Slicing polarized additive normalization

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Abstract

To attack the problem of "computing with the additives", we introduce a notion of sliced proof-net for the polarized fragment of linear logic. We prove that this notion yields computational objects, sequentializable in the absence of cuts. We then show how the injectivity property of denotational semantics guarantees the "canonicity" of sliced proof-nets, and prove injectivity for the fragment of polarized linear logic corresponding to simply typed λ-calculus with pairing.

1 Introduction

The question of equality of proofs is important in the "proofs-as-programs" paradigm. Traditional syntax (sequent calculus, natural deduction,...) distinguish proofs which are clearly the same as computational processes. On the other hand, denotational semantics identifies "too many" proofs. The seek of an object sticking as much as possible to the computational nature of proofs led to the introduction of a new syntax for logic: proof-nets, a graph-theoretic presentation which gives a more geometric account of proofs (see [5]). This discovery was achieved by a sharp (syntactical and semantical) analysis of the cut-elimination procedure.

Any person with a little knowledge of the multiplicative framework of LL, has no doubt that proof-nets are the canonical representation of proofs. But as soon as one moves from such a fragment, the notion of proof-net appears "less pure". A reasonable solution for the multiplicative and exponential fragment of LL (with quantifiers) does exist (combining [2] and [7], like in [16]), but trying to add the additive connectives means entering a true jungle. Of course, it is possible to survive in this jungle: it is shown in [5] how to compute with the additives, and in [16] an even (apparently much) more savage way of computing is defined and shown to yield a unique cut-free object in some important cases. So what? The problem is that the objects (the proof-nets) used are definitely not *canonical*.

Some better ("more" canonical) solutions have been proposed in [9] and [15]. Recently, a new fragment of LL appeared to have a great interest: in [6] and [3] the *polarized fragment* of LL is shown to be enough to translate faithfully classical logic. A study of proof-nets for such a fragment was undertaken in [13], and the the notion of [9] drastically simplified. In [14] a proof of strong normalization and confluence of the cut-elimination procedure is given for polarized LL, using the syntax of [5] (notice that for full LL confluence is wrong and strong normalization is still not completely proven).
Despite these positive results, the notion of proof-net still appears as (more or less desperately, depending on the cases) non canonical.

The first contribution of the present paper is the proposal of a mathematical definition of the term "canonical". And here is where denotational semantics comes into the picture: in [17], the question of *injectivity* of denotational semantics is addressed for proof-nets. Roughly speaking, denotational semantics is said to be injective when the equivalence relation it defines on proofs coincide with the one defined by the cut-elimination procedure. Our proposal is to let semantics decide on the canonicity of some notion of proof-net: this is canonical when there exists a (non contrived, obvioulsy!) denotational semantics which is injective with respect to the would-be canonical notion of proof-net.

Notice that this is a rather severe notion of canonicity. Indeed, proof-nets for multiplicative LL are canonical, but the previously mentioned extension to multiplicative and exponential LL is not guaranteed to be canonical: the time being we only know that coherent (set and multiset based) semantics *is not* injective for such proof-nets (see [17]). Of course, none of the previously mentioned syntax for the additives is canonical w.r.t. our definition.

The notion of *slice* was first introduced in [5]. The idea is very simple: instead of dealing with both the components of an additive box "at the same time", what about working with the two slices of such a box separately? This attitude is tempting because it ignores the superimposition notion underlying the connective & (which is precisely the difficult point to understand). It is shown in [9] that the correctness of the slices of a proof-structure does not imply the correctness of the proof-structure itself.

However, this naïve (and usually completely wrong) view of the additives, turns out to be correct in a polarized and cut-free framework (theorem 3).

We then define, in section 2, a notion of *sliced proof-structure* for polarized LL (definition 3), and show how to translate sequent calculus proofs into sliced proof-structures. To obtain canonical objects, we deal with atomic axioms and proof-structures in the style of the "nouvelle syntaxe" of Danos and Regnier. We also introduce particular formulæ which do not occur in sequent calculus, but are very useful in our framework: a formula $\flat A$ is necessarily the premise of an ?-link. The notation (and the meaning) of $\flat A$ is clearly very much inspired from Girard's works on ludics ([10], [12] and [11]) and on light linear logic ([8]).

We introduce in section 3 the three denotational semantics we will consider in the paper: coherent (set and multiset based) and relational semantics. The translation of formulas by coherent spaces (and sets) is standard and we don't recall it. We adapt the definition of experiment of [5] to our framework, and we define the interpretation of a sliced proof-structure (definition 8). Particular experiments coming from [17] are also introduced (injective 1-experiments), to be used later in section 7.

Section 4 is devoted to define and study the notion of "correct" sliced proof-structure (or sliced *proof-net*). The polarization constraints allow to extend to our framework the *correctness criterion* of [13]. We define a *sliced* cut-elimination procedure (definition 12), we prove that correctness is preserved by our sliced cut-elimination steps (theorem 1) and that our semantical interpretation is sound (theorem 2). Our sliced proof-nets are thus proven to be computational objetcs.

In section 5, we prove that in the absence of cuts, the correctness criterion (plus some obviously necessary conditions on sets of slices) is enough to "glue" in a unique way different slices: a sliced proof-net comes from a sequent calculus proof (theorem 3).

Section 6 explains and justifies in details our method: the use of injective denotational semantics as a witness of canonicity of our sliced proof-nets.

The reader should notice that this is the very first time a notion of proof-net containing the additives can really pretend to be canonical.

Finally, section 7 shows that our method makes sense: there exist interesting fragments of polarized LL for which denotational semantics is injective (and thus the corresponding proof-nets are

canonical): the λ-calculus with pairing. The result that we prove is an extension of the result of [17]. Thanks to a remark of L. Regnier on the λ-calculus (expressed by proposition 3), we could avoid to reproduce the entire proof. We thus get injectivity only for relational semantics, but in a quick and simple way.
Let us conclude by stressing the fact that the last section is simply an example to illustrate the method explained in section 6, and it is (very) likely that injectivity for coherent and relational semantics hold for the whole polarized fragment. This would give canonical proof-nets for polarized LL, that is for classical logic (see [14]).

2 Sliced proof-structures

We define sliced proof-structures and give the translation of sequent calculus proofs.

Definition 1
A *polarized formula* is a linear propositional formula verifying the following constraints:

$$\begin{array}{rclcccccc} N & ::= & X & \mid & N \parr N & \mid & N \,\&\, N & \mid & ?P \\ P & ::= & X^{\perp} & \mid & P \otimes P & \mid & P \oplus P & \mid & !N \end{array}$$

or a positive formula P prefixed by the symbol $\flat$ (considered as a negative formula).
$\mathsf{LL}_{\mathrm{pol}}$ [13] is the fragment of LL using only polarized formulæ.

Lemma 1
Every sequent $\vdash \Gamma$ provable in $\mathsf{LL}_{\mathrm{pol}}$ contains at most one positive formula.

Proof. See [13]. □

Definition 2 (Proof-structure)
A *proof-structure* is an oriented graph whose nodes are called links, and whose edges are typed by formulæ of LL. When drawing a proof-structure we represent edges oriented up-down so that we may speak of moving upwardly or downwardly in the graph, and of links or edges "above" or "under" a given link/edge. Links are defined together with an arity and a coarity, i.e. a given number of incident edges called the premises of the link and a given number of emergent edges called the conclusions of the link.

