and not $\Gamma\backslash A, A[C/C \wp D], ?D$). To apply the induction hypothesis, let's first check that the proof-net thus obtained (let's call it R'') still satisfies (P): this is true, because (by hypothesis) in any case $D \neq ?F$, for every formula F. We then obtain an injective experiment of R'', and by lemma 3.3.2 (which will be proven in section 3.3), this yields an injective experiment of R. $\square$

3.1.4. Proposition. *If R is a $(?\wp)$LL proof-net, then there exists an injective 1-experiment of R.*

Proof: By definition of $(?\wp)$LL, the linearized $L(R)$ of R satisfies the hypotheses of lemma 3.1.3: there exists then an injective experiment of $L(R)$. To conclude, we simply have to apply proposition 3.3.4 (which will be proven in section 3.3). $\square$

3.1.5. Remark. Notice that we have implicitly proven (in the proof of lemma 3.1.3) that if R is a $(?\wp)$LL proof-net, there always exists a proof-net of $L(R)^{\wp}$ containing only terminal $?co$ links.

The two following sections are devoted to prove the three results mentioned at the beginning of chapter 3 and used to prove the previous proposition (proposition 3.2.4 of section 3.2, lemma 3.3.2 of section 3.3, proposition 3.3.4 of section 3.3).

3.2 The case of terminal contraction links

We prove that for every proof-net without boxes, without weakenings, without $\wp$ links, and such that it contains only terminal contraction links, there exists an injective experiment (proposition 3.2.4).
We would like the reader to notice the role played in the proof by the connectivity of our proof-nets (see also remark A.2.6).

In the present section, a proof-net will always be a proof-net without boxes, without weakenings, and without $\wp$ links.

If α is an edge of atomic type conclusion of the axiom link n of the proof-net R, we will denote in the sequel by $\alpha^{\perp}$ the edge conclusion of n different from α.

3.2.1. REMARK. Let a and a' be two (different) edges of the same type of the proof-net R. Because there are no weakening links in R, case (3) of lemma 2.4.4 has to be excluded. Moreover, if $a \in G_{a'}$ or if $a' \in G_a$, then (because there are no boxes in R) necessarily a or a' is the conclusion of a $?co$ link. The lemma just mentioned tells us that for every injective experiment e of R, $e(a) \neq e(a')$. This means that we have only two possibilities for the coherence relation between $e(a)$ and $e(a')$: $e(a) \smile e(a')$ or $e(a) \frown e(a')$.
An injective experiment of a proof-net R associates then with a pair of different edges of the same type one of the two elements of the set $\{\frown, \smile\}$. We will denote by $e(a, a')$ the value "$\frown$" or "$\smile$" of e on the pair of edges of the same type $\{a, a'\}$. By definition of experiment (1.1.1) the value of the injective experiment e on a pair $\{a, a'\}$ of edges depends only on the value of e on the pairs of *atomic* edges of the same type (actually on the pairs of "similar" atomic edges, see definition 3.2.9) $\{\alpha, \alpha'\}$, where $\alpha \in G_a$ and $\alpha' \in G_{a'}$. The data of an injective experiment of a proof-net R can be seen as the data of a function which associates with every pair of edges of the same atomic type one of the two elements of the set $\{\frown, \smile\}$, with the *only* constraint that if α and α' are two edges of the same atomic type, the value of the function on the pair $\{\alpha, \alpha'\}$ is different from the value of the function on the pair $\{\alpha^{\perp}, \alpha'^{\perp}\}$.
Notice also that one can define a would-be experiment e' from an experiment e by modifying the coherence relation between the edges of the same atomic type α and α', with the only effect that we also modify the coherence relation between the edges $\alpha^{\perp}$ and $\alpha'^{\perp}$.

3.2.2. REMARK. We will use in the sequel the term "pair" always meaning "ordered pair", (unless we write "unordered pair"). Actually, the use of unordered pairs would be more suitable for our purposes, but it turns out that to handle ordered pairs is easier.

3.2.3. REMARK. Let R be a proof-net. With every axiom link l with conclusions α_l and $\alpha_l^\perp$ one associates an element x_l of the web of the coherent space $\mathcal{A}$ in such a way that if $l \neq l'$ then $x_l \neq x_{l'}$.
We want to know whether there exists an experiment e of R s.t. for every axiom link l one has $e(\alpha_l) = e(\alpha_l^\perp) = x_l$. In other terms, we wonder whether or not the labeling of R's edges induced by the previous assignment of labels to the atomic edges of R is an experiment.
We only have to beware $?co$ links: if we call e this labeling of R, e is an experiment of R *if and only if* for every $?co$ link of R with premises $a_1, \ldots, a_k$ one has $e(a_i) \smile e(a_j)$, $\forall i, j \in \{1, \ldots, k\}$.

We want to prove the following

3.2.4. PROPOSITION. *If R is a proof-net having only terminal $?co$ links, then there exists an injective experiment of R.*

Proof: It is a consequence of proposition 3.2.5. □

The following proposition is only apparently stronger than the previous one: the reader can easily see that they are in fact equivalent. We will prove this second statement at the very end of the present section.

3.2.5. PROPOSITION. *Let R be a proof-net having only terminal $?co$ links. There exists an injective experiment e of R such that for every pair (a, a') of edges of the same type A and conclusions of R, one has $e(a) \smile e(a')(\mathcal{A})$.*

3.2.6. REMARK. If R is a proof-structure satisfying condition (AC) of definition A.2.5 but not condition (ACC) of remark A.2.6, then the previous proposition is wrong (in general). A simple counterexample is given by the proof-structure consisting in two axiom links whose conclusions have the same types.
The failure of proposition 3.2.5 in the absence of connectivity will be used in section 4.2.

Convention: *In the rest of the present section*, every proof-net will contain no contraction links. In other terms, in our proof-nets we now have only axiom links, dereliction links and tensor links.

Let's start by familiarizing a little with the objects that are now our proof-nets. In the sequel, we will constantly switch from a proof-net to its "tree-like representation", introduced in the following remark.

3.2.7. REMARK. **(Tree-like representation).** A proof-net with conclusions the edges $a_1, \ldots, a_n$ is made of n "blocks" (the graphs $G_{a_1}, \ldots, G_{a_n}$), connected by some axiom links (and their conclusions). This case looks like the purely multiplicative one of section 1.3, except the fact that in our proof-nets there is a unique correctness graph (see definition A.2.4), which is connected: the proof-net itself.
This allows to represent our proof-nets in an alternative way: every block is a node, and every axiom link with conclusions α and $\alpha^{\perp}$ is an edge connecting two nodes. If R is a proof-net, we will refer to this representation as the "tree-like representation" of R, and we will denote it by $R^{\star}$. The motivation for this way of representing nets lies in the behaviour of the coherence relation w.r.t. the connectives $\otimes$ and ?, as explained in remark 3.2.11.
Clearly, for every proof-net R, the graph $R^{\star}$ is a tree (i.e. it is acyclic and connected).
If a and b are two edges conclusions of R, if α and $\alpha^{\perp}$ are two edges of atomic type conclusions of the same axiom link such that $\alpha \in G_a$ and $\alpha^{\perp} \in G_b$, then we will still denote by a and b the nodes of $R^{\star}$ corresponding to the blocks G_a and G_b, and we will denote by $\alpha\alpha^{\perp}$ (resp. $\overrightarrow{\alpha\alpha^{\perp}}$) the unoriented (resp. oriented) edge of $R^{\star}$ connecting a and b (resp. with source a and target b).
If Λ and Λ' are two oriented paths of $R^{\star}$ s.t. the last node of Λ is the first node of Λ', then we will denote by $\Lambda \star \Lambda'$ the oriented path of $R^{\star}$ having as first (resp. last) node the first (resp. the last) node of Λ (resp. Λ').

3.2.8. DEFINITION. **(Paths of $R^{\star}$).** If a and b are two edges conclusions of R, then there exists a unique unoriented path (which will be denoted by $\Theta_{a,b}$) of $R^{\star}$ connecting a and b. We will denote by $\overrightarrow{\Theta_{a,b}}$ (resp. $\overrightarrow{\Theta_{b,a}}$) the oriented path of $R^{\star}$ having as first node a (resp. b) and as last node b (resp. a).

If a_1 and a_2 are two edges of a proof-net, G_{a_1} and G_{a_2} are now trees (instead of equivalence classes of trees). This implies that when a_1 and a_2 have the same type, there exists an obvious correspondence between G_{a_1} and G_{a_2}. Two edges "corresponding each other" are said to be similar. The following (horrible) definition makes this notion more precise.

3.2.9. DEFINITION. **(Similar edges).** Let A be a formula containing only the connectives "?" and "$\otimes$", and let A' be any formula obtained from A by changing the name of its propositional variables in such a way that every propositional variable of the language occurs at most once in A'.
Let a_1 and a_2 be two edges of type A of the proof-net R. Let G'_{a_1} be the tree obtained from G_{a_1} by changing the types of the atomic edges of G_{a_1} in such a way that now the edge a_1 is of type A'. We consider here that the edges of G'_{a_1} are the same as the ones of G_{a_1}, but with different types. We define G'_{a_2} in the same way.
Let c_1 (resp. c_2) be an edge of G_{a_1} (resp. of G_{a_2}). We will say that c_1 and c_2 are similar when, as edges of G'_{a_1} and G'_{a_2}, c_1 and c_2 have the same type.

