Additives of linear logic and normalization- Part I: a (restricted) Church-Rosser property

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Abstract

We define a generalized cut-elimination procedure for proof-nets of full linear logic (without constants), for which we prove a (restricted) Church-Rosser property.

Keywords: additive connectives, cut-elimination, proof-nets with jumps, confluence.

1 Introduction

All the connectives of linear logic (LL) come from a semantical decomposition of the usual intuitionistic and classical ones. But the additive fragment of LL is much less studied than its multiplicative brother, because of its unsatisfactory syntax (which was only recently somewhat improven in [Girard 95a]).
However, the additives are the natural candidates to type functions like "predecessor" and "if...then...else..." (see [Girard 87], [Girard 95b] and [DJ 99]).
Another peculiarity of the additive connectives is that, while any fragment of LL without additives enjoys confluence property, this is no longer true when one adds the (full) additive fragment (even though denotational soundness still holds: two different reducts of the same proof have the same coherent semantics).

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This paper is a contribution to the study of proof-nets normalization in presence of the additives. The main problem is how to deal with the additive commutative elementary reduction step (e.r.s.). The only paper really dealing with this e.r.s. is [Girard 87]. Following Girard's original definition of cut-elimination, during this step an additive box swallows and duplicates a particular subproof-net having an occurrence of the cut formula among its conclusions (called the empire of this occurrence of formula). As we do not see any serious reason to duplicate precisely the empire, we propose the natural generalization of this e.r.s. (which is, obviously, semantically sound): we allow *any* subproof-net having this occurrence of formula among its conclusions to be swallowed and duplicated. Such a reduction step cannot (in general) be simulated in sequent calculus.

Let us now be more precise on the contents of the paper.
In sections 2 and 3, we define proof-nets and their normalization. This is not trivial (in the sense that we cannot rely on previous work), because one cannot ignore the several improvements in the technology of proof-nets which have been introduced since [Girard 87] (in particular [Danos 90] and [Girard 91]). These two sections are thus devoted to:

1. define a notion of proof-net for full LL, generalizing the improvements of [Danos 90] and [Girard 91]

2. adapt (or modify) such a notion in order to get a sharp enough definition of subproof-net, yielding a correct additive commutative e.r.s..

The presence of weakenings (which "disconnect" the graph) suggests to take the option of [Danos 90] for the definition of proof-nets: a proof-net is a proof-structure s.t. every correctness graph is acyclic. But this definition (definition 2.8) yields an additive commutative e.r.s. which is not correct (see the discussion following theorem 2.9). Following an idea mentioned in the appendix of [Girard 95a], we then introduce a notion of "jump" for the weakening links, which makes the proof-nets (more precisely the "correctness graphs" of the proof-nets) connected again (see definition 2.16), and allows to define a correct commutative e.r.s.. Observe that jumps are introduced here to perform cut-elimination (and not to sequentialize). Each proof-net comes then equipped with a jump function (which is not considered part of the structure), and we have to describe how this function is modified by the different cut-elimination steps (definition 3.14). Theorem 3.15, proven in [Tortora 00], shows that our definitions are sound.

In section 4, we prove a key property of the additives with respect to cut-elimination (the separation property). Like some others (rare indeed) properties and definitions introduced in this paper, the separation property is not used here, but it is essential in [Tortora 00b]. We nevertheless decided to include its proof in the paper, because on the one hand it suggested us the main theorem of section 5, and on the other hand it helps to distinguish clearly the duplication involved in a contraction e.r.s. from the one involved in an additive commutative e.r.s.. We have the feeling that (despite its simplicity) the separation property expresses an important feature of the additives.

The last section contains the main result of the paper: we prove that *every* normal form of a proof-net with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions can be reached without performing any additive commutative e.r.s.. The normal form of such proof-nets is then shown to be unique (theorem 5.12).
This is actually an important case: usual data types (integers, etc...) are formulas without & nor $\exists^2$, *but* "&" and "$\exists^2$" appear during computations (see also [Girard 95b] and [DJ 99]).
This syntactical result is connected to an ongoing semantical research (see remark 5.15 and [Tortora 00]).