-an *axiom* link or *ax*-link has no premise and two conclusions typed by dual atomic formulæ,

-a *cut* link has two premises typed by dual formulæ (which are also called the active formulæ of the cut link) and no conclusion,

-a $\parr$- (resp. $\otimes$-) link has two premises and one conclusion. If the left premise is typed by the formula A and the right premise is typed by the formula B, then the conclusion is typed by the formula $A \parr B$ (resp. $A \otimes B$),

-an !-link has exactly one conclusion of type ! and some conclusions of type $\flat$,

-a $\flat$-link has one premise of type A and one conclusion of type $\flat A$,

-a ?-link has $k \geq 0$ premises of type $\flat A$ and one conclusion of type $?A$.

Let G be a set of links s.t.:

(α)every edge of G is the conclusion of a unique link;

(β)every edge of G is the premise of at most one link.

We say that the edges which are not premise of a link are the conclusions of G.
We will say that such a graph is a proof-structure if with every !-link with conclusion $!A, \flat\Gamma$ is associated a proof-structure with conclusion $A, \flat\Gamma$ (called its *box*).

Remark 1
Notice that, by definition, the boxes of a proof-structure satisfy a *nesting condition:* two boxes are either disjoint or contained one in the other.
Notice also that the type of every conclusion of a box is a negative formula.

Definition 3 (Sliced proof-structure)
A *sliced proof-structure* is a set S of slices such that all the slices have the same conclusion, up to the ones of type $\flat$.
If S contains n slices, and if $\Gamma, \flat\Delta_i$ is the conclusion of the slice s_i of S, then $\Gamma, \flat\Delta_1, \ldots, \flat\Delta_n$ is the conclusion of S.

A *slice* s is a proof-structure possibly containing some unary $\&_1$-, $\&_2$- (resp. $\oplus_1$-, $\oplus_2$-) links, whose premise has type A, B and whose conclusion has type $A \mathbin{\&} B$ (resp. $A \oplus B$). With every !-link n of s is now associated a sliced proof-structure S_n. This means, in particular, that the !-conclusion of n appears in every slice of S_n, while every $\flat$-conclusion of n appears in exactly one slice of S_n.

Definition 4 (Single-threaded slice)
A *single-threaded slice* is a slice s such that the sliced proof-structures associated with the !-links of s contain only one slice, which is itself a single-threaded slice.

Remark 2
With every sliced proof-structure S is naturally associated a set of single-threaded slices, to which we will refer as the set of the "single-threaded slices of S (or associated with S)".

Definition 5 (Links and depths)
A link (resp. an edge) of a sliced proof-structure S is a link (resp. an edge) of one of the single-threaded slices of S.
Let s be a single-threaded slice of S. Every link of the proof-structure s has depth 0 in s. Let m_1 be an !-link of the proof-structure s and let n be a link of the single-threaded slice s_{m_1} associated with m_1. If n has depth k in s_{m_1}, then n has depth $k+1$ in s.
The depth of a link n of S is the depth of n in the single-threaded slice containing n. The depth of an edge a is the depth of the link of which a is conclusion.
The depth of a sliced proof-structure S is the maximum of the depths of the single-threaded slices of S.

Remark 3
Every formula A of a sliced proof-structure is a conclusion of a unique link introducing A. (Notice that this is of course not the case in any version of proof-nets for the full propositional fragment of LL).

We are now going to associate with every linear sequent calculus proof a sliced proof-structure with the same conclusions.

Definition 6 (Translation of the sequent calculus)
Let R be the last rule of the linear sequent calculus proof π. We define the sliced proof-structure S_π by induction on π.

- If R is an axiom with conclusion $X, X^\perp$, then the unique slice of S_π is an axiom link with conclusion $X, X^\perp$.

- If R a $\parr$- or a $\oplus$-rule, having as premise the subproof π', then S_π is obtained by adding to every slice of $S_{\pi'}$ the link corresponding to R.

- If R is a $\otimes$- or a *cut* rule with premises the subproofs π_1 and π_2, then S_π is obtained by connecting every slice of S_{π_1} and every slice of S_{π_2} by means of the link corresponding to R. Notice that if S_{π_1} (resp. S_{π_2}) contains k_1 (resp. k_2) slices, then S_π contains $k_1 k_2$ slices.

- If R is a &-rule with premises the subproofs π_1 and π_2, then S_π is obtained by adding a $\&_1$- (resp. $\&_2$-) link to every slice of S_{π_1} (resp. S_{π_2}).

- If R is a dereliction rule on A having as premise the subproof π', then S_π is obtained by adding to each slice a $\flat$-link with premise A and conclusion $\flat A$ and a unary ?-link with premise $\flat A$ and conclusion $?A$.

- If R is a weakening rule on $?A$, then S_π is obtained by adding a ?-link with arity 0 and conclusion $?A$.

- If R is a contraction rule on $?A$ having as premise the subproof π', then by induction hypothesis, every slice of $S_{\pi'}$ has two formulæ $?A$ among its conclusions. By remark 3, these two formulæ are both conclusion of a ?-link. We replace them by a unique ?-link with the required arity, and thus obtain the slices of S_π.

- If R is a promotion rule with conclusion $!C, ?A_1, \ldots, ?A_n$ having as premise the subproof π', then let s_i be one of the $p \geq 1$ slices of $S_{\pi'}$. For every slice s'_i of $S_{\pi'}$ with conclusion $C, ?A_1, \ldots, ?A_n$, we call s_i the graph obtained by erasing the ?-links with conclusions $?A_1, \ldots,$ $?A_n$. s_i is a slice with conclusion $C, \flat A^1_{1,i}, \ldots, \flat A^{q_{1,i}}_{1,i}, \ldots, \flat A^1_{n,i}, \ldots, \flat A^{q_{n,i}}_{n,i}$, with $q_{j,i} \geq 0$. The unique slice of S_π is an !-link with conclusion $!C, \flat A^1_{1,1}, \ldots, \flat A^{q_{n,1}}_{n,1}, \ldots, \flat A^1_{1,p}, \ldots, \flat A^{q_{n,p}}_{n,p}$, to which we add for every $1 \leq j \leq n$ a ?-link having as premises $\flat A^k_{j,i}$ ($1 \leq i \leq p$ and $1 \leq k \leq q_{j,i}$). The sliced proof-structure associated with the unique !-link of S_π is the set of the s_i ($i \in \{1, \ldots, p\}$).

Remark 4
Let's try to give a more informal (but, hopefully clearer) description of the last case of the previous definition. For every formula $?A_i$, we replace the ?-links introducing it in each slice by a unique ?-link in the (unique) slice of S_π.

3 Semantics

We consider the concrete semantics of experiments introduced in [5]. Such an approach suits very well three kinds of semantics: coherent set-based and multiset-based and relational semantics (see [16]).
Our results (like the existence of an injective 1-experiment used in the proof of lemma 7, or proposition 2) will be completely proven only in the relational case, but the extension to the coherent

semantics is just a matter of checking some minor details, consisting in the extension to our framework of the results proven in [17] without the additives.
The space interpreting a formula A will be denoted in the sequel by $\mathcal{A}$. We define the space $\flat\mathcal{A}$ as the space $?\mathcal{A}$.

Definition 7 (Experiment)
An experiment e of a sliced proof-structure S is an experiment e of one of the slices of S.
An experiment e of a slice s of S, is an application which associates with every edge a of type A of the proof-structure s an element $e(a)$ of $|\mathcal{A}|$, called *label* of a. We define such an application by induction on the depth p of s.
If $p = 0$, then:

- If $a = a_1$ is the conclusion of an axiom link with conclusions the edges a_1 and a_2 of type X and $X^\perp$ respectively, then $e(a_1) = e(a_2)$.
- If a is the conclusion of a $\parr$ (resp. $\otimes$) link with premises a_1 and a_2, then $e(a) = (e(a_1), e(a_2))$.
- If a is the conclusion of a link $\oplus_i$ (resp. $\&_i$), $i \in \{1, 2\}$ with premise a_1, then $e(a) = (i, e(a_1))$.
- If a is the conclusion of a dereliction link with premise a_1, then $e(a) = \{e(a_1)\}$.
- If a is the conclusion of a ?-link of arity $k \geq 0$, with premises $a_1, \ldots, a_k$, then $e(a) = e(a_1) \cup \ldots \cup e(a_k)$, and $e(a) \in |?\mathcal{C}|$. If $k = 0$ we have $e(a) = \emptyset$.
- If a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.