3.2.10. REMARK. Let a_1 and a_2 be two edges conclusions of the proof-net R. If c_1 (resp. c_2) is an edge of G_{a_1} (resp. of G_{a_2}), and if c_1 and c_2 are similar, then a_1 and a_2 are also similar (i.e. a_1 and a_2 are two conclusions of the same type).

3.2.11. REMARK. By definition of the coherence relation in the spaces $\mathcal{A} \otimes \mathcal{B}$ and $?\mathcal{B}$, for every proof-net R and for every injective experiment e of R, we have:
for every pair (a, a') of conclusions of the same type of R, $e(a) \smile e(a')$ iff there exists a pair $(\alpha, \alpha') \in G_a \times G_{a'}$ of similar edges of atomic type s.t. $e(\alpha) \smile e(\alpha')$.

3.2.12. DEFINITION. **((C)-pair).** Let R be a proof-net and let a and a' be two conclusions of R of the same type.
We will say that the pair $(\alpha, \alpha') \in G_a \times G_{a'}$ of similar edges of atomic type is a (C)-pair for (a, a') when the path $\Theta_{a,a'}$ of $R^\star$ connecting a and a' contains the edge $\alpha\alpha^\perp$ or/and the edge $\alpha'\alpha'^\perp$.

3.2.13. REMARK. For every pair (a, a') of conclusions of the same type of a proof-net R, there always exists a (C)-pair for (a, a'), and there exist at most two.

Notice that when the type of a and a' contains no occurrence of the connective $\otimes$, there will be a unique (C)-pair for (a, a').
Notice also that if (α, α') is the unique (C)-pair for (a, a'), then $(\alpha^\perp, \alpha'^\perp)$ cannot be a (C)-pair (for any pair of conclusions of R). This remark is important: it will be used in the proof of proposition 3.2.17.

3.2.14. DEFINITION. **(Pair of similar paths).** Let R be a proof-net and let a, a' be two edges of the same type and conclusions of R. Let n be a strictly positive integer. Let Φ (resp. Φ') be an oriented path of $R^\star$ with starting node a (resp. a') and whose edges are $\overrightarrow{\alpha_1\alpha_1^\perp}, \ldots, \overrightarrow{\alpha_n\alpha_n^\perp}$ (resp. $\overrightarrow{\alpha'_1\alpha'^\perp_1}, \ldots, \overrightarrow{\alpha'_n\alpha'^\perp_n}$).
We will say that (Φ, Φ') is a pair of similar paths starting from (a, a') when:
- the edges α_i and α'_i are similar $\forall i \in \{1, \ldots, n\}$
- the edges $\alpha_i^\perp$ and $\alpha'^\perp_i$ are similar $\forall i \in \{1, \ldots, n-1\}$.
We will say that the pair of similar paths (Φ, Φ') is maximal, when the edges $\alpha_n^\perp$ and $\alpha'^\perp_n$ are not similar.

The following proposition is a consequence of the connectivity of the tree $R^\star$, and it is the essential ingredient of the proof of proposition 3.2.17.

3.2.15. PROPOSITION. **(Maximal pair of similar paths).** Let a and a' be two conclusions of the proof-net R of the same type, and suppose that there exist for (a, a') two (C)-pairs.
There exists then a maximal pair of similar paths (Φ, Φ') starting from (a, a') and s.t. Φ is a prefix of the oriented path $\overrightarrow{\Theta_{a,a'}} \star \Phi'$.

Proof: Let (α, α') be a (C)-pair for (a, a') and suppose that $\overrightarrow{\alpha\alpha^\perp}$ is the first edge of $\overrightarrow{\Theta_{a,a'}}$. Let's then call $\overrightarrow{\alpha_n\alpha_n{}^\perp}$ the last edge of $\overrightarrow{\Theta_{a,a'}}$: because there are two (C)-pairs for (a, a'), one has $\alpha' \neq \alpha_n{}^\perp$.
Let b (resp. b') be the edge conclusion of R s.t. $\alpha^\perp \in G_b$ (resp. $\alpha'^\perp \in G_{b'}$). If $\alpha^\perp$ and $\alpha'^\perp$ are not similar, then we are done. If they are similar, then b and b' are two conclusions of the same type of R (by remark 3.2.10). Because $\alpha' \neq \alpha_n{}^\perp$, the node b' is not a node of $\overrightarrow{\Theta_{a,a'}}$ and the edge $\overrightarrow{\alpha'\alpha'^\perp}$ is not an edge of $\overrightarrow{\Theta_{a,a'}}$. More precisely, we have $\overrightarrow{\Theta_{a,a'}} \star \overrightarrow{\alpha'\alpha'^\perp} = \overrightarrow{\alpha\alpha^\perp} \star \overrightarrow{\Theta_{b,b'}}$.
Let $\overrightarrow{\beta\beta^\perp}$ be the first edge of $\overrightarrow{\Theta_{b,b'}}$: we have $\beta \neq \alpha^\perp$. Let $\beta' \in G_{b'}$ be the edge similar to β: we have $\beta' \neq \alpha'^\perp$. The pair (β, β') is a (C)-pair for (b, b'), and

there exist two (C)-pairs for (b, b'): (β, β') and $(\alpha^\perp, \alpha'^\perp)$. We are then back to the original situation, this time with the pair (b, b'). The reason why we will have to stop one day, is that $R^\star$ is a tree.
The situation is represented in figure 1, where the nodes of the tree $R^\star$ $(a, a', b, b', \ldots)$ are (in general) connected with several nodes, but we only drew the ones we were concerned with.

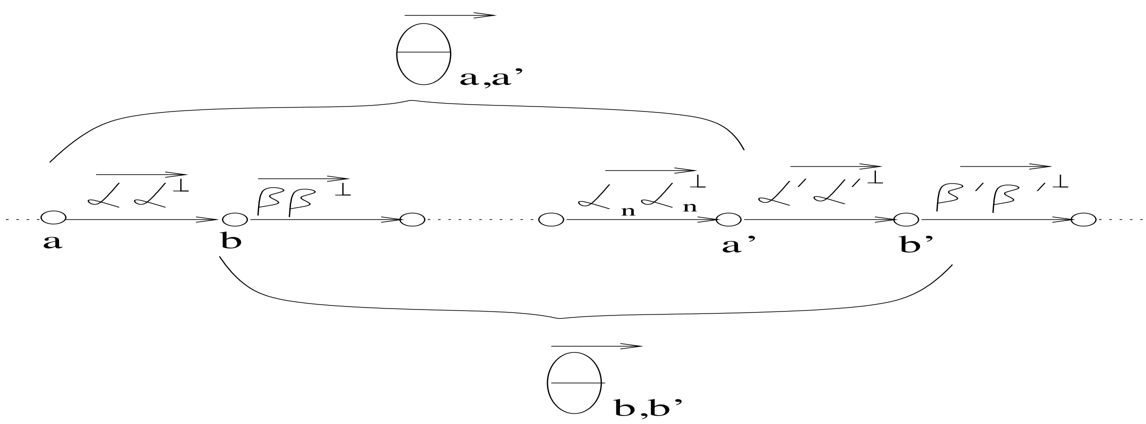


Figure 1. Construction of the maximal pair of similar paths starting from (a, a').

It is (very) easy to give a formal version of the previous proof: one can for example define a size $||\cdot||$ on the oriented paths of $R^\star$ and argue by induction on this size, proving that $|| \overrightarrow{\Theta_{a,a'}} || > || \overrightarrow{\Theta_{b,b'}} ||$. □

We will use the following remark in the proof of the following proposition.

3.2.16. Remark. Let's use the notations of the previous proposition. Let c and c' be two conclusions of the same type of R. If $(\gamma, \gamma') \in G_c \times G_{c'}$ is a pair of similar edges of atomic type s.t. $\overrightarrow{\gamma\gamma^{\perp}}$ (resp. $\overrightarrow{\gamma'\gamma'^{\perp}}$) is an edge of Φ (resp. Φ'), then (γ, γ') is a (C)-pair for (c, c').

The following proposition entails proposition 3.2.5 (and then proposition 3.2.4 which we want to prove). It shows that the solution to our problem for a pair of conclusions of the same type (a, a') is given *precisely* by the pairs of edges "participating to the connection between a and a'": the (C)-pairs.

3.2.17. PROPOSITION. *Let R be a proof-net. There exists an injective experiment e of R such that:*
for every pair (a, a') of edges of the same type and conclusions of R, there exists a (C)-pair (α, α') for (a, a') satisfying $e(\alpha) \smile e(\alpha')$ (and then $e(a) \smile e(a')$ by remark 3.2.11).