2 Proof-nets

This section is devoted to the definition of proof-nets for second order LL, introduced in [Girard 87]. Roughly speaking, the multiplicatives and the exponentials are treated as in [Danos 90] and the quantifiers as in [Girard 91]. To handle weakenings we introduce a jump function, the definition of which is rendered delicate by the presence of the additive connective &.
We decided to spend some time to give a precise definition of such proof-nets, because after Girard's first paper on the subject several improvements have been suggested but never written, at least for the very general framework we are going to deal with.
The section ends with the sequentialization theorem, proven in [Tortora 00].

In the whole paper, we will usually speak of "a formula A" instead of "an occurrence of the formula A".
We deal with second order linear formulas, but quantifiers are not our main concern: once proof-nets and their normalization have been defined, they are rarely mentioned in the paper.

2.1 The sequent calculus LL_2 and LL'_2

In this paragraph, Γ and Δ are multisets of linear formulas.
The last rule below is called *mix*: it is the unique rule of LL'_2 which is not a rule of LL_2.

Axiom and cut:

$$(\mathrm{Ax}) \quad \vdash A, A^{\perp} \qquad\qquad (\mathrm{cut})\ \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}$$

Multiplicative rules:

$$(\otimes)\ \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad\qquad (\wp)\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$$

Additive rules:

$$(\oplus)\ \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

$$(\&)\ \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B, \Delta}$$

Structural rules:

$$(?\mathrm{W}\ )\frac{\vdash \Gamma}{\vdash \Gamma, ?A}$$

$$(?\mathrm{C})\ \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

Promotion or of course rule:

$$(!)\ \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$$

Dereliction or why not rule:

$$(?\mathrm{de})\ \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$$

Second (or first) order quantifiers (Y is not free in Γ):

$$(\forall)\ \frac{\vdash \Gamma, A[Y/X]}{\vdash \Gamma, \forall X\ A} \qquad\qquad (\exists)\ \frac{\vdash \Delta, A[T/X]}{\vdash \Delta, \exists X\ A}$$

Hypothesis rule:

$$(\mathrm{Hyp})\ \frac{}{\vdash \Gamma}$$

Mix rule:

$$(\text{mix}) \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

2.2. DEFINITION. **(variant of [DanReg 95])** A **pseudoproof-structure** is an oriented graph whose nodes are called links, and whose edges are labeled by formulas of LL. When drawing a proof-structure we represent edges oriented up-down so that we may speak of moving upwardly or downwardly in the graph. Links are defined together with an arity and a coarity, i.e. a given number of incident edges called the premises of the link and a given number of emergent edges called the conclusions of the link.

- A **Hypothesis** or H link has $n \geq 1$ conclusions, each of them labeled by a formula, and no premise
- an **axiom** link has no premise and two conclusions labeled by dual formulae
- a **cut** link has two premises labeled by dual formulae (which are also called the active formulas of the cut link) and no conclusion
- a **par** or $\wp$ (resp. **times** or $\otimes$) link has two premises and one conclusion. If the left premise is labeled by the formula A and the right premise is labeled by the formula B, then the conclusion is labeled by the formula $A \wp B$ (resp. $A \otimes B$)
- an **of course** link has one premise and one conclusion labeled by the **of course** of the premise
- a **dereliction** link has one premise and one conclusion labeled by the **why not** of the premise
- a **weakening** link has no premise and one conclusion labeled by $?A$ for some formula A
- a **contraction** link has two premises and one conclusion, all labeled by $?A$ for some formula A
- a **pax** link has one premise and one conclusion, both labeled by $?A$ for some formula A
- a **1-plus** or $\oplus_1$ (resp. a **2-plus** or $\oplus_2$) link has one premise and one conclusion such that if A is the label of the premise, then $A \oplus B$ (resp. $B \oplus A$) is the label of the conclusion, for some formula B. We also

speak of a **plus** or $\oplus$ link when we do not want to specify whether the link is a $\oplus_1$ or $\oplus_2$ link.

- a **with** or & link has two premises and one conclusion labeled by the **with** of the two premises

- a **coad** link has two premises and one conclusion, all labeled by the same formula

- a **forall** or $\forall$ (resp. an **exists** or $\exists$) link has one premise and one conclusion such that if A (resp. $A[T/X]$) is the label of the premise, then $\forall XA$ (resp. $\exists XA$) is the label of the conclusion. We will say that the variable X is the *eigenvariable* of the link $\forall$.