If the conclusions of S are the edges $a_1, \ldots, a_n$ of type, respectively, $A_1, \ldots, A_n$, and e is an experiment of S s.t. $\forall i \in \{1, \ldots, n\}$ $e(a_i) = x_i$, then we shall say that $(x_1, \ldots, x_n) \in |\mathcal{A}_1 \parr \ldots \parr \mathcal{A}_n|$ is the *result* of the experiment e of S. We shall also denote it by $x_1, \ldots, x_n$.
If $p > 0$, then e satisfies the same conditions as in case $p = 0$, and for every !-link n with depth 0 in s and with conclusion c of type $!C$ and $a_1, \ldots, a_n$ of type, respectively, $\flat A_1, \ldots, \flat A_n$, there exist $k \geq 0$ experiments $e_1, \ldots, e_k$ of the sliced proof-structure S' associated with n s.t.

- $e(c) = \{x_1, \ldots, x_k\}$, where x_j is the label associated with the edge of type C by e_j,
- If s' is the (unique!) slice of S' containing the edge a'_j with the same type as a_j, then $e(a_j)$ is the union of the labels associated with a'_j by the k experiments of s'

Of course, we require that $e(c) \in |!\mathcal{C}|$, and $e(a_j) \in |\flat\mathcal{A}_j|$.

Definition 8 (Interpretation)
The *interpretation* or *the semantics* of a sliced proof-structure S with conclusion Γ is the set $[\![S]\!] := \{\gamma \in |\parr\Gamma| :$ there exists an experiment e of S with result $\gamma\}$, where $\parr\Gamma$ is the space interpreting the $\parr$ of the formulæ of Γ.

Remark 5
The interpretation of a sliced proof-structure S depends on the interpretation chosen for the atoms of the formulæ of S. Once this choice is made, $[\![S]\!]$ is (by definition) the union of the interpretation of the slices of S.
The reader should notice that the union of the interpretations of the single-threaded slices of S *is not* enough to recover $[\![S]\!]$ (except in some particular cases, when S is a cut-free proof-net, see section 7). This is a crucial point (behind which hide the complex relations between the additive and multiplicative worlds) showing the impossibility of working only with single-threaded slices.

The following notion of 1-experiment is a particular case of the more general notion of n-obsessional experiment introduced in [17].

Definition 9 (1-experiments, [17])
An experiment e of a sliced proof-structure S is a 1-*experiment*, when with every !-link of S one has (using the notations of definition 7) $k = 1$, and e_1 is an 1-experiment.

Remark 6
Let S be a sliced proof-structure.

1. Let e be a 1-experiment of S. If a is any edge of S of type A, then with a the experiment e associates a unique element of $|\mathcal{A}|$, *whatever the depth of a is.* In case e is not a 1-experiment, this is (in general) the case only for the edges with depth 0.

2. The 1-experiments of S are exactly the 1-experiments of the single-threaded slices of S.

3. We say that a 1-experiment e of a single-threaded slice s of S is *injective* when for every pair of (different) axiom links n_1 and n_2 of s, if x_1 (resp. x_2) is the (unique) label associated by e with n_1 (resp. n_2), then $x_1 \neq x_2$.

4. If S contains no cut links, then there always exists an injective *relational* experiment of any single-threaded slice of S. This is not that obvious in the coherent case (due to the presence of ?-links): it is actually wrong in a non polarized framework, even for sliced proof-structures coming from sequent calculus proofs (see [17]).

4 Proof-nets and cut-elimination

We now define a notion of correct sliced proof-structure: a *proof-net* is a sliced proof-structure satisfying some geometrical condition. For these sliced proof-nets, a "sliced" cut-elimination procedure is given: a cut-elimination step is a step in *one* of the slices.
We show that the cut-elimination steps preserve the correctness of the structures, and that the intepretation given by definition 8 is sound (i.e. invariant w.r.t. these steps).

4.1 Definitions

Definition 10 (Acyclic sliced proof-structure)
The *correction graph* (see [13]) of a single-threaded slice (or of a box) B is the directed graph obtained by erasing the edges conclusion of B, forgetting the sliced proof-structure associated with every !-link with depth 0 in B and by orienting negative (resp. positive) edges downwardly (resp. upwardly).
A single-threaded slice of S satisfies (AC) when its correction graph, so as the correction graph of all its boxes, is acyclic.
A sliced proof-structure S is *acyclic*, when every single threaded slice associated with S satisfies (AC).

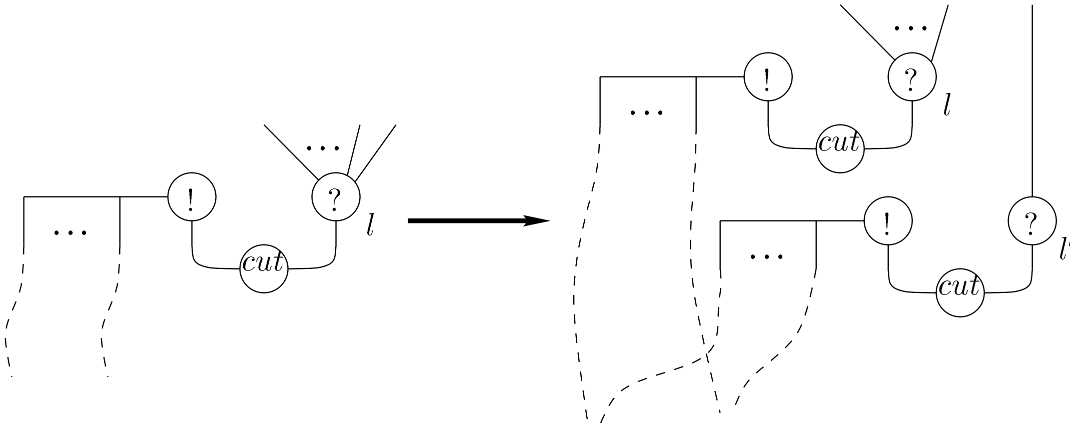
Definition 11 (Proof-net)
Let S be an acyclic sliced proof-structure without any $\flat$-conclusion. S is a *proof-net* if every slice of S has exactly one $\flat$-link or one positive conclusion (at depth 0). Moreover, we require that the sliced proof-structures $S_1, \ldots, S_k$, recursively associated with the !-links of S also satisfy these properties.

Remark 7
It is easy (and standard) to show, by induction on the sequent calculus proof, that the sliced proof-structure associated by definition 6 with a sequent calculus proof is a proof-net.
Notice that the condition given by definition 11 is nothing but the proof-net version of lemma 1.

We come now to the definition of the cut-elimination procedure. If the cut link c has depth n in the sliced proof-structure S, the cut elimination step associated with c will be a step for the sliced proof-structure associated with the !-link (of depth $n-1$) the box of which contains c.

Definition 12 (Cut-elimination)
Let S be a sliced proof-structure without $\flat$-conclusions. We define a one step reduct S' of S. Let $s \in S$ and c be a cut link of s. We define $\{s'_i\}_{i\in I}$, obtained by applying some transformations to s. S' is the set of the slices obtained from S by substituting $\{s'_i\}_{i\in I}$ for s.