Proof: Let h be the number of unordered pairs of edges of the same type and conclusions of the proof-net R. Let e be any injective experiment of R (which obviously exists because there are no $?co$ links in R). Let k_e be the number of unordered pairs of edges of the same type $\{c, c'\}$ conclusions of R such that there exists a (C)-pair $(\alpha_c, \alpha'_c) \in G_c \times G_{c'}$ s.t. $e(\alpha_c) \smile e(\alpha'_{c'})$.
We prove, by induction on $h - k_e$, that there exists an injective experiment e' of R satisfying the conclusion of the proposition.
If $h - k_e = 0$, then e is the injective experiment of R we look for.
Otherwise, there exists a pair (a, a') of conclusions of the same type of R s.t. for the (C)-pair(s) $(\alpha_a, \alpha_{a'})$ for (a, a'), one has $e(\alpha_a) \frown e(\alpha'_{a'})$. We now build, starting from e, an injective experiment e' of R s.t. $k_{e'} > k_e$. We will then conclude by applying the induction hypothesis.
Let's fix a (C)-pair (α, α') for (a, a'). If (α, α') is the unique (C)-pair for (a, a'), then (see remark 3.2.13) $(\alpha^{\perp}, \alpha'^{\perp})$ cannot be a (C)-pair. In this case, it is enough to define the experiment e' as the experiment e except on $\{\alpha, \alpha'\}$ and on $\{\alpha^{\perp}, \alpha'^{\perp}\}$: in this case we define $e'(\alpha) \smile e'(\alpha')$ (and then $e'(\alpha^{\perp}) \frown e'(\alpha'^{\perp})$). We indeed have $k_{e'} = k_e + 1$.
If (α, α') is not the unique (C)-pair for (a, a'), then one can apply proposition 3.2.15: let (Φ, Φ') be the maximal pair of similar paths starting from (a, a') s.t. Φ is a prefix of $\overrightarrow{\Theta_{a,a'}} \star \Phi'$.
Suppose that Φ and Φ' contain $k + 1$ different edges ($k \geq 0$). Let's call $(b_1, b'_1), \ldots, (b_k, b'_k)$ the pairs of edges conclusions of R and $(\beta_1, \beta'_1), \ldots, (\beta_k, \beta'_k)$ the pairs of atomic edges of R s.t.:
- the nodes crossed by Φ' are, successively, $a', b'_1, \ldots, b'_k, b'_{k+1}$
- the nodes crossed by Φ are, successively, $a, b_1, \ldots, b_k, b_{k+1}$

- $\beta'_i \in G_{b'_i}$ (resp. $\beta_i \in G_{b_i}$) and $\beta'^{\perp}_i \in G_{b'_{i+1}}$ (resp. $\beta^{\perp}_i \in G_{b_{i+1}}$), for $i \in \{1, \ldots, k\}$
- $\forall i \in \{1, \ldots, k\}$ (resp. $\forall i \in \{1, \ldots, k-1\}$), β'_i and β_i (resp. $\beta'^{\perp}_i$ and $\beta^{\perp}_i$) are similar, while $\beta'^{\perp}_k$ and $\beta^{\perp}_k$ are not similar
- $\overrightarrow{\alpha\alpha^{\perp}}$ (resp. $\overrightarrow{\alpha'\alpha'^{\perp}}$) is the edge of Φ' (resp. of Φ) connecting a to b_1 (resp. a' to b'_1)
- $\forall i \in \{1, \ldots, k\}$, $\overrightarrow{\beta'_i\beta'^{\perp}_i}$ (resp. $\overrightarrow{\beta_i\beta^{\perp}_i}$) is the edge of Φ' (resp. of Φ) connecting b'_i and b'_{i+1} (resp. b_i and b_{i+1}).

We then define e' on every unordered pair of atomic edges of the same type $\{\delta, \delta'\}$ of R in the following way:

- if $\{\delta, \delta'\} \notin \{\{\alpha, \alpha'\}, \{\alpha^{\perp}, \alpha'^{\perp}\}\} \cup \{\{\beta_i, \beta'_i\}, \{\beta^{\perp}_i, \beta'^{\perp}_i\} : i \in \{1, \ldots, k\}\}$, then $e'(\delta, \delta') = e(\delta, \delta')$
- $e'(\alpha) \smile e'(\alpha')$ (then $e'(\alpha^{\perp}) \frown e(\alpha'^{\perp})$)
- $\forall i \in \{1, \ldots, k\}$, $e'(\beta_i) \smile e'(\beta'_i)$ (then $e'(\beta^{\perp}_i) \frown e'(\beta'^{\perp}_i)$).

Let's show now that $k_{e'} > k_e$.
If $\{c, c'\} \notin \{\{a, a'\}, \{b_1, b'_1\}, \ldots, \{b_k, b'_k\}\}$, then $\forall \delta \in G_c$ and $\forall \delta' \in G_{c'}$ where δ and δ' are similar atomic edges, we have $e'(\delta, \delta') = e(\delta, \delta')$ (remember that $\beta'^{\perp}_k \in G_{b'_{k+1}}$ and $\beta^{\perp}_k \in G_{b_{k+1}}$ are not similar). This means that if (δ, δ') is a (C)-pair for (c, c') s.t. $e(\delta) \smile e(\delta')$, it will still be the case for e'.
If $\{c, c'\} \in \{\{b_1, b'_1\}, \ldots, \{b_k, b'_k\}\}$, then by remark 3.2.16 we know that $\forall i \in \{1, \ldots, k\}$ the pair (β_i, β'_i) is a (C)-pair for (b_i, b'_i), and by definition of e' we have $e'(\beta_i) \smile e'(\beta'_i)$.
If $(c, c') = (a, a')$, then we have by definition of e': $e(\alpha) \smile e(\alpha')$ (where (α, α') is a (C)-pair).
In any case, we clearly have $k_{e'} \geq k_e + 1$. □

Proof (of proposition 3.2.5): Let R be a proof-net having only terminal $?co$ links, and let R' be the subproof-net of R obtained from R by erasing all the (terminal) $?co$ links of R and their conclusions.
By proposition 3.2.17, there exists an injective experiment e' of R' satisfying the conclusion of proposition 3.2.5. This experiment can be straightforwardly extended into an experiment e of R satisfying the conclusion of proposition 3.2.5. □

3.3 Adding par links and boxes

We proved in section 3.1 (remark 3.1.5) that with every proof-net R of $(?\wp)$LL, one could associate a proof-net of $L(R)^{\wp}$ containing only terminal $?co$ links. The previous section 3.2 (and precisely proposition 3.2.4) allows to conclude that for such a net there exists an injective experiment. The point is now to follow the path leading from R to $L(R)^{\wp}$ in the opposite direction, and prove that along this "reverted" path the property we are interested in (the existence of an injective experiment) is preserved.
In other terms, we fill the last two holes in the proof of proposition 3.1.4: we show that if there exists an injective experiment of an element of $L(R)^{\wp}$, then there exists an injective experiment of $L(R)$ (lemma 3.3.2), and that in this last case there exists also an injective 1-experiment of R (proposition 3.3.4).

Let R be a proof-net and $L(R)$ its linearized. Remember that because $L(R)$ is without boxes, an experiment of $L(R)$ associates with every edge of $L(R)$ a unique label: it is a labeling of the edges of $L(R)$ (like in the multiplicative case).

3.3.1. REMARK. If R is a proof-net without boxes, one can associate with every axiom link l with conclusions α_l and $\alpha_l^{\perp}$ an element x_l of the web of the coherent space $\mathcal{A}$ in such a way that if $l \neq l'$ then $x_l \neq x_{l'}$.
We can extend remark 3.2.3, but not straightforwardly, due to the possible presence, in R, of some weakening links. The labeling e of R's edges induced by the previous assignment of labels to the atomic edges of R is an experiment of R *if and only if* for every $?co$ link of R with premises $b_1, \ldots, b_h$ one has $e(b_i) \asymp e(b_j)$, $\forall i, j \in \{1, \ldots, h\}$.

3.3.2. LEMMA. *Let R be a proof-net without boxes, and let R' be a proof-net obtained from R by mutilation of a $\wp$ formula.*
If there exists an injective experiment e' of R', then there exists an injective experiment e of R.

Proof: We use in the proof the same notations as in the definition 3.1.2 of the procedure of $\wp$-mutilation.
In particular, we will suppose that the sequent conclusion of R (resp. R') is Γ (resp. $\Gamma \backslash A, A[D/(C \wp D)], ?C$, or $\Gamma \backslash A, A[D/(C \wp D)], C$). We will also assume that none of the c_i is the conclusion of a $?w$ link (remember the

footnote in the definition of the procedure of $\wp$-mutilation), leaving it to the reader to extend the proof to this case.

Every edge b of R different from $a_1, \ldots, a_k$ is an edge of R' which will be denoted by b'. The reader should notice that the edges b and b' are not necessarily of the same type: if B is the type of the edge b' of R', then the edge b of R is either of type B or of type $B[(C \wp D)/D]$.

We are going to show that the experiment e of R that we look for is nothing but "the experiment e' defined on the edges of R in the only possible way".

To define correctly the experiment e, we introduce a terminology which will only be used in the present proof. We will say that an edge b of R s.t. $b \notin \{a_1, \ldots, a_k\}$ is "special", when there exists a path (of course, a straight path following definition A.3.2) containing b, starting from an edge among the a_i ($i \in \{1, \ldots, k\}$) and going downwards to a conclusion of R. (Actually, this conclusion is always the edge a of the procedure 3.1.2).

Notice that every special edge b of R is also an edge b' of R', having a type different from the type of b.

For every edge b of R:

- if b is not special and $b \notin \{a_1, \ldots, a_k\}$, then we define $e(b) = e'(b')$
- if $b = a_i$ for some $i \in \{1, \ldots, k\}$, then we define $e(a_i) = (e'(c_i'), e'(d_i'))$
- if b is a special edge, then the definition of $e(b)$ follows from the previous ones (by definition 1.1.1).