Let G be an oriented graph obtained by using the previous links, and s.t.:
(α) every edge of G is the conclusion of a unique link
(β) every edge of G is the premise of at most one link.
We say that the edges which are not premise of a link are the conclusions of G.
We will say that such a graph is a pseudoproof-structure if the three following conditions are satisfied:

(1) !-box condition:

(1.i) with each of course link n is associated a (unique) subgraph $B^!$ of G (satisfying (α) and (β)), s.t. one among the conclusions of $B^!$ is the conclusion of n and any other conclusion of $B^!$ (there might be no other conclusion) is the conclusion of a pax link. $B^!$ is called an exponential box and it is represented by a rectangular frame (see fig.1), and n is called the of course door of $B^!$

(1.ii) with each pax link n is associated an exponential box $B^!$ of G, s.t. one among the conclusions of $B^!$ is the conclusion of n (see fig.1). n is called a pax door of $B^!$

(2) &-box condition:

(2.i) with each *with* link n is associated a (unique) subgraph B of G (satisfying (α) and (β)), s.t. one among the conclusions of B is the conclusion of n and any other conclusion of B (there might be no other conclusion) is the conclusion of a coad link. B is called an additive box: it is still represented by a rectangular frame (see fig.2), and n is called the front door of B

(2.ii) with each coad link n is associated an additive box B, s.t. one among the conclusions of B is the conclusion of n (see fig.2). n is called an auxiliary door of B

(3) nesting condition:

two boxes, no matter of which kind (additive or exponential), are either disjoint or included one in the other.

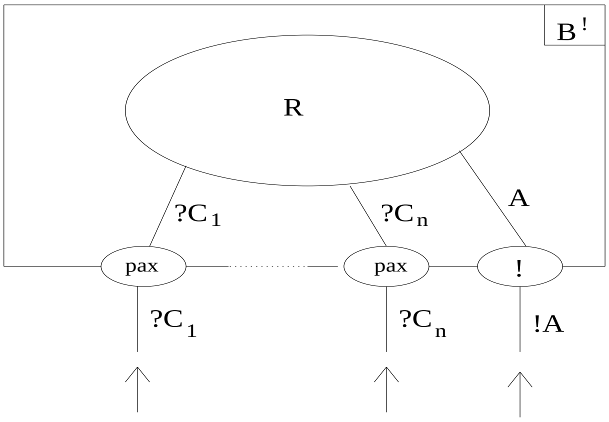


Figure 1. The exponential box $B^!$ of the proof-structure G.

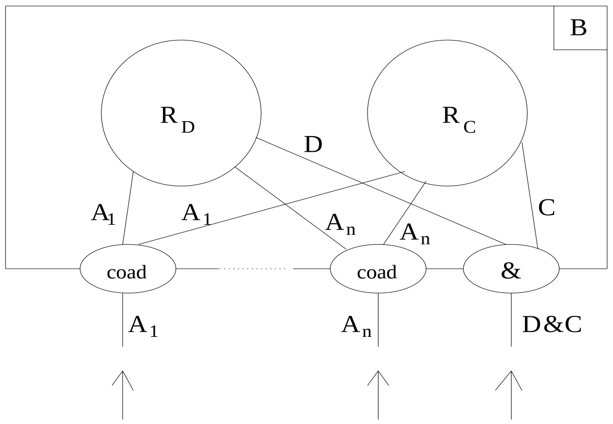


Figure 2. The additive box B of the proof-structure G.

The following links are called **logical links**: par, times, of course, dereliction, 1-plus, 2-plus, with, forall, exists. Par, times, with, 1-plus and 2-plus are also called **propositional logical links**.
We will often speak of a box, a link or an edge of a proof-structure R contained in a box B (additive or exponential) of R. In case of links, we will not consider the doors of B as links contained in B. We will also speak of "a link l (resp. an edge a) of a box B (additive or exponential)" of a given proof-structure, meaning that l (resp. a) is contained in B or it is a door (resp. a conclusion) of B. If B is a box (additive or exponential) of a proof-structure R, then the biggest (resp. the smallest) box of R containing B is clearly well-defined, thanks to the nesting condition of definition 2.2.
We shall say that a link or an edge of a given proof-structure R has exponential (resp. additive) **depth** n in R, if it is contained in exactly n exponential (resp. additive) boxes of R. For a box B (additive or exponential), we shall say that B has exponential (resp. additive) **depth** n in R, if it is contained in exactly n exponential (resp. additive) boxes of R, all different from B. When R is a proof-net, the same definition will extend to the case of a subproof-net of R (as defined in 2.19).