- If c is a cut link of type ax, then $\{s'\}$ is obtained by erasing the axiom link and the cut link and by merging the other premise of the cut with the not cut conclusion of the axiom link.
- If c is a cut link of type $⅋/\otimes$, let A and B (resp. $A^\perp$ and $B^\perp$) be the premises of the $⅋$-link (resp. $\otimes$-link). $\{s'\}$ is obtained by erasing the $⅋$-link, the $\otimes$-link and the cut link and by putting two new cut links between A and $A^\perp$, and B and $B^\perp$.
- If c is a cut link of type $\&_i/\oplus_i$, then $\{s'\}$ is obtained by erasing the two links and by moving up the cut link to their premises.
- If c is a cut link of type $\&_1/\oplus_2$ (or $\&_2/\oplus_1$), then $I=\emptyset$ (we simply erase s). Moreover, if s is the unique slice of the sliced proof-structure S_n associated with the !-link n, we also erase the slice containing n (and so on recursively...).
- If c is a cut link of type $!/?$ with a 0-ary ?-link, then the box associated with the !-link is erased and the edges between its $\flat$-conclusions and the ?-links under them are also erased.
- If c is a cut link of type $!/?$ with a 1-ary ?-link under a $\flat$-link, let T be the sliced proof-structure associated with the !-link. With each slice t_i of T, we associate the slice s'_i defined by erasing the ?-link and the $\flat$-link, by replacing in s the !-link by t_i and by cutting the main conclusion of t_i with the premise of the $\flat$-link.
- If c is a cut link of type $!/?$ with a 1-ary ?-link whose premise is a $\flat$-conclusion of an !-link l', let T be the sliced proof-structure associated with l' and l be the cut !-link. Let $?A/!A^\perp$ be the cut formula. $\{s'\}$ is obtained by erasing l and its conclusion and by replacing the conclusion $\flat A$ of l' by all the $\flat$-conclusions of l. And with this new !-link (which we still denote by l') is associated a sliced proof-structure T' obtained by replacing the (unique) slice t of T having $\flat A$ among its conclusions by the slice obtained adding to the conclusion of type $\flat A$ of t a unary ?-link and cutting its conclusion (of type $?A$) with the conclusion of type ! of l. (The sliced proof-structure associated with l remains unchanged).
- If c is a cut link of type $!/?$ with a n-ary ?-link l with $n>1$, then $\{s'\}$ is obtained by creating a new unary ?-link l' having as premise one of the premises of l (and erasing the corresponding edge above l), by duplicating the !-link and by cutting the copy with the conclusion of l', every $\flat$-conclusion of the copy of the !-link is premise of the same links as the edge it is a copy of.



Remark 8
The attentive reader certainly noticed that there are two particular cases in which the previous definition is problematic:

1. when the two premises of a cut link are both conclusions of the same axiom link,

2. when the two premises of a cut link are both conclusions of the same !-link (in fact, in our framework, this means that the premise of type ? of the cut link is the conclusion of a ?-link whose premise is a conclusion of type $\flat$ of the !-link).

In these two cases the cut elimination procedure is not defined. By the following section, the acyclicity of a sliced proof-structure is a sufficient condition to ensure that these configurations never appear.

4.2 Preservation of correctness

Proposition 1 (Preservation of acyclicity)
If S' is a sliced proof-structure obtained from the acyclic sliced proof-structure S (without $\flat$ conclusions) by performing some steps of cut-elimination, then S' is acyclic.

Proof. We study every cut-elimination step, using the notations of definition 12:

- For the $\&_i/\oplus_j$ $(i \neq j)$ and !/0-ary ? steps, we erase a part of the graphs, such an operation cannot create cycles.
- For the ax and $\&_i/\oplus_i$ steps, some paths are replaced by shorter ones changing nothing to cycles.
- For the $⅋/\otimes$ step, if p is a path containing a cycle in S', it must use one of the two new cut links starting from the premise A of the $⅋$-link and going to the premise $A^\perp$ of the $\otimes$-link, for example. If p exists, then replacing in S the part from A to $A^\perp$ by the path going from A through the $⅋$-link, the cut link and the $\otimes$-link to $A^\perp$ would give a cycle in S.
- For the !/1-ary ? step with a $\flat$-link just above the ?-link, if s'_i contains a cycle p, either it is inside t_i and thus comes from a cycle in S or it goes outside t_i, but due to the orientation, it is impossible for a path to go outside t_i and to come inside t_i since t_i has only emergent edges (since it has only negative conclusions from remark 1).

- For the $!/1$-ary ? step with an !-link just above the ?-link: at depth 0, some paths are just replaced by shorter ones, and this cannot create any cycle. At depth 1, adding a cut and an !-link to an acyclic graph cannot create any cycle.
- For the $!/n$-ary ? step ($n > 1$), if p is a cycle in S', it has to cross one of the two residues of the cut link of S. But identifying the two ?-links, the two cut links and the two !-links in p would give a cycle in S thus p doesn't exist.

□

Theorem 1 (Preservation of correctness)
If S' is a sliced proof-structure obtained from the proof-net S by performing some steps of cut-elimination, then S' is a proof-net.

Proof. S' is acyclic by proposition 1. To conclude, we now prove that if S' is a one-step reduct of S, then (whatever reduction step has been performed) S' has exactly one positive conclusion or one $\flat$-link at depth 0 and in every slice of every box:

- The multiplicative and additive steps are straightforward and the $!/0$-ary ? step, too.
- For the $!/1$-ary ? step with a $\flat$-link just above the ?-link, the $\flat$-link at depth 0 is erased and replaced by the one coming from the slice of the box of the !-link (which necessarily exists by remark 1 and definition 11).
- For the $!/1$-ary ? step with an !-link just above the ?-link, the $\flat$-links and the positive conclusions at depth 0 are not modified and at depth 1, we just add an !-node to a slice.
- For the $!/n$-ary ? step ($n > 1$), some links are duplicated but the $\flat$-links (and the positive conclusions) are unchanged.

□

4.3 Soundness of the interpretation

We are going to prove that the cut-elimination procedure previously defined preserves the semantical interpretation, both for the coherent (set and multiset-based) semantics and for the relational semantics. We use exactly the same technique as in [5], and give the details of the proof only in the more relevant cases.

Proposition 2 (see [5])
If S is an acyclic sliced proof-structure with conclusion Γ (where Γ contains no $\flat$ formula), then $[\![S]\!]$ is an element of the space $\invamp\Gamma$.

Sketch of proof: There is nothing to prove in the relational case. In the coherent case, the result has been proven for the multiplicative fragment in [5] (in the connected case), for pure proof-nets in [4], and for the multiplicative and exponential fragment in [1]. There is no real difficulty to extend this last result to our framework. □

Remark 9
By induction on the sequent calculus proof π, one can check that the semantics of π (as defined for example in [5]) is the semantics of the sliced proof-structure S_π of definition 6. This remark gives also a proof of the previous proposition for the proof-nets coming from sequent calculus proofs.

Theorem 2
Let S' be a sliced proof-structure obtained from the acyclic proof-structure S without $\flat$-conclusions by performing some steps of cut-elimination. Then $[\![S]\!] = [\![S']\!]$.

Proof. Let Γ be the conclusion of S and S' and γ an element of $|\parr\Gamma|$. We show that there exist a slice s of S and an experiment e of s with result γ, iff there exist a slice s' of S' and an experiment e' of s' with result γ.
One has to check this is the case for each one of the cut-elimination steps defined in definition 12. We will use for these steps the notations of definition 12. Let c be a cut link of a slice s of S. Notice that our claim is obvious for the slices which are not concerned by the cut-elimination step that we consider, and we then restrict to the other ones: we prove that there exists an experiment e of s with result γ, iff there exist a slice s' of $\{s'_i\}_{i\in I}$ and an experiment e' of s' with result γ.
By induction on the depth of c in s, we can restrict to the case where c has depth 0. The steps associated with the ax and $\parr/\otimes$ cut links are the same as in [5].