What we have to prove is that the labeling e of R thus defined is indeed an experiment of R. By remark 3.3.1, the only possibility for our labeling not to be an experiment is the presence of a contraction link m of R having as conclusion the edge q and as premises the edges $q_1, \ldots, q_h$, such that $e(q_l) \frown e(q_s)$ for some $l, s \in \{1, \ldots, h\}$. If none of the edges $q_1, \ldots, q_h$ is special, then because for every $l, s \in \{1, \ldots, h\}$ we have $e'(q_l') \stackrel{\smile}{\frown} e'(q_s')$, we also have (by definition of e), $e(q_l) \stackrel{\smile}{\frown} e(q_s)$. Otherwise, *all* the edges $q_1, \ldots, q_h$ (so as q) are special: we then simply have to show that in this case $\forall l, s \in \{1, \ldots, h\}$ one has $e(q_l) \stackrel{\smile}{\frown} e(q_s)$. Notice that the edges $q_1', \ldots, q_h'$ are premises of a $?co$ link of R'.

We are going to prove that $\forall i, j \in \{1, \ldots, k\}$, we have the following property $(\star)$:

($\star$.1) if $e'(d_i') \smile e'(d_j')$, then $e(a_i) \smile e(a_j)$

($\star$.2) if $e'(d_i') = e'(d_j')$, then $e(a_i) \stackrel{\smile}{\frown} e(a_j)$.

Let us first show that this implies that $\forall l, s \in \{1, \ldots, h\}$ we have $e(q_l) \overset{\smile}{\frown} e(q_s)$ (and we are done, because the only problematic case, mentioned above, cannot occur).

Let's fix two distinct edges t_1 and t_2 of the same type T, both special, and such that $G_{t_1} \cap G_{t_2} = \emptyset$. Intuitively, $(\star)$ says that if $e'(t'_1) \overset{\smile}{\frown} e'(t'_2)$ and if the coherence relation between $e'(d'_i)$ and $e'(d'_j)$ has anything to do with the fact that $e'(t'_1) \overset{\smile}{\frown} e'(t'_2)$, then the coherence relation between $e(a_i)$ and $e(a_j)$ will play the same role, and we will then be able to claim that $e(t_1) \overset{\smile}{\frown} e(t_2)$. More precisely, we prove that:

- if $e'(t'_1) \smile e'(t'_2)$, then $e(t_1) \smile e(t_2)$
- if $e'(t'_1) = e'(t'_2)$, then $e(t_1) \overset{\smile}{\frown} e(t_2)$.

From $e'(q'_l) \overset{\smile}{\frown} e'(q'_s)$, we will then be able to deduce that $e(q_l) \overset{\smile}{\frown} e(q_s)$.

What we want to prove is rather clear, but the only way we see to give a convincing proof is to argue by induction. For every edge b of R, let $\sharp\Phi_b$ be the number of edges of the path having b as first edge and a conclusion of R as terminal edge. We make a proof by induction on $p = \Sigma_{i=1}^{k} \sharp\Phi_{a_i} - (\sharp\Phi_{t_1} + \sharp\Phi_{t_2})$. The case $p = 0$ has to be excluded. Let then $p > 0$.

One has to check all the possible cases for T, and we won't. As an example, we consider the case $T = U \wp S$. Let u_1 and u_2 be the premises of type U and let s_1 and s_2 be the premises of type S of the links $\wp$ whose conclusions are, respectively, t_1 and t_2. By definition $e(t_1) = (e(u_1), e(s_1))$ and $e(t_2) = (e(u_2), e(s_2))$. Because t and t' are special, exactly one among u_1 and s_1 and one among u_2 and s_2 is a special edge of R. Suppose for example that u_1 (and then u_2) are special edges. Then s_1 and s_2 aren't, and we have that $e(s_1) = e'(s'_1)$ and $e(s_2) = e'(s'_2)$. If u_1 is not one among $a_1, \ldots, a_k$, then u_2 neither. And the result is a straightforward application of the induction hypothesis. If u_1 is one among $a_1, \ldots, a_k$, then u_2 also is one among $a_1, \ldots, a_k$ (and this is the interesting case). In this case, one of the two premises of each of the two $\wp$ links of R' having t'_1 and t'_2 as conclusions is then one of the edges d'_i $(i \in \{1, \ldots, k\})$: the premises of these links are the edges d'_{i_1} and s'_1 (for t'_1) and d'_{i_2} and s'_2 (for t'_2), where of course $i_1, i_2 \in \{1, \ldots, k\}$. The coherence relation $(\frown, \smile, =)$ between $e(s_1)$ and $e(s_2)$ is then the same as the one between $e'(s'_1)$ and $e'(s'_2)$. The property $(\star)$ allows then to conclude. Indeed, if $e'(t'_1) \smile e'(t'_2)$ then either $e'(d'_{i_1}) \smile e'(d'_{i_2})$ and $e'(s'_1) \overset{\smile}{\frown} e'(s'_2)$, or $e'(d'_{i_1}) \overset{\smile}{\frown} e'(d'_{i_2})$ and $e'(s'_1) \smile e'(s'_2)$: in both cases we have by $(\star)$ that $e(t_1) \smile e(t_2)$. While if $e'(t'_1) = e'(t'_2)$, we have that $e'(d'_{i_1}) = e'(d'_{i_2})$

and $e'(s'_1) = e'(s'_2)$, and then by $(\star)$ $e(a_{i_1}) \stackrel{\smile}{\frown} e(a_{i_2})$ and $e(s_1) = e(s_2)$: then $e(t_1) \stackrel{\smile}{\frown} e(t_2)$.
To conclude, it is then enough to prove $(\star)$. Remember that $\forall i \in \{1, \dots, k\}$ the edges c_i and d_i (premises of the link n_i with conclusion a_i) are not special, which means that $e(c_i) = e'(c'_i)$ and $e(d_i) = e'(d'_i)$.
The property $(\star)$ is actually a consequence of the fact that $\forall i, j \in \{1, \dots, k\}$ the edges c'_i and c'_j of R' are premises (up to a $?de$ link) of a $?co$ link: we have $e'(c'_i) \stackrel{\smile}{\frown} e'(c'_j)$ and then $e(c_i) \stackrel{\smile}{\frown} e(c_j)$. Let then $i, j \in \{1, \dots, k\}$. By definition of the coherence relation in the space $\mathcal{C} \wp \mathcal{D}$, we have:
$(\star.1)$ if $e'(d'_i) \smile e'(d'_j)$, then $e(d_i) \smile e(d_j)$ and $e(c_i) \stackrel{\smile}{\frown} e(c_j)$: then $e(a_i) \smile e(a_j)$.
$(\star.2)$ if $e'(d'_i) = e'(d'_j)$, then $e(d_i) = e(d_j)$ and $e(c_i) \stackrel{\smile}{\frown} e(c_j)$: then $e(a_i) \stackrel{\smile}{\frown} e(a_j)$.
□

We have proven that "reverting" the operation of $\wp$ mutilation preserves the existence of injective experiments. We are going to proceed in a similar way with the operation of linearization (defined in 3.1.1).

3.3.3. LEMMA. *Let R be a proof-net, and let a and a' be two different edges of the same type A of R. Let e_L be an injective experiment of $L(R)$ and e the delinearized of e_L (this means that we are supposing the existence of e).*
If $e_L(L(a)) \smile e_L(L(a'))(\mathcal{L}(\mathcal{A}))$, then $e(a) \smile e(a')(\mathcal{A})$.

Proof: Like in case of the lemmas of section 2.4, the proof is a simple application of the definition of coherence in the spaces interpreting LL formulas. One argues, as usual, by induction on $s(G_a^R) + s(G_{a'}^R)$, and one uses lemma 2.4.4. The details are left to the reader. □

3.3.4. PROPOSITION. *Let R be a proof-net. If there exists an injective experiment e_L of $L(R)$, then there exists an injective 1-experiment e of R: it is the delinearized of e_L.*

Proof: Like in the proof of proposition 2.4.6, we argue by induction on a sequentialization π of R. The more significant case will again be when the last rule of π is a contraction rule, and we will apply in this case lemma 3.3.3 (and lemma 2.4.4). □

4 Positive and negative results

At the beginning of section 2.4, we replaced our original problem (1.2.3) by the question of the existence of a 1-injective experiment for a given set of proof-nets.
We gave a positive answer to this question for $(?\,\wp)$LL in the previous chapter. We now prove that this yields a positive answer to problem 1.2.3 for $(?\,\wp)$LL (section 4.1). In section 4.2, we give a negative answer to the question above for $MELL$, and we show how this yields a counterexample allowing to answer negatively to problem 1.2.3.
We end the paper by a table summing up the results obtained and the (seemingly) interesting conjectures (section 4.3).

4.1 Fragments of injectivity

We now plug together the results of the previous chapters and present the positive outcomes of the paper: injectivity of multiset-based coherent semantics is proven for $(?\,\wp)$LL (theorem 4.1.2) and for the "weakly polarized" fragment of LL (theorem 4.1.5). This last result yields a proof of injectivity of the coherent model of the simply typed λ-calculus, with a bound on the cardinality of the model which separates two λ-terms that are not $\beta\eta$-equivalent (theorem 4.1.7).
Appendix B guarantees that all the injectivity results just mentioned for multiset-based coherent semantics also hold in the relational case.

We forget (only in the present section) the conventions used up to now for proof-nets.
Let R be an $MELL$ proof-net. We denote by R_0 the standard proof-net associated with the normal form of R (definition 1.2.1). We use the notation $\simeq_{\beta\eta}$ introduced in 1.2.3.