2.3. Remark. Let B be a box (additive or exponential) of the pseudoproof-structure R. The content of B is (by definition 2.2) itself a pseudoproof-structure.

2.4. DEFINITION. **(proof-structure)** Let R be a pseudoproof-structure. We shall say that R is a **proof-structure**, when:
(i) every (occurrence of) $\forall$ link makes use of a different eigenvariable
(ii) if a variable occurs free in the label of a conclusion of R, then it is not the eigenvariable of a $\forall$ link of R
(iii) if X is the eigenvariable of a $\forall$ link contained in a box B of R, then X cannot occur outside B
(iv) for every box B of R, the pseudoproof-structure contained B is a proof-structure.

2.5. REMARK. Behind the conditions of the previous definition, hides a phenomenon of renaming of variables: if we want for example to connect two proof-structures R_1 and R_2 by means of a $\otimes$ link, in order to get a proof-structure we might have to rename some variables occurring free in some formulas of R_1/R_2.

2.6. DEFINITION. **(graph with pairs 1)** We will say that two edges of an oriented graph are coincident when they have the same target. The couple $(G, App(G))$ is called a **graph with pairs** when G is an oriented graph and $App(G)$ is a set of n-tuples ($n \geq 2$) of coincident edges.
Let R be a proof-structure and $B_1, \ldots, B_k$ the boxes of R with additive and exponential depth zero. We are going to associate with R a set $App(R)$ and a graph with pairs $R_{ap} = (G_R, App(R))$.
The graph G_R is obtained from R in the following way:
- to each box B_i with p_i conclusions ($i \in \{1, \ldots, k\}$), substitute a link H with p_i conclusions
- if n is a $\forall$ link with eigenvariable X_n, if the link m of R is not a link of one among the boxes B_i and if one premise of m has a label containing a free occurrence of X_n, then add an edge having as source m and as target n
- if n is a $\forall$ link with eigenvariable X_n, if the link m of R is a link of one among the B_i and if one premise of m has a label containing a free occurrence of X_n, then add an edge having as source the H link substituting the box and as target n.
The set $App(R)$ contains the following (and only the following) m-tuples:
- the couples of premises of every $\wp$ and $?co$ link of R with additive and exponential depth zero
- the p-tuples of the incident edges of every $\forall$ link of G_R (among which there is of course at least the premise of the $\forall$ link considered).

2.7. Definition. **(correctness graph 1)** Let R be a proof-structure and $B_1, \ldots, B_k$ the boxes with additive and exponential depth zero in R. Let $R_{ap} = (G_R, App(R))$ be the graph with pairs associated with R by definition 2.6.
A **switching S** of R is the choice of an edge for every n-tuple of $App(R)$. With each switching S is associated an unoriented graph $S(R)$, called **correctness graph**: for every n-tuple of $App(R)$, erase the edges of G_R which are not selected by S, and then forget the labels and the orientation of the edges of the graph. The correctness graph of R associated with S will be denoted by $S(R)$.

2.8. Definition. **(proof-net -preliminary-)** Let R be a proof-structure and $B_1, \ldots, B_k$ the boxes with additive and exponential depth zero in R. We say that R is a **proof-net** when the following conditions are satisfied:
- R satisfies (AC): for every switching S of R, the correctness graph $S(R)$ is acyclic (there is no cycle in $S(R)$)
- for every exponential box $B_i \in \{B_1, \ldots, B_k\}$, the proof-structure R_i contained in B_i is a proof-net
- for every additive box $B_i \in \{B_1, \ldots, B_k\}$, there exist two disjoint proof-nets R_i^1 and R_i^2 contained in B_i, s.t. every link contained in B_i is either a link of R_i^1 or a link of R_i^2, every conclusion of R_i^1 (resp. R_i^2) is the premise of a door of B_i, and every door of B_i has two premises: a conclusion of R_i^1 and a conclusion of R_i^2.