- If c is a cut link of type $\&_i/\oplus_i$, and e is an experiment of s, let (i,x) be the element of $|\mathcal{A}\&\mathcal{B}| = |\mathcal{A}^\perp \oplus \mathcal{B}^\perp|$ associated by e with the two edges premises of c. Then the experiment e' of s' we look for is the "restriction" of e to s': the label associated by e with the two premises of the unary $\&_i$ and $\oplus_i$ links of s is x, and x is also the label associated by e' with the two premises of the "residue" of c in s'. For the converse, one clearly proceeds in the same way.

- If c is a cut link of type $\&_1/\oplus_2$ (or $\&_2/\oplus_1$), then there exists no experiment of s (remember the condition of definition 7 on the label of the premises of a cut link), and no experiment of $\{s'_i\}_{i\in I}$ (remember $I = \emptyset$).

- If c is a cut link of type $!/?$ with a 0-ary ?-link, then we are simply applying the weakening step of [5]

- If c is a cut link of type $!/?$ with a 1-ary ?-link whose premise is the conclusion of a $\flat$-link, let T be the sliced proof-structure associated with the !-link. With each slice t_i of T, this step associates a slice s'_i.

  Let e be an experiment of s, let $\{x\}$ be the element of $|!\mathcal{A}| = |?\mathcal{A}^\perp|$ associated by e with the two edges premises of c. By definition of experiment, because the label of the conclusion of the !-link is a singleton, there is a unique slice t_i of the sliced proof-structure T (associated with the !-link), from which e is built. Let e_i be this experiment: the label associated with the conclusion of type A of t_i will be $x \in |\mathcal{A}|$. Again by definition of experiment, the label associated by e with the premise of type A of the $\flat$ link is $x \in |\mathcal{A}^\perp|$. We can then build (from e_i) an experiment e'_i of s'_i with the same result as e. For the converse, one proceeds in the same way: an experiment e' of some slice s'_i induces an experiment e_i of t_i, and then an experiment e of s.

  (The *cognoscenti* might have noticed that in the set-based coherent semantics, this is a consequence of some "uniformity property", as mentioned in [5]).

- If c is a cut link of type $!/?$ with a 1-ary ?-link whose premise is a conclusion (of type $\flat$) of the box associated with the !-link l' (different from l), then there is nothing new w.r.t. the commutative step of [5].

- If c is a cut link of type $!/?$ with a n-ary ?-link l with $n > 1$, then let e be an experiment of s, let $\{x_1, \ldots, x_k\} = a_1 \cup \ldots \cup a_n$ be the element of $|!\mathcal{A}| = |?\mathcal{A}^\perp|$ associated by e with the two edges premises of c. Suppose that a_1 is the label of the one among the premises of the ?-link of arity n, which becomes the conclusion of the new unary ?-link. We have $\{x_1, \ldots, x_k\} = a_1 \cup \{y_1, \ldots, y_h\}$. This splitting is actually a splitting of the k experiments of the sliced proof-structure associated with the !-link. This remark is enough to conclude the existence of an experiment e' of s' with the same result as e. Conversely, let e' be an experiment of s'. Because the sliced proof-structure associated with the two !-links is the same *and it is acyclic*, we know (in the coherent case) that the results of any two experiments of such a structure are coherent (proposition 2). This allows to build an experiment e of s with the same result as e'.

□

5 Sequentialization for (cut-free) slices

We show that the conditions on sliced proof-structures given in definitions 10 and 11 yield a *correctness criterion* for cut-free proof-structures (theorem 3): they allow to characterize those proof-structures coming from sequent calculus proofs.
A novelty due to our sliced presentation is that we have to be able to glue together slices. Thanks to the polarization constraint this will be possible, provided one restricts to cut-free proof-structures. In the whole section, all our proof-structures will be cut-free.

Definition 13 (Equivalence of links)
Let s_1, s_2 be two slices of a sliced proof-structure S. Let n_1 and n_2 be two links at depth 0 having the same negative non-$\flat$ conclusion A. We define, by induction on the number of links under A in s_1, the meaning of n_1 *and* n_2 *are equivalent links* denoted by $n_1 \equiv n_2$.
If A is a conclusion of s_1 then it is also a conclusion of s_2 and $n_1 \equiv n_2$ if they are the links introducing A in s_1 and s_2.
Let A be the premise of the unary link m_1 (resp. m_2) of s_1 (resp. s_2) and the conclusion of n_1 (resp. n_2): if $m_1 \equiv m_2$, then $n_1 \equiv n_2$.
Let A be the left or right premise of the binary link m_1 (resp. m_2) of s_1 (resp. s_2) and the conclusion of n_1 (resp. n_2): if $m_1 \equiv m_2$, then $n_1 \equiv n_2$.
It is clear that $\equiv$ is an equivalence relation on the negative links at depth 0 of S.

Remark 10
If $n_1 \equiv n_2$ then n_1 and n_2 are links of the same kind except if $n_1 = \&_1$ and $n_2 = \&_2$.

Definition 14 (Weights)
Let S be a sliced proof-structure and $\&^1, \ldots, \&^k$ be the equivalence classes for $\equiv$ of the &-links at depth 0 of S. We associate to each $\&^i$ an *eigen weight* p_i that is a boolean variable. The *weight* of a slice s of S is (with an empty product equal to 1 by convention):

$$w(s) = \prod_{\&_1^i \in s} p_i \prod_{\&_2^i \in s} \bar{p}_i$$

and the weight of the set S is:

$$w(S) = \sum_{s \in S} w(s)$$

The sliced proof-structure S is *full* if $w(S) = 1$ and *compatible* if $w(s)w(t) = 0$ for $s \neq t$.

Remark 11
We can now be more precise than in remark 7: the sliced proof-structure associated by definition 6 with a cut-free sequent calculus proof is a (cut-free) proof-net, which is full and compatible.

Theorem 3 (Sequentialization)
Let S be a cut-free sliced proof-structure, S is the translation of a sequent calculus proof if and only if S is a full and compatible proof-net.

Proof. Since S has no $\flat$-conclusions, the conclusions of the slices of S are the same. We prove the second implication by induction on the size of S (the first one is remark 11. The size of a slice s is the triple (depth(s),number of ?-links with arity at least 2 and depth 0,number of links with depth 0), lexicographically ordered, and the size of S is the sum (component by component) of the sizes of the slices of S.
Let s be a slice of S. We shall say that a link of s is terminal when its conclusion is a conclusion of s.

- If s has a terminal $⅋$-link, a corresponding link appears in each slice since they have the same conclusions. We can remove these links in each slice and we obtain a sliced proof-structure S' verifying the hypothesis of the theorem.
- If s has a terminal &-link, a corresponding link appears in each slice. For some slices this link will be a $\&_1$-link (we call S_1 the set of slices obtained by erasing the $\&_1$-links in these slices) and for some others a $\&_2$-link (we call S_2 the corresponding set without the $\&_2$-links). We have to show that S_1 and S_2 are full and compatible. The weight of S_1 (resp. S_2) is obtained by taking $p = 1$ (resp. $p = 0$) in $p.w(S)$ (resp. $\bar{p}.w(S)$) thus 1. Let s and t be two slices of S_1 with weights $w_1(s)$ and $w_1(t)$, their weights in S are $p.w_1(s)$ and $p.w_1(t)$ thus $w_1(s)w_1(t) = 0$ (idem for S_2). We can now apply the induction hypothesis to S_1 and S_2.

Now, s has no terminal $⅋$-links and &-links thus it has no such links at depth 0 by polarization. This entails that s is the only slice of S by compatibility.