4.1.1. THEOREM. *Let R be an $MELL$ proof-net such that R_0 contains no weakening links and such that there exists an element of $L(R_0)^{\wp}$ having only terminal contraction links.*
If R' is a proof-net with the same conclusions as R and if $[\![R]\!] = [\![R']\!]$, then $R \simeq_{\beta\eta} R'$.

Proof: Let $L(R_0)^-$ be the element of $L(R_0)^{\wp}$ which contains only terminal contraction links. By proposition 3.2.4, there exists an injective experiment of $L(R_0)^-$. By lemma 3.3.2, there exists an injective experiment of $L(R_0)$.

By proposition 3.3.4, there exists an injective 1-experiment of R_0. Let R'_0 be the standard proof-net associated with R'. By theorem 2.4.8, we have $R_0 = R'_0$, i.e. $R \simeq_{\beta\eta} R'$. □

4.1.2. THEOREM. *The multiset-based coherent semantics is injective for* $(?\wp)$*LL proof-nets.*

Proof: Let R and R' be two semantically equivalent $(?\wp)$LL proof-nets. We know by proposition 3.1.4 that there exists an injective 1-experiment of R_0. By theorem 2.4.8, we have $LPS(R_0) = LPS(R'_0)$.
Notice now that because R_0 and R'_0 are $(?\wp)$LL (and standard) proof-nets, every $?w$ link of R_0 or R'_0 which is not terminal is the premise of a $\wp$ link; and every $\wp$ link of these two proof-nets has at least one premise which is not the conclusion of a $?w$ link.
Well, *in this very particular case*, even in presence of weakenings, the characterization of boxes given by proposition 2.4.7 is still valid for R_0 and R'_0. We can then deduce $R_0 = R'_0$, i.e. $R \simeq_{\beta\eta} R'$. □

We now define the notion of "weakly polarized formula", which is related to the one of "polarized formula", widely studied in the last 10 years (see [Gir91], [DJS97], [QTdF96],[TdF97],[Lau99],[LQTdF00],...).

4.1.3. DEFINITION. **(Weakly polarized formulas).** A propositional formula P (resp. N) of LL is weakly positive (resp. weakly negative) when it is built in the following way (where X is an atomic formula):

$$\begin{array}{rcccc} P ::= & X & \mid P \otimes P & \mid P \otimes !N & \mid !N \otimes P \\ N ::= & X & \mid N \wp N & \mid ?P \wp N & \mid N \wp ?P \end{array}$$

We will say that a formula is weakly polarized when it is weakly positive or weakly negative.
A proof-net of $MELL$ is weakly polarized when the types of its conclusions are all subformulas of weakly polarized formulas.

4.1.4. REMARK. The difference between the weakly polarized formulas and the (strongly) polarized ones (coming from [Gir91]) is that every atomic formula is both weakly positive and weakly negative: we do not suppose anything on the atoms for weakly polarized formulas.
In particular, a weakly polarized formula A is not equivalent to $?A$ nor to $!A$ (contrary to polarized formulas, see [DJS97] or [TdF00b]).

4.1.5. THEOREM. *The multiset-based coherent semantics is injective for the set of weakly polarized proof-nets.*

Proof: Simply notice that a weakly polarized proof-net is a $(?\,\wp)$LL proof-net. □

4.1.6. COROLLARY. *The coherent multiset-based semantics is injective for the intuitionistic fragment ILU of Girard's unified logic ([Gir93]).*

Proof: The system ILU is actually the t-fragment of LK^{η} (defined in [DJS97]), and Danos, Joinet and Schellinx proved that for this fragment Girard's translation $A \to B = !A \multimap B$ yields a denotational semantics for ILU. It then suffices to notice that this translation uses only weakly polarized formulas, and to apply theorem 4.1.5. □

4.1.7. THEOREM. *Let t_1 and t_2 be two terms of the simply typed λ-calculus, and let R_0^1 (resp. R_0^2) be the proof-net associated with t_1 (resp. t_2) by Girard's translation ($A \to B = !A \multimap B$). Let k_1 (resp. k_2) be the number of axiom links of R_0^1 (resp. R_0^2), and $k \geq max(k_1, k_2)$.*
If t_1 and t_2 are not $\beta\eta$-equivalent, then there exists a coherent space $\mathcal{X}$ such that $card(|\mathcal{X}|) = k$, and the model of the simply typed λ-calculus obtained by interpreting every atomic type by the space $\mathcal{X}$ distinguishes t_1 and t_2.

Proof: The standard proof-net R_0^1 (resp. R_0^2) is weakly polarized, and then it is also a $(?\,\wp)$LL proof-net. By proposition 3.1.4, there exists an injective experiment e_1^1 of R_0^1. Now choose $\mathcal{X}$ in such a way that the k elements of the web $|\mathcal{X}|$ satisfy the coherence relations required by the existence of e_1^1, and let $n > max(h(R_0^1), h(R_0^2))$ (remember that $h(R)$ is the $?co$-size of the proof-net R, defined in 2.3.4). We can then apply proposition 2.4.6: there exists (with the chosen interpretation for the atoms) an injective n-obsessional experiment e_n^1 of R_0^1 (the one induced by e_1^1).
Because t_1 and t_2 are not $\beta\eta$-equivalent, we have $R_0^1 \neq R_0^2$, and then there cannot be in the model an experiment of R_0^2 with the same result as e_n^1. □

4.2 Counterexamples

We prove that there does not exist (in general) an injective experiment for a given proof-net without boxes. This leads immediately to a negative answer to our original question (1.2.3), thus corroborating the pertinence of our approach.

We then give another counterexample to the injectivity of coherent semantics for $MELL$, of a (slightly) different nature.
Both our counterexamples also hold in a coherent set-based framework.

4.2.1 The first counterexample

In order to prove injectivity for $MELL$, the previous chapters (and specially theorem 2.4.8 and proposition 3.3.4) suggest to show the existence of an injective experiment for every $MELL$ (standard) proof-net without boxes. But for both the standard proof-nets of figure 2, such an experiment does not exit.
The digits that we have associated with the different edges of the proof-nets indicate which are the requests of the contraction links: if with the two edges a and a' of R (risp. R') is associated the same integer, then every experiment e of R (risp. R') must satisfy $e(a) \asymp e(a')$.
Let $x, y \in |\mathcal{X}|$ be the labels that the experiment e of, say, R associates with the conclusions of the two axiom links of R. We see very well that one must have on the one hand $x \asymp y(\mathcal{X})$ (request of the square $?co$ link in the figure) and on the other hand $x \asymp y(\mathcal{X}^{\perp})$ (request of the triangular $?co$ link in the figure), that is $x = y$. This precisely means that there exist no injective experiment of R.
The reader probably noticed that one could find a simpler example of proof-net for which there is no injective experiment (as shown in remark 2.4.10), but the reason why we chose the previous one appears clearly if one crosses the edges conclusions of the two axiom links: because denotational semantics is unable to distinguish between the two axiom links, by crossing the edges one gets a proof-net R', with the same semantics as R but different from R. The previous proof-net was chosen in order to obtain $R \neq R'$ (with R and R' both standard).
To convince herself/himself that $R \neq R'$ the reader might notice (for example) that there exists a subproof-net of R which is not a subproof-net of R'. The fact that R and R' are semantically equivalent, both for the set-based and for the multiset-based coherent semantics, is an immediate consequence of the absence of an injective experiment. Notice that (of course) $OLPS(R) = OLPS(R')$, following theorem 2.3.12.

It is interesting to notice that if one "opens" the two contraction links of R and R' (i.e. if one erases the $?co$ links and their conclusions), then one gets two proof-nets R_1 and R_1' with the same conclusions, which are subproof-nets of R and R' respectively. The proof-nets R_1 and R_1' *do not* have the same semantics: the (uniformity) condition imposed by the erased contraction links has disappeared, and one can now find injective experiments which distinguish (semantically speaking) the two proof-nets.

4.2.2 The second counterexample

We are now going to show another phenomenon (still due to the "uniformity of coherent semantics"), which also leads to answer negatively to the question 1.2.3.

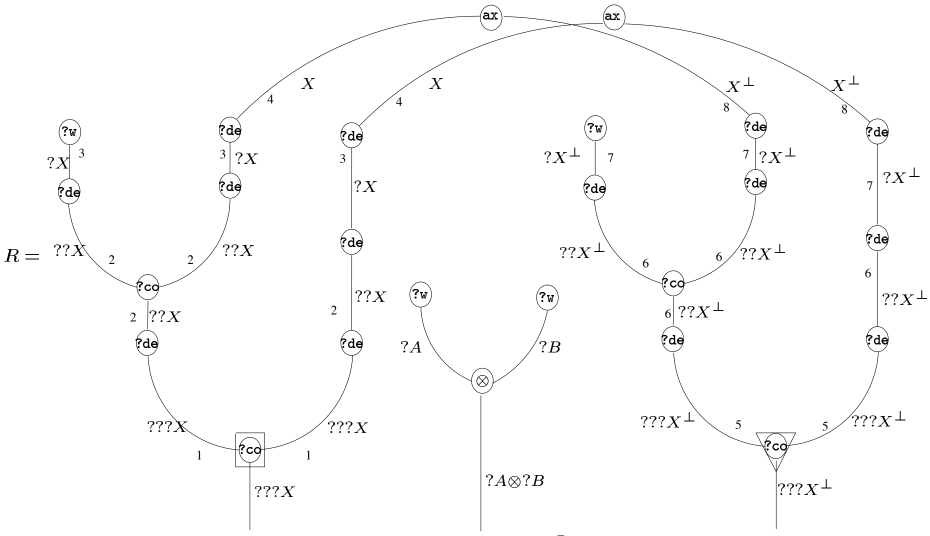
In presence of weakenings, the characterization of the boxes given by proposition 2.4.7 is obviously wrong (in general). We show that the semantics cannot uniquely determine the connexions between the different doors of the boxes of a proof-net. In other terms, we give two semantically equivalent proof-nets R and R' satisfying $LPS(R) = LPS(R')$, and such that $R \neq R'$.