If B is an additive box of a proof-net T, then (by definition of proof-net) there is a a subproof-structure R of T which is a proof-net and s.t. B has additive and exponential depth zero in R. If the conclusion of the front door of B is labeled by $E\&F$, then we will often speak of the first (resp. the second) **component of** B meaning (with the notations of the definition) the one among R_i^1 and R_i^2 which has E (resp. F) among its conclusions, and we will denote it by $\gamma_1(B)$ (resp. $\gamma_2(B)$). We will sometimes write $\gamma_i(B)$, always meaning $i \in \{1, 2\}$; and we will also refer to $\gamma_i(B)$ as the component i of B.
Let LL'_2 denote the usual second order linear sequent calculus with the mix rule. It is easy to see that with each proof in LL'_2 is associated a (unique) proof-net.
The converse is known to be a difficult result. However, the reader acquainted with proof-nets will find no difficulty to believe that by generalizing the results of Danos thesis ([Danos 90]) to LL'_2 one can prove the following theorem

2.9. THEOREM. *If R is a proof-net, then there exists a proof π of LL'_2 s.t. R is the proof-net associated with π. π is then called a sequentialization of R.*

Definition 2.8 is a very general definition of proof-net, and one could try to prove properties like strong normalization and confluence refering to this definition. Without additives, this can be done by adapting to this framework the proofs of [Girard 87] and of [Danos 90]. But as soon as we add the additive connectives, we have to deal with a (general or restricted) notion of *subproof-net*, because of the additive commutative e.r.s.. The presence of disconnected correctness graphs makes the notion of subproof-net delicate to define; the naïve one (a subproof-net is a subproof-structure which is a proof-net), leads to incorrect proof-structures: the proof-structure obtained from a proof-net by eliminating an additive commutative cut link might not be a proof-net any more. The reader who whishes to convince herself/himself can consider the additive box B having among the conclusions of its *coad* links an edge labeled by A, and the proof-net T obtained by cuting two axiom links, both with conclusions $A, A^\perp$. She/he can then consider the proof-net R obtained by cuting B and T. The two axiom links (and their conclusions) are a subgraph T' of T which *is* a proof-net (in the sense of definition 2.8): but if we duplicate T' (that is we apply an elementary reduction step [*ccad*], see definition 3.9 of next section) we get an additive box s.t. two conclusions of two different doors are connected by a *cut* link. This *is not* a proof-net (in the sense of definition 2.8) any more. One has then to find a way to define a proof-net (and thus a subproof-net) as a proof-structure whose correctness graphs are not only acyclic but also connected.

The disconnectedness of the correctness graph is only due to the presence of weakening links. We then introduce the notion of **jump**, the intuitive meaning of which is to "connect" every weakening link of a proof-net to one of its boxes (additive or exponential) or to one of its axiom links. The idea of introducing the notion of jumps for weakenings is not new: as far as we know it is an idea of J.-Y. Girard and has been explicitely mentioned in the appendix of [Girard 95a]. However, the works making a precise use of this idea are very rare. Indeed, we are only aware of the recent work [Laurent 99], which makes use of a notion of jump in the restricted polarized case, mainly for a sequentialization purpose.

2.10. DEFINITION. Let R be a proof-net and B an additive box of R. Let l_1 and l_2 (resp. a_1 and a_2, B_1 and B_2) two links (resp. edges, boxes) of R. We shall say that B **separates** l_1 and l_2 (resp. a_1 and a_2, B_1 and B_2), when they are contained in two different components of B. We shall also say that l_1 and l_2 (resp. a_1 and a_2, B_1 and B_2) are **separated** by B, or simply **separated**.

2.11. REMARK. If two links, edges or boxes of a proof-net R are separated, there obviously exists a unique (additive) box separating them, to which we can then refer to as *the* box which separates them.

In the following definition, condition (1) guarantees that after any cut-elimination step (see the next section) every weakening link is still "connected" to a box or an axiom link, condition (2) guarantees that there are not "too many" jumps, and condition (3) is clearly necessary for the sequentailization theorem (theorem 2.17).

2.12. DEFINITION. Let T be a proof-net, let W be the set of the weakening links of T, let Ax be the set of the axiom links of T and $\mathcal{P}(Ax)$ the power set of Ax. Let j_T be a function from W to $\mathcal{P}(Ax)$.
We say that (T, j_T) is a **proof-structure with jumps** iff for every link n of W, there exists an integer $k \geq 1$ s.t. $j_T(n) = \{y_1, \ldots, y_k\}$, and the following contitions hold:
(1) if $y \in j_T(n)$ is a link of an additive box B of T which does not contain n, then there exists $y' \in j_T(n)$ s.t. B separates y and y'.
(2) if $i, j \in \{1, \ldots, k\}$ and $i \neq j$, then there exists an additive box B_{ij} of T, which does not contain n and separates y_i and y_j.
(3) if B is a box (of any kind) and n is a link of B, then every link of $j_T(n)$ is a link of B.
The axiom links of $j_T(n)$ are called the **jumps** of n.