- If s has a terminal 0-ary ?-link, we can remove it and it corresponds to a weakening rule.
- If s has a terminal n-ary ?-link with $n \geq 2$, we break it into n unary links, we apply the induction hypothesis and perform $n - 1$ contraction rules in the sequent calculus proof thus obtained.
- If s has a unary ?-link under a $\flat$-link, we remove both of them, and this corresponds to a dereliction rule. (Notice that we can apply the induction hypothesis, because when removing the two links we replace a $\flat$-link at depth 0 by a positive conclusion).
- If none of the previous conditions is satisfied and if s has a terminal $\otimes$-link, it is a splitting link, and we can apply the induction hypothesis to the two sub-proof-structures.
- If s has a terminal $\oplus$-link, we just remove it and apply the induction hypothesis.

If s doesn't correspond to any of the cases above, either it is an axiom link (straightforward) or it is reduced to an !-link with a unary ?-link under each $\flat$-conclusion. Let S' be the sliced proof-structure associated with the !-link, let π' be the proof obtained by sequentializing S' with unary ?-links under each $\flat$-conclusion, the sequentialization π of S is obtained by adding a promotion rule to π'.

□

Remark 12
In fact, a(n apparently) stronger version of theorem 3 could be given: the reader certainly noticed that nowhere in the proof of the theorem we have used the acyclicity property of our proof-nets. This is simply due to the fact that every cut-free sliced proof-structure S is acyclic. Indeed, a path starting from a positive edge of S upwardly goes to an axiom link or an !-link and then goes down to a conclusion stopping there; while a path starting from a negative edge goes directly down to a conclusion and stops.

6 Computing with slices

We now introduce a general method, allowing to use denotational semantics in order to guarantee the "canonicity" of our proof-nets. More precisely, we introduce the notion of *injective* semantics (coming from [16]), and show how the existence of such a semantics is a witness of the canonicity of our sliced proof-nets as computational objects.

Remark 13
We will use in the sequel the strong normalization property for proof-nets w.r.t. the cut-elimination procedure. We do not give the proof of such a result, which is proven in [14] (for $\mathsf{LL}_{\mathsf{pol}}$) in the framework of polarized proof-nets with additive boxes.

Let F be a subsystem of our sliced proof-structures, and let $[\![\ ]\!]$ be an interpretation of the sliced proof-structures of F (satisfying theorem 2 and) injective: if S_1 and S_2 are two cut-free sliced proof-nets s.t. (for every interpretation of the atomic formulæ) $[\![S_1]\!] = [\![S_2]\!]$, then $S_1 = S_2$.
Another way to speak of injectivity is the following: $[\![\ ]\!]$ is injective when the semantical equivalence class of every proof-net contains a unique cut-free proof-net. In this (strong) sense our objects are canonical. In particular, such a property entails confluence: if S_1^0 and S_2^0 are two normal forms of the proof-net S, then by theorem 2 and injectivity $S_1^0 = S_2^0$.
Another crucial point is that injectivity allows to compute with the sliced proof-structures of F coming from sequent calculus proofs. Indeed, let π be any linear propositional sequent calculus proof, let S_π be the sliced proof-structure associated with π by definition 6, and let S_0 be the normal form of S_π. Now compute a normal form π_0 of π semantically correct (i.e. satisfying $[\![\pi]\!] = [\![\pi_0]\!]$), which can be done by performing cut-elimination directly in sequent calculus in several different ways.
By remark 9, $[\![\pi]\!] = [\![S_\pi]\!]$ and $[\![\pi_0]\!] = [\![S_{\pi_0}]\!]$, by theorem 2 $[\![S_\pi]\!] = [\![S_0]\!]$, and we know that $[\![\pi]\!] = [\![\pi_0]\!]$. By injectivity, we can then conclude that $S_{\pi_0} = S_0$.

$$\begin{array}{ccc} \pi & \longrightarrow & \pi_0 \\ \downarrow & & \downarrow \\ S_\pi & \longrightarrow & S_{\pi_0} \end{array}$$

Notice that the mentioned argument holds for *any existing syntax for* LL_{pol} (like proof-nets with additive boxes see [5] and [14], multiboxes see [15], proof-nets with weights see [9] and [13]): let R be a proof in such a system, it will always be possible to translate R by a sliced proof-structure S_R with the same semantics as R (in the previously mentioned syntax, this is straighforward because R is sequentializable). Let S_0 be the normal form of S_R. Let R_0 be a normal form of R and let S_{R_0} be the sliced proof-structure associated with R_0. As before, we have $S_0 = S_{R_0}$.
We are claiming that our proof-nets are canonical computational objects: they are actually the first example of such objects in presence of the additive connectives. Indeed, (sliced) proof-nets

are computational objects by theorem 1, and they are canonical by the injectivity property (as we already explained).
Notice that none of the previously mentioned syntax can really claim to yield a canonical representation of proofs: denotational semantics is not injective for proof-nets with boxes nor multiboxes (even though this last syntax realizes a much greater quotient on proofs), and it is well-known that with a proof-net with weights can be associated several sequent calculus proofs (and the cut-elimination procedure is not always defined for such proof-nets).

We then have a new canonical syntax, independent from sequent calculus, allowing to make correct computations. And if we start from a sequentializable proof-net S, we eventually reach a normal form S_0 which is itself sequentializable. This means on the one hand that nothing is lost, and on the other hand that the new objects which naturally appear (and which are not necessarily sequentializable) have a clear and well-structured computational behaviour.
The equivalence relation on sequent calculus proofs defined by our (sliced) proof-nets can be very well compared to the one defined by ordinary proof-nets in the multiplicative fragment of linear logic.

But do there exist some (interesting) subsystems F of sliced proof-structures with an injective semantics?
Such systems and semantics certainly exist in the absence of the additives (see [17]). The next section gives a positive answer to the previous question in presence of the additives. We want to mention here that this is just a first (limited) result, and it is very likely that it can be extended to full $\mathsf{LL}_{\mathrm{pol}}$.

7 An application: λ-calculus with pairing

We prove that (relational) semantics is injective for the fragment $\lambda\mathsf{LL}_{\mathrm{pol}}$ of $\mathsf{LL}_{\mathrm{pol}}$, which corresponds to the simply typed λ-calculus with pairing.

Definition 15 (λ-calculus with pairing)

$$t \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad (t)t \quad | \quad \pi_1 t \quad | \quad \pi_2 t \quad | \quad <t,t>$$

Definition 16 (Girard's translation)
The types of the λ-calculus with pairing are translated as negative formulas as follow:

$$\begin{array}{ccc} X & \rightsquigarrow & X \\ A \to B & \rightsquigarrow & ?A^{\perp} \parr B \\ A \wedge B & \rightsquigarrow & A \,\&\, B \end{array}$$

and terms are translated by the straightforward extension of Girard's translation [2] for the λ-calculus.
Let $\lambda\mathsf{LL}_{\mathrm{pol}}$ be the sub-system of $\mathsf{LL}_{\mathrm{pol}}$ containing only the following formulas:

$$\begin{array}{rclcccc} N & ::= & X & | & N \,\&\, N & | & ?P \parr N \\ P & ::= & X^{\perp} & | & P \oplus P & | & !N \otimes P \end{array}$$

and such that all the conclusions of proofs are negative formulas.
The translations of terms are proof-nets of $\lambda\mathsf{LL}_{\mathrm{pol}}$. The constraint that axiom links are introducting only atomic formulæ entails that the translation contains an implicit η-expansion of terms.