This counterexample shows also that even for the system ELL (defined in [Gir95] and simplified in [DJ01]) both the coherent set-based semantics and the coherent multiset-based semantics are not injective. To be precise one should slightly modify it (see [TdF00b] for a more precise discussion).

The two proof-nets R and R' of figure 3 are (clearly) different and they have the same coherent (set-based and multiset-based) semantics. Let's show that R and R' are semantically equivalent. We have associated with some edges a digit, following the same convention as in the previous counterexample: if with the two edges a_1 and a_2 of R (resp. of R') is associated the same integer, then every experiment e of R (resp. of R') must satisfy $e(a_1) \asymp e(a_2)$. Notice that this notation is meaningful, because the edges with which we have associated a digit all have depth zero, and following definition 1.1.1 every experiment associates with these edges a unique label. We identify here, as we did for 1-experiments, $e(a)$ with the unique element of $e(a)$, for every experiment e of R (resp. R') and for every edge *a with depth zero* in R (resp. R').

Let c_1 and c_2 (resp. c'_1 and c'_2) be the conclusions of the two pal doors with depth zero in R (resp. in R'). The uniformity condition coming from the $?co$ link with depth zero in R (resp. in R') requires that for every experiment e of R (resp. e' of R') $e(c_1) \asymp e(c_2)$ (resp. $e'(c'_1) \asymp e'(c'_2)$). But every element of $e(c_i)$ (resp. of $e'(c'_i)$), for $i \in \{1, 2\}$, is of the form $\{n_i[\emptyset]\}$ (resp. $\{n'_i[\emptyset]\}$): the only possibility is then that $n_1 = n_2$ (resp. $n'_1 = n'_2$). In the set-based case we have $n_1 = n_2 = n'_1 = n'_2 = 1$.

Observe now that the unique way for the semantics to distinguish between R and R' is to be able to express the fact that the subgraph T of R and R' is



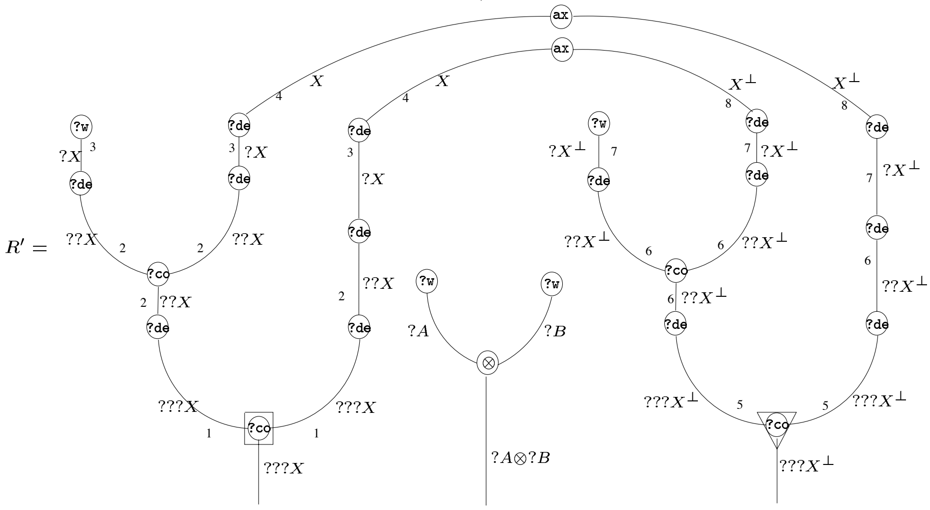


Figure 2. Counterexample 1: for the proof-nets R and R' above, one has $\|R\| = \|R'\|$ and $R \neq R'$.

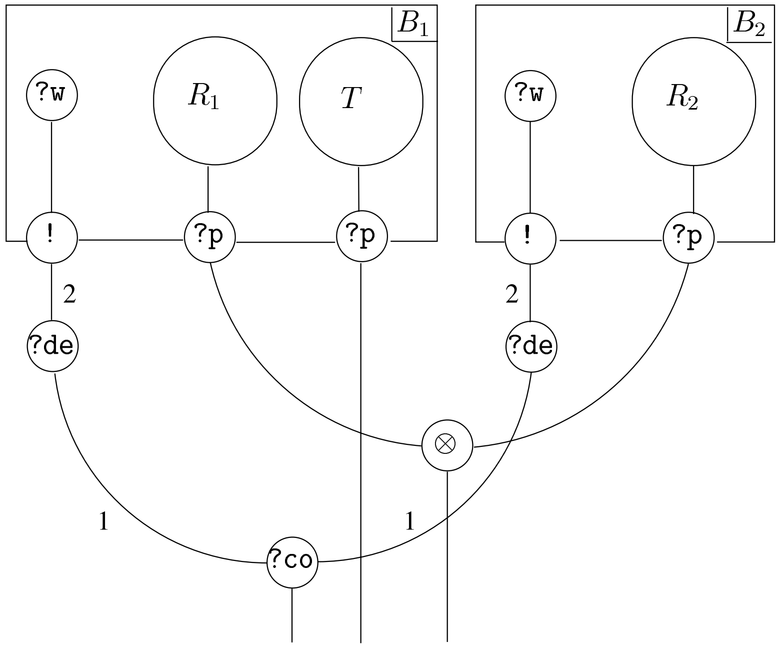
in a given box and not in the other one: this is precisely what the presence of the $?co$ link forbids. The semantics cannot then tell us in which box T is, and this allows conclude: with every experiment of R one can associate an experiment of R' with the same result (and conversely, of course).

We did never mention in the present paper the neutral elements of LL. Let us simply point out that the previous counterexample is also a counterexample to the injectivity of coherent multiset-based semantics for the multiplicative and exponential fragment of LL, without axiom links but with the links introducing the multiplicative constants 1 and $\perp$. In the set-based case, the non injectivity is obvious (see [TdF00b] for more details).

4.2.3. Remark. Our counterexamples show also that the result of [Sta83] on the maximality of the $\beta\eta$-equivalence for the simply typed λ-calculus does not extend to $MELL$ (see [TdF00b] for more details).

Let us conclude the present section by observing that in both our counterexamples the presence of weakenings is crucial, and that none of our proof-nets is *polarized*.

$R =$



$R' =$

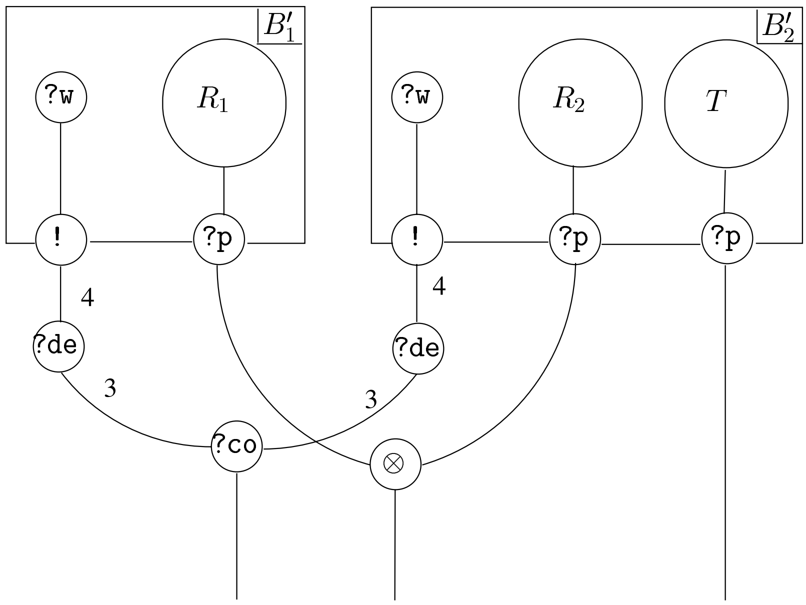


Figure 3. Counterexample 2: for the proof-nets R and R' above, one has $\|R\| = \|R'\|$ and $R \neq R'$.

4.3 Summing up

We use in this section the content of appendix B.

We denote by:

- $MELL\backslash\{?W\}$, the subsystem of $MELL$ containing all the proof-nets of $MELL$ whose normal forms do not contain any weakening link
- LL_{pol} the system of polarized proof-nets: an $MELL$ proof-net is polarized when the types of its conclusions are all subformulas of a positive (P) or of a negative (N) formula, where:

  $$\begin{array}{llll} P ::= & !X & \mid P \otimes P & \mid !N \\ N ::= & ?X^{\perp} & \mid N \wp N & \mid ?P \end{array}$$

- $[\![\ ]\!]_{cohs}$ the set-based coherent semantics
- $[\![\ ]\!]_{cohm}$ the multiset-based coherent semantics
- $[\![\ ]\!]_{rel}$ the relational semantics.