2.13. REMARK. We use the notations of the previous definition.
(i) If $k \geq 2$ then there exists an additive box of T containing all the links of $j_T(n)$. In this case we denote by B_n the smallest among these boxes. One can check that B_n does not contain n.
(ii) If y is a jump of the weakening link n, then $j_T(n) = \{y\}$ iff n and y are contained in the same additive boxes of T.

Let us now give an intuitive explanation of the previous definition.
Let π be a proof of LL_2 without the rule *Hyp*. We can associate with π a (unique) proof-net (in the sense of definition 2.8) R_π. We are going to describe how π defines also a jump function j_{R_π} which makes (R, j_{R_π}) a proof-structure with jumps (actually a proof-net with jumps, following now

definition 2.16). The following description clearly shows that in general j_{R_π} is not unique.
Let S be a weakening rule of π and w_S be the the weakening link of R_π associated with S. We have to define a set $j_{R_\pi}(w_S)$ of axiom links of R_π. We start from the rule S of π and move upwards in π "until we reach some axioms". More precisely: when we meet a rule with one premise we continue moving upwards, when we meet a rule with two premises (i.e. a &-rule or a $\otimes$-rule), if it is a $\otimes$-rule we choose (arbitrarily) one of the premises and we continue moving upwards, while if it is a &-rule we move upwards in both the premises. The process ends in a certain number (greater than zero, because there is no Hyp-rule in π) of axiom rules. The set of axiom links of R_π associated with these rules is $j_{R_\pi}(w_S)$.

Conversely, suppose now that T is not only a proof-structure with jumps but also a proof-net (following the final definition, see 2.16 and theorem 2.17).
If $j_T(n) = \{y\}$, and y and n are contained in the same boxes (additive and exponential) of T, then for every sequentialization of T the weakening rule corresponding to n will "follow" the axiom rule corresponding to y.
If $j_T(n) = \{y\}$, and y is contained in a box which does not contain n, then (by condition (1)) every such box is an exponential box. Let $B^!$ be the biggest among these boxes and m be the of course door of $B^!$. For every sequentialization of T, the weakening rule corresponding to n will "follow" the of course rule corresponding to m.
If $k \geq 2$, then let m be the front door or the of course door of the biggest box B of T which does not contain n and which contains $\{y_1, \ldots, y_k\}$ (B *might* be the additive box B_n). For every sequentialization of T, the weakening rule corresponding to n will "follow" the with or the of course rule corresponding to m.

Thanks to the notion of jump, we have "connected" each weakening link to a box or an axiom link: we now have a notion of proof-net for which the correctness graphs are not only acyclic but also connected (see definition 2.16). This gives a sharper notion of subproof-net (see definition 2.19) which allows to define a correct additive commutative e.r.s. (see theorem 3.15).

2.14. Definition. **(graph with pairs 2)** Let (R, j_R) be a proof-structure with jumps. We are going to associate with (R, j_R) a set $App(R)$ and a graph with pairs, which we still denote by $R_{ap} = (G_R, App(R))$, obtained from the graph G_R defined in 2.6 by adding an edge for each weakening link n of R with additive and exponential depth zero:

- if $j_R(n) = \{y\}$ and y has additive and exponential depth zero in R, then we add an edge having as source n and as target y
- otherwise, we add an edge having as source n and as target the H link substituting the box with additive and exponential depth zero which contains all the links of $j_R(n)$.

2.15. DEFINITION. **(correctness graph 2)** The definition of switching and of correctness graph is the same as the one given in 2.7, except the fact that R_{ap} (then G_R) will be the graph of definition 2.14 instead of the one of definition 2.6.

The following is the final definition of proof-net to which we will refer in the sequel, except in section 5 (where we will distinguish between proof-nets in the sense of definition 2.8, and proof-nets with jumps following the next definition). PN_2 will denote the set of all proof-nets.