In the present section, we will restrict to proof-structures, slices and sliced proof-structures of $\lambda\mathsf{LL}_{\mathrm{pol}}$ *without cut links.*

Definition 17
Let s be a single-threaded slice. We denote by $L(s)$ (the "linearized" of s) the graph obtained by replacing every !-link n by the associated slice. More precisely, if n is an !-link having a conclusion of type of type $!A$ with an associated slice s_n, we replace n by a modified unary !-link with as premise the conclusion A of s_n and the $\flat$-conclusions of n are replaced by the corresponding $\flat$-conclusions of s_n.

Remark 14
If e is a 1-experiment of s, then with every edge a of type A of $L(s)$ is associated a unique label $e(a)$ of $|\mathcal{A}|$.
For the 1-experiment e, we will denote by $e|_{L(s)}$ the labeling of the edges of $L(s)$ associated with e.

Lemma 2
Let s and s' be two single-threaded slices. Let e (resp. e') be an injective 1-experiment of s (resp. s') with result γ (resp. γ').
If $\gamma = \gamma'$, then $L(s) = L(s')$ and $e|_{L(s)} = e'|_{L(s')}$.

Proof. Our claim is that the graph $L(s)$ so as the labels of its edges are completely determined by the types of the conclusions of s and by the result of an injective 1-experiment of s. Indeed, let's start from some edge a of $L(s)$, with its type A and its label $x \in |\mathcal{A}|$. There are exactly three cases in which either the type A of a is not enough to determine the link of $L(s)$ having a as conclusion or the link is known but the bottom-up propagation of the labels is not obviously deterministic:

1. $A = C \,\&\, D$: then a might be conclusion of a $\&_1$- or of a $\&_2$-link. But the label of a tells us which of these two cases holds, and which is the label of the premise of the &-link.

2. $A = C \oplus D$: exaclty like in the previous case.

3. $A = ?C$: then, *because e is a 1-experiment*, the cardinality of the label of a is the arity of the ?-link with conclusion a. This also implies that there is a unique way to determine the labels of the premises of the ?-link.

To conclude, notice that the fact that e is injective allows to uniquely determine the axiom links of $L(s)$. □

7.1 Recovering boxes in $\lambda\mathsf{LL}_{\mathrm{pol}}$

We are now going to use in a strong way the particular shape of the (sliced) proof-nets of $\lambda\mathsf{LL}_{\mathrm{pol}}$. We show that for a single-threaded slice s of this fragment, the graph $L(s)$ contains as much information as s. In other terms, once $L(s)$ is known, the fact that s is a single-threaded slice of a sliced proof-structure which is the translation of a term, uniquely determines the way to "put" the boxes on the graph $L(s)$.

Lemma 3
If s is a slice of a $\lambda\mathsf{LL}_{\mathrm{pol}}$ proof-net, there is exactly one $\flat$-link with depth 0 in every box of s.

Proof. Just the correctness criterion (theorem 3). □

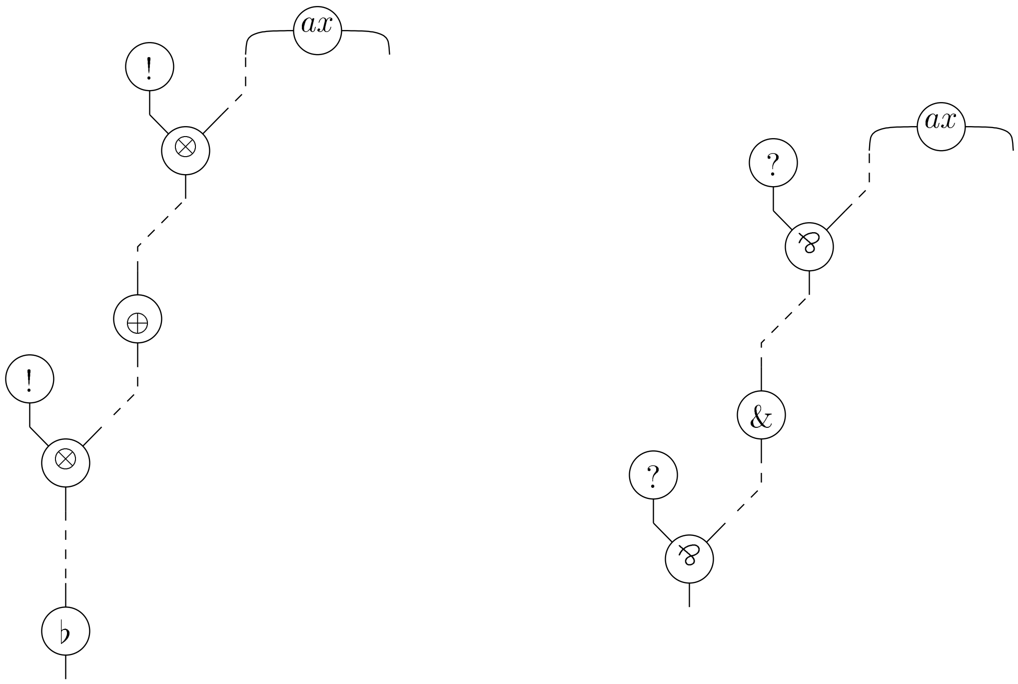


Figure 1: Combs

Lemma 4
Let s be a single-threaded slice, and a an edge of type A of s.
If A is a negative (resp. positive) formula, then the graph above a is a comb (see figure 1):

- the teeth of the "negative comb" are edges of type ?, while the outside edge is made of unary &-nodes and of ⅋-nodes and moving upwards along it one necessarily ends in the unique (negative) atomic edge of the comb.
- dually, the teeth of the "positive comb" are edges of type !, while the outside edge is made of ⊕-nodes and ⊗-nodes and moving upwards along it one necessarily ends in the unique (positive) atomic edge of the comb.

We will speak of *the comb associated with a*. Notice that a is considered as an edge of the comb.

Proof. Immediate consequence of the definition of $\lambda\mathsf{LL}_{\mathrm{pol}}$ and of the definition of single-threaded slice. □

Remark 15
As a consequence of the previous lemma, with every negative edge a of a single-threaded slice s, is associated an oriented path Φ_a of s (see figure 2): it is the path with starting edge a (oriented upwardly), following the outside edge of the negative comb up to the negative atomic edge X of the comb, crossing the axiom link and its positive conclusion $X^\perp$ (oriented now downwardly) and moving downwardly along the outside edge of a positive comb (crossing ⊕- and ⊗-links) until a ♭-link is reached (there are no other possibilities).
Clearly, a similar remark holds for positive edges (in this case, moving downwardly we will necessarily stop in an !-link or a conclusion).

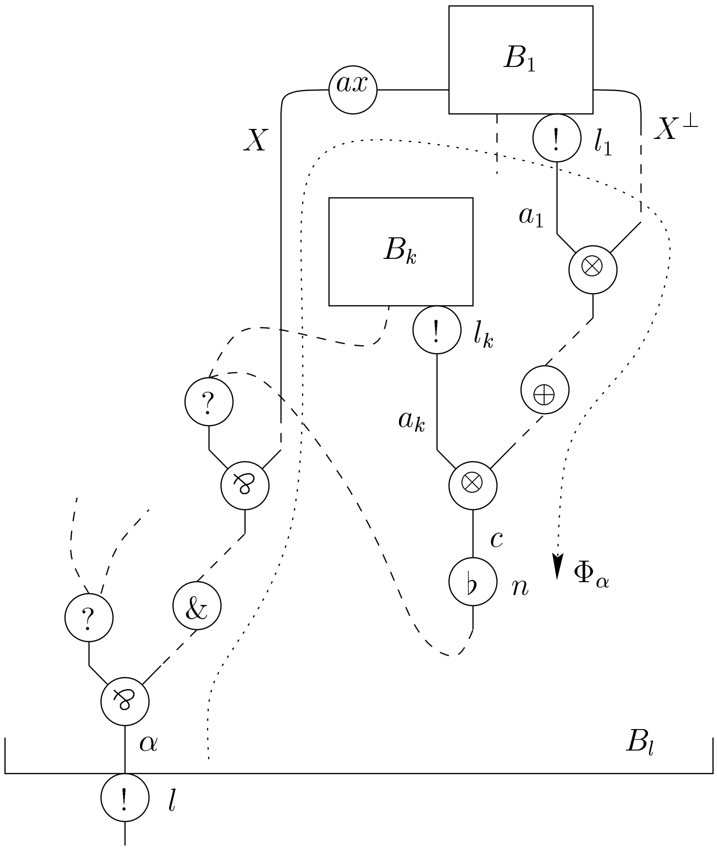


Figure 2: Lemma 5 (dashed lines are given as examples)

We will refer to Φ_a (in the sequel of the paragraph) as the oriented path associated with the negative (resp. positive) edge a.