The following table sums up the state of the art concerning the question of injectivity: we wrote in capital letters the answers that we do have (the results of the present paper), and in small letters the conjectures (which are actually open problems).

	$[\![\ ]\!]_{cohs}$	$[\![\ ]\!]_{cohm}$	$[\![\ ]\!]_{rel}$
$MELL$	NO	NO	? (yes)
$MELL\backslash\{?W\}$	? (yes)	? (yes)	? (yes)
LL_{pol}	? (yes)	? (yes)	? (yes)
(? $\wp$)LL	? (yes)	YES	YES

Let us comment a bit on this table. The choice of the subsystems of $MELL$ mentioned above is easy to justify: we do not comment on (? $\wp$)LL (for obvious reasons!), $MELL\backslash\{?W\}$ seems interesting because a positive answer for this subsystem (in the coherent case) would probably help to understand more precisely the relation between connectivity and coherence, and LL_{pol} is certainly interesting, because it allows to encode classical logic (see [Gir91], [DJS97], [LQTdF00]).
Notice that (thanks to appendix B) any positive answer in the coherent (multiset-based) case gives immediately a positive answer in the relational

one. Let us also mention the paper [BE97], relating the semantic equivalence relation induced by the set-based coherent semantics and the one induced by the multiset-based coherent semantics. In particular, it is shown in this work that two equivalent proofs w.r.t. the multiset-based coherent semantics are always equivalent w.r.t. the set-based coherent semantics. And the converse does not hold, as soon as one adds the constants 1 and $\bot$.

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A Proof-nets and coherent semantics

In this paper, we deal both with syntax and semantics, and more precisely with proof-nets and their semantics. For the sake of self-containment, we recall here the notions of proof-net and of coherent space. Notice that while the latter is standard, this is not the case of the former: there are several variants of LL proof-nets in the literature. In [TdF00b] (see also [TdF00a]), one can find a detailed description of proof-nets and their normalization for second order LL. We refer to this notion of proof-net, and recall it here for the multiplicative and exponential fragment of LL.

A.1 Coherent spaces

We give the definition of coherent space (see for example [Gir87]), and of the *multiset* based interpretation of the exponential connectives (see [Gir91]).

A.1.1. Definition. **(coherent space)** A **coherent space** $\mathcal{A}$ is the data of a set $|\mathcal{A}|$ (the **web** of $\mathcal{A}$) and of a binary reflexive and symmetric relation denoted by $\,\overset{\frown}{\smile}\,$ (the **coherence** relation on $|\mathcal{A}|$). If $x, y \in |\mathcal{A}|$ and if (x, y) is an element of the relation $\overset{\frown}{\smile}$, one says that x and y are coherent and one writes $x \overset{\frown}{\smile} y(\mathcal{A})$. We often use the following notations: $x \frown y(\mathcal{A})$ (when $x \overset{\frown}{\smile} y(\mathcal{A})$ and $x \neq y$), $x \smile y(\mathcal{A})$ (when x and y are not coherent) and $x \underset{\frown}{\smile} y(\mathcal{A})$ (when $x \smile y(\mathcal{A})$ or $x = y$). The elements of $\mathcal{A}$, called **cliques**, are the multisets of elements of $|\mathcal{A}|$ pairwise coherent. In the original set-based definition (see for example [Gir87]), the cliques are the sets of elements of $|\mathcal{A}|$ pairwise coherent.
The interpretation of LL formulas is defined by induction on their complexity. One associates some arbitrary coherent spaces with atomic formulas (which means that for every such choice one gets a different interpretation). Then the coherent spaces associated with compound formulas are defined as follows:

- $|\mathcal{A}^\perp| = |\mathcal{A}|$, and for every $x, y \in |\mathcal{A}|$, one has $x \overset{\frown}{\smile} y(\mathcal{A}^\perp)$ iff $x \underset{\frown}{\smile} y(\mathcal{A})$
- $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$, and for every $x, x' \in |\mathcal{A}|$ and $y, y' \in |\mathcal{B}|$ one has $(x, y) \overset{\frown}{\smile} (x', y')(\mathcal{A} \otimes \mathcal{B})$ iff $x \overset{\frown}{\smile} x'(\mathcal{A})$ and $y \overset{\frown}{\smile} y'(\mathcal{B})$
- $|!\mathcal{A}| = \mathcal{A}_f$, whose elements are the finite elements of $\mathcal{A}$ (notice that here the set-based and the multiset-based webs are different), and for every $x, y \in |!\mathcal{A}|$ one has $x \overset{\frown}{\smile} y(!\mathcal{A})$ iff $x \cup y \in \mathcal{A}$ (i.e. $x \cup y \in \mathcal{A}_f$).

A.2 Proof-nets

A.2.1. Definition. A **proof-structure** is an oriented graph whose nodes are called links, and whose edges are labeled by formulas of LL. When drawing a proof-structure we represent edges oriented up-down so that we may speak of moving upwards or downwards in the graph. Links are defined together with an arity and a coarity, i.e. a given number of incident edges called the premises of the link and a given number of emergent edges called the conclusions of the link.

- A **Hypothesis** or H link has $n \geq 1$ conclusions, each of them labeled by a formula, and no premise
- an **axiom** link has no premise and two conclusions labeled by dual formulas
- a **cut** link has two premises labeled by dual formulas (which are also called the active formulas of the cut link) and no conclusion
- a **par** or $\wp$ (resp. **times** or $\otimes$) link has two premises and one conclusion. If the left premise is labeled by the formula A and the right premise is labeled by the formula B, then the conclusion is labeled by the formula $A \wp B$ (resp. $A \otimes B$)
- an **of course** link has one premise and one conclusion labeled by the **of course** of the premise
- a **dereliction** or $?de$ link has one premise and one conclusion labeled by the **why not** of the premise
- a **weakening** or $?w$ link has no premise and one conclusion labeled by $?A$ for some formula A
- a **contraction** or $?co$ link has $k \geq 2$ premises and one conclusion, all labeled by $?A$ for some formula A
- a **pax** link has one premise and one conclusion, both labeled by $?A$ for some formula A.

Let G be a set of links s.t.:
(α) every edge of G is the conclusion of a unique link
(β) every edge of G is the premise of at most one link.
We say that the edges which are not premise of a link are the conclusions of G.

We will say that such a graph is a proof-structure if the two following conditions are satisfied:

(1) !-box condition:

(1.i) with each of course link n is associated a (unique) sub-graph $B^!$ of G (satisfying (α) and (β)), s.t. one among the conclusions of $B^!$ is the conclusion of n and every other conclusion of $B^!$ (there might be no other conclusion) is the conclusion of a pax link. $B^!$ is called an exponential box and it is represented by a rectangular frame, and n is called the front door or the pal door of $B^!$

(1.ii) with each pax link n is associated an exponential box $B^!$ of G, s.t. one among the conclusions of $B^!$ is the conclusion of n. The link n is called a pax door of $B^!$

(2) nesting condition:

two boxes are either disjoint or included one in the other.

We will often speak of a box, a link or an edge of a proof-structure R contained in a box B of R. In case of links, we will not consider the doors of B as links contained in B. We will also speak of "a link l (resp. an edge a) of a box B" of a given proof-structure, meaning that l (resp. a) is contained in B or it is a door (resp. a conclusion) of B. If B is a box of a proof-structure R, then the biggest (resp. the smallest) box of R containing B is clearly well-defined, thanks to the nesting condition of definition A.2.1.
We shall say that a link or an edge of a given proof-structure R has **depth** n in R, if it is contained in exactly n boxes of R. For a box B, we shall say that B has **depth** n in R, if it is contained in exactly n boxes of R, all different from B. When R is a proof-net, the same definition will extend to the case of a subproof-net of R (as defined in A.2.7). Clearly, one can define in the same way the depth of a subproof-structure S (a subgraph which is itself a proof-structure) of the proof-structure R.
The depth of a proof-structure is the maximal depth of its boxes.

A.2.2. REMARK. Notice that (contrary to the premises of a $\otimes$ or of a $⅋$ link) the premises of a contraction link *are not* ordered. This means that a proof-structure is defined up to the order of the premises of the $?co$ links: we are actually dealing with an equivalence class of graphs rather than with a graph.

A.2.3. DEFINITION. **(graph with pairs)** We will say that two edges of an oriented graph are coincident when they have the same target. The couple $(G, App(G))$ is called a **graph with pairs** when G is an oriented graph and $App(G)$ is a set of n-tuples $(n \geq 2)$ of coincident edges.
Let R be a proof-structure and let $B_1, \ldots, B_k$ be the boxes of R with depth zero. We are going to associate with R a set $App(R)$ and a graph with pairs $R_{ap} = (G_R, App(R))$.
The graph G_R is obtained from R in the following way:
- substitute for each box B_i with p_i conclusions $(i \in \{1, \ldots, k\})$, a link H with p_i conclusions
The set $App(R)$ contains the following (and only the following) m-tuples:
- the couples of premises of every $\wp$ link of R with depth zero
- the p-tuples of premises of every $?co$ link of R with depth zero.