Until the end of this section 7.1, we will fix the following notations according to figure 2:

- α is the (negative) edge premise of an !-link l of the single-threaded slice s
- B_l is the box associated with l
- n is the last link of Φ_α which is a $\flat$-link (by remark 16)
- c is the (positive) premise of n
- $l_1, \ldots, l_k$ are the $k \geq 0$!-links of s whose conclusions are the teeth of the positive comb associated with c
- $B_1, \ldots, B_k$ are the boxes associated with $l_1, \ldots, l_k$

Remark 16
Every edge of s "above α" is contained in B_l. Moreover, all the links of Φ_α (including n) are contained in B_l.

Lemma 5
Every edge with depth 0 of B_l is either an edge of Φ_α, or the conclusion of the $\flat$-link n or a tooth of one of the two combs of Φ_α.

Proof. See figure 2.
Let G be the correction graph of B_l (see definition 10). The initial nodes of G are n and the 0-ary ?-links. Every link in G is accessible by an oriented path from an initial link, but any 0-ary ?-link is the premise of a $\parr$-link (in $\lambda\mathsf{LL}_{\mathsf{pol}}$) that must be also accessible through its other premise. By induction on the number of links above this $\parr$-link we easily show that it is accessible from a non ?-link. So that every link (except 0-ary ?-links) at depth 0 in B_l is accessible from n. We said that every conclusion of a 0-ary ?-link is the premise of a $\parr$-link, and we just proved that this $\parr$-link is accessible from n: the conclusions of the 0-ary ?-links must then be teeth of the negative comb of Φ_α.

□

We are now going to define a (partial) order relation on the !-links of a single-threaded slice.

Definition 18
Let s be a single-threaded slice, let l and m be two !-links of s. We define the relation $<_1$ on the !-links of s as follows: $m <_1 l$ iff the oriented path Φ_α crosses the $\otimes$-link of s having m as premise. We define the relation $\leq$ as the reflexive and transitive closure of $<_1$.

Lemma 6
Let s be a single-threaded slice. If l (resp. m) is an !-link of s and B_l (resp. B_m) is the box associated with l (resp. m), then $m \leq l$ iff B_l contains B_m.
In particular, this implies that the relation $\leq$ is indeed a partial order relation.

Proof. Suppose that $m \leq l$. From the nesting condition, it is clearly enough to show that if $m <_1 l$, then B_l contains m, and this is a consequence of remark 16.
Conversely, suppose that B_l contains B_m. It is again enough to consider the case in which B_l is the smallest box containing B_m. By lemma 5, the conclusion of m is one of the teeth of the positive comb of Φ_α. We just proved that $m <_1 l$. □

Proposition 3
Let s and s' be two single-threaded slices of $\lambda\mathsf{LL}_{\mathsf{pol}}$. If $L(s) = L(s')$, then $s = s'$.

Proof. The reason why this holds is that the paths of $L(s)$ are the same as the paths of s.
We show, by induction on the number of !-links of $L(s)$ smaller (w.r.t. $\leq$) than l (noting that it is a finite number by lemma 6, since our graphs are finite), that once $L(s)$ is known, we know how to recover B_l. By induction hypothesis, we know how to recover $B_1, \ldots, B_k$. By lemma 5, every edge with depth 0 of B_l is either an edge of one of the two combs of Φ_α, or the conclusion of n. By remark 16, all the just mentioned edges are edges of B_l. Then B_l can only be the graph containing $B_1, \ldots, B_k$, the two combs of Φ_α (including the ?-links) and the conclusion of n. □

7.2 Injectivity for $\lambda\mathsf{LL}_{\mathsf{pol}}$

We prove the following lemma for relational semantics. It certainly holds in the coherent case too, but a detailed proof would require some more intermediate results.
In the sequel, we will write $[\![S]\!] = [\![S']\!]$, always meaning that the equality holds *for evey interpretation of the atoms* of the formulæ of S and S'.

Lemma 7
Let S and S' be two sliced proof-structures with the same conclusion.
If $[\![S]\!] = [\![S']\!]$, then $sgth(S) = sgth(S')$, where $sgth(S)$ (resp. $sgth(S')$) is the set of the single-threaded slices of S (resp. S').

Proof. By contradiction, suppose that $sgth(S) \neq sgth(S')$. There exists a single-threaded slice s of S which is different from all the single-threaded slices of S'. Let e be an injective 1-experiment of s with result γ. Such an experiment obviously exists, at least in the case of relational semantics. From $[\![S]\!] = [\![S']\!]$, there exists an experiment e' of S' with result γ. It is easy to convince oneself that e' is a 1-experiment of S' (see [17] for a proof without additives). From remark 6, e' is then a 1-experiment of a single-threaded slice s' of S'.
By lemma 2, we obtain that $L(s) = L(s')$, and then by proposition 3 $s = s'$, which is a contradiction. $\square$

Definition 19 ($\flat$-free subgraph)
The *$\flat$-free subgraph* of a slice s is the graph obtained by keeping only the part of s at depth 0 and by replacing every !-link by an !-link without any $\flat$-conclusion. This erases some $\flat$-edges that are premises of ?-links.

Definition 20 (Non-contradiction of slices)
Let s and s' be two single-threaded slices with the same non-$\flat$ conclusions, the fact that s and s' are *non-contradictory* is defined by induction on the depth d of s:

- If $d = 0$, s and s' are non-contradictory if either there exists a $\&_i$ (resp. $\&_j$) link n (resp. n') of s (resp. s') such that $n \equiv n'$ and $i \neq j$, or s and s' have the same $\flat$-free subgraph.
- If $d > 0$, s and s' are non-contradictory if the same condition occurs and the boxes of s and s' are non-contradictory.

A sliced proof-structure S is *non-contradictory* if for every pair of single-threaded slices s and s' of S, s and s' are non-contradictory.

Theorem 4
Let S and S' be two non contradictory proof-nets with the same conclusion.
If $[\![S]\!] = [\![S']\!]$, then $S = S'$.

Proof. By lemma 7, $sgth(S) = sgth(S')$. For a given set of non-contradictory single-threaded slices, there is only one way to reconstruct a sliced proof-structure: to glue (recursively with respect to the depth) the single-threaded slices with the same part at depth 0. $\square$

7.3 Computing with the λ-calculus slices

To apply the content of section 6 to $\lambda\mathsf{LL}_{\mathrm{pol}}$, just notice that if S_{π_0} is the sliced proof-structure associated with the cut-free sequent calculus proof π_0, then by remark 11 S_{π_0} is full and compatible (thus non contradictory). If π is a sequent calculus proof of $\lambda\mathsf{LL}_{\mathrm{pol}}$ and S_0 is a normal form of S_π, then $[\![S_{\pi_0}]\!] = [\![S_0]\!]$ and by theorem 4 $S_0 = S_{\pi_0}$.

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