A.2.4. DEFINITION. **(correctness graph)** Let R be a proof-structure and let $B_1, \ldots, B_k$ be the boxes with depth zero in R. Let $R_{ap} = (G_R, App(R))$ be the graph with pairs associated with R by definition A.2.3.
A **switching S** of R is the choice of an edge for every n-tuple of $App(R)$. With each switching S is associated an unoriented graph $S(R)$, called **correctness graph**: for every n-tuple of $App(R)$, erase the edges of G_R which are not selected by S, and then forget the labels and the orientation of the edges of the graph. The correctness graph of R associated with S will be denoted by $S(R)$.

A.2.5. DEFINITION. **(proof-net)** Let R be a proof-structure which contains no occurrences of the link H, and let $B_1, \ldots, B_k$ be the boxes with depth zero in R. We say that R is a **proof-net** when the following conditions are satisfied:
- R satisfies (AC): for every switching S of R, the correctness graph $S(R)$ is acyclic (there is no cycle in $S(R)$)
- for every box $B_i \in \{B_1, \ldots, B_k\}$, the proof-structure R_i contained in B_i is a proof-net.

The just given notion of proof-net corresponds to the standard notion of sequent calculus proof, *provided* one adds to the multiplicative and exponential linear sequent calculus the two following rules:

1. the so-called "mix" rule:

$$(\text{mix})\ \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

2. the following "proof" of $?A$ for every formula A:

$$(?\ \text{Hyp})\ \frac{}{\vdash ?A}$$

A.2.6. Remark. In [TdF00a], we used the notion of "jump" and modified the previous definition, in order to get the correspondence between proof-nets and the usual multiplicative and exponential linear sequent calculus without the mix rule and without the (? Hyp) rule. The notion of proof-net thus obtained is also proven to be stable w.r.t. the usual cut-elimination steps.
We will not develop here such a notion: the reader can refer to [TdF00a] (and also to [TdF00b] for a more detailed version), but also simply to definition A.2.5, keeping in mind that all the proof-nets considered in the present paper satisfy (AC) and can be sequentialized in the usual multiplicative and exponential sequent calculus (without adding any rule).
It is however important to stress the fact that in the absence of weakenings, a proof-net R can be sequentialized in the usual multiplicative and exponential linear sequent calculus (without mix nor (? Hyp)) iff R satisfies definition A.2.5, where the (AC) condition is substituted by the following one:
(ACC) for every switching S of R, the correctness graph $S(R)$ is acyclic and connected (i.e. $S(R)$ is a tree).

A.2.7. Definition. Let T be a proof-net. A subgraph R of T is a **subproof-net** of T, when R is a proof-net.

A.2.8 Cut-elimination for LL Proof-nets

We are only concerned with the multiplicative and exponential fragment of LL proof-nets, for which the cut-elimination procedure is standard. We won't define here the elementary reduction steps (see [Gir87], [Dan90], [TdF00a], [TdF00b]).

A.3 Conventions and notations

A.3.1 Some conventions

We restrict in the paper (except for subsection A.2), the term "label" to the elements of the web of some space associated with an edge of a proof-net by an experiment (see definition 1.1.1). The formula (which is called "label" in subsection A.2) associated with a given edge of a proof-net is said to be **the type** of the edge.

There is the obvious remark to do concerning formulas: we simply say "formula" actually meaning "occurrence of formula". Which of these two notions one refers to is in general clear from the context. However, it might be absolutely crucial to handle occurrences of formulas (and not formulas): in these cases we sometimes stress the difference.

A sequent is (as usual) a multiset of formulas, sometimes prefixed by the symbol $\vdash$.

A last link of a proof-structure S is a link whose conclusion(s) is(are) a conclusion(s) of S. A cut link of S is a last link if it has depth 0. We say that n is a terminal link of a proof-net when there exists a sequentialization having as last rule the one with which is associated the link n. Of course, there are *some* links for which the two notions coincide.

If A is a formula and E is an occurrence of subformula of A, we sometimes speak of "the complexity of $A\backslash E$": we mean the integer $c_A - c_E$, where c_A (resp. c_E) is the number of occurrences of connectives of A (resp. E).

We always work with **multisets** (unless explicitly mentioned), for which we use the same notations as for sets.

The notion of "path" used in the paper (unless explicitly mentioned) is the one of "straight path" introduced in [DR95]: a straight path is an oriented path changing direction only when crossing a cut link or an axiom link.
In the whole paper, we simply write "path" always meaning "straight path".

A.3.2. DEFINITION. (**[DR95]**) Let R be a proof-net. A path of R is a sequence of edges or reverted edges (i.e. a path may take an edge from its goal to its source). We sometimes make the abuse of considering links as part of paths (the idea is that the node associated with the link is crossed by the path). We denote by $\alpha, \beta, \ldots$ the edges of a proof-net (oriented as usual, following definition A.2.1) and by $\alpha^*, \beta^*, \ldots$ the previous edges "reverted" (i.e. oriented now in the opposite direction). Let now a (resp. b) be an edge or a reverted edge whose goal (resp. source) is the link n; we denote by ab

the path consisting of the edge a (followed by the link n, itself) followed by the edge b.
We say that the path Φ of R is a **straight path** if:
(i) Φ does not contain any $\alpha^*\alpha$ nor any $\alpha\alpha^*$
(ii) if α and β are two distinct premises (and then, according to our notations, two edges) of a same link n, and if $\alpha\beta^*$ is a subpath of Φ, then n is a cut link.

A.3.3 Notations

We denote in the paper:
- a, b, c, ... the edges of a proof-net,
- A, B, C, ... the types of these edges,
- $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... the structures interpreting these types,
- $|\mathcal{A}|$, $|\mathcal{B}|$, $|\mathcal{C}|$, ... the webs of these structures,
- α, β, α', β', ... the edges of atomic type of a proof-net,
- e, e', e_i, ... the experiments,
- γ, γ', δ, δ',... the results of some experiments,
- Γ, Γ', Δ, Δ',... multisets of formulas,
- We denote by $\wp\ \Gamma$ the formula obtained by performing the $\wp$ between the formulas of Γ. For simplicity, we still denote by $\wp\ \Gamma$ the structure interpreting the formula $\wp\ \Gamma$,
- R_B the biggest subproof-net of the proof-net R which is contained in the box B of R (in case R is a proof-structure, R_B is the biggest subproof-structure contained in B),
- $card(y)$ the cardinality of the multiset y,
- $n[z]$ the repetition n times of the element z: for example $\{n[z]\}$ is the multiset containing n occurrences of the element z.

We use the standard notations for coherence and incoherence (strict and strict or equal), introduced by definition A.1.1. When the context makes this unambiguous, we simply write $x \stackrel{\smile}{\frown} y$, instead of $x \stackrel{\smile}{\frown} y(\mathcal{A})$, for $x, y \in |\mathcal{A}|$.

B About injectivity for relational semantics

Several results presented in the paper are valid also for the relational semantics, which can be roughly defined as "coherent semantics without coherence".
We define the relational interpretation of LL formulas, and state a result (proposition B.2), whose immediate consequence is the fact that when the

multiset-based coherent semantics is injective, so is also the relational semantics (remark B.3).

B.1 Definition

Let $|\ |$ be a function which associates with every propositional variable a set. We define the extension of $|\ |$ to LL formulas as follows:

- $|\mathcal{A}^\perp| = |\mathcal{A}|$
- $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$
- $|!\mathcal{A}| = M_f(|\mathcal{A}|)$, where $M(|\mathcal{A}|)$ is the free commutative monoid generated by $|\mathcal{A}|$ and $M_f(|\mathcal{A}|)$ is the set of the finite multisets of elements of $|\mathcal{A}|$.

For every LL formula A, the elements of the space $\mathcal{A}$ associated with A by the relational semantics are the multisets of elements of $|\mathcal{A}|$.

Definitions 1.1.1, 1.1.4 straightforwardly extend to relational semantics, and the analogue of theorem 1.1.6 can be proven. The same holds for the notion of obsessional experiment, and a study of these experiments in the relational framework has been undertaken in [TdF00b].
Nevertheless, several crucial results do not extend to relational semantics: contrary to the coherent case, there exist several experiments with the same result, and proposition 2.2.7 does not hold in the relational case.
However, there is a very natural and useful property, allowing to extend all our injectivity results to the relational case. The statement of the following proposition (proven in [TdF00b]) was suggested us by Thomas Ehrhard.

In the following, we will denote by $|\mathcal{A}|_{cohm}$ the multiset-based web of the coherent space $\mathcal{A}$, and by $[\![R]\!]_{cohm}$ (resp. $[\![R]\!]_{rel}$) the multiset-based coherent semantics (resp. the relational semantics) of the proof-net R.
Let's choose a relational and a coherent interpretation of the propositional variables of the language, s.t. if $\mathcal{X}_{rel}$ (resp. $\mathcal{X}_{cohm}$) is the space interpreting the propositional variable X, then $|\mathcal{X}|_{cohm} = |\mathcal{X}|_{rel}$.

B.2 Proposition

Let R be a proof-net with conclusion Γ. We have that $[\![R]\!]_{cohm} = [\![R]\!]_{rel} \cap |\wp\, \Gamma|_{cohm}$.

B.3 Remark

Proposition B.2 implies that if R and R' are two proof-nets with the same conclusions, then from $[\![R]\!]_{rel} = [\![R']\!]_{rel}$ one deduces that $[\![R]\!]_{cohm} = [\![R']\!]_{cohm}$. This means that for every fragment F of LL, if the coherent (multiset-based) semantics is injective for F, then the relational semantics is injective too.