2.16. DEFINITION. **(proof-net -final-)** Let (R, j_R) be a proof-structure with jumps and $B_1, \ldots, B_k$ the boxes with additive and exponential depth zero in R. We say that (R, j_R) is a **proof-net** when the following conditions are satisfied:
- R satisfies (ACC): for every switching S of R, the correctness graph $S(R)$ is acyclic and connected (i.e. $S(R)$ is a tree)
- for every exponential box $B_i \in \{B_1, \ldots, B_k\}$, let R_i be the proof-structure contained in B_i, and let j_{R_i} be the restriction of j_R to the weakening links of R_i. We require that (R_i, j_{R_i}) is a proof-net
- for every additive box $B_i \in \{B_1, \ldots, B_k\}$, let R_i^1 and R_i^2 be the two proof-structures of definition 2.8 (the two components of B_i), and let $j_{R_i^1}$ (resp. $j_{R_i^2}$) be the restriction of j_R to the weakening links of R_i^1 (resp. R_i^2). We require that $(R_i^1, j_{R_i^1})$ and $(R_i^2, j_{R_i^2})$ are two proof-nets.

In the sequel, we will often omit to mention the function j_R and we will simply speak of "a proof-net R", always meaning "a proof-net (R, j_R)".
Let now LL_2 denote the usual second order linear sequent calculus (without the mix rule). Like in the case of LL_2', it is easy to see that with each proof in LL_2 is associated a proof-net. The reader acquainted with proof-nets will probably believe that the following theorem holds. The proof can be found in [Tortora 00].

2.17. THEOREM. *If R is a proof-net, then there exists a proof π of LL_2 s.t. R is a proof-net associated with π. π is then called a sequentialization of R.*

2.18. REMARK. A proof-net associated with a proof of LL_2 which does not make use of the rule Hyp does not contain any link H, and conversely. In the sequel, we will then rather refer to these (more standard!) notions of proof and proof-net.

2.19. DEFINITION. Let (T, j_T) be a proof-net and let R be a subgraph of T. We will say that R is a **subproof-net** of T, when (R, j_R) is a proof-net, where j_R is the restriction of j_T to the set of the weakening links of R.

2.20. REMARK. (i) If R is a subproof-net of the proof-net T and n is a weakening link of R, then the links of $j_R(n) = j_T(n)$ are links of R.
(ii) If B is an additive box of the proof-net (R, j_R), then by definition 2.16 $(\gamma_i(B), j_{\gamma_i(B)})$ (resp. (B, j_B)), where $j_{\gamma_i(B)}$ (resp. j_B) is the restriction to $\gamma_i(B)$ (resp. B) of j_R, is a subproof-net of (R, j_R).

3 Cut-elimination for proof-nets

In this section we define the elementary reduction steps (e.r.s.) associated with the different kinds of cut links of PN_2. If (R, j_R) is a proof-net with jumps and x is a cut link of R, an e.r.s. associated with x (and denoted by $[x]$) is a deformation of the graph R into a graph R' and a "modification" of the function j_R into the function $j_{R'}$. The function $j_{R'}$ *is not* unique (see definition 3.14), and this makes the $[x]$-reduct of (R, j_R) not unique. However, provided $[x]$ is not an additive commutative e.r.s., the graph R' is uniquely determined by x and R (whatever the jump function j_R is). Otherwise stated, the jump function interfere with the cut-elimination process only during the $[ccad]$ e.r.s.. This remark (stated in 3.11) will play an important role in section 5.
Theorem 3.15, proven in [Tortora 00], shows that our definition of these e.r.s. is compatible with the definition 2.16 of proof-net given in the previous section.

We first introduce the notions of straight and direct path of a proof-net. A direct path is (a bit roughly speaking) an oriented path moving from a subformula to a conclusion or a cut.

3.1. DEFINITION. (**[DanReg 95]**) Let R be a proof-net. A path of R is a sequence of edges or reverted edges (i.e. a path may take an edge from its goal to its source). We denote by $\alpha, \beta, \ldots$ the edges of a proof-net (oriented as usual, following definition 2.2) and by $\alpha^*, \beta^*, \ldots$ the previous edges "reverted" (i.e. oriented now in the opposite direction). Let now a (resp. b) be an edge or a reverted edge whose goal (resp. source) is the link n; we denote by ab the path consisting of the edge a followed by the link n, itself followed by the edge b.
We will say that the path Φ of R is a **straight path** if:
(i) Φ does not contain any $\alpha^*\alpha$ nor any $\alpha\alpha^*$
(ii) Φ does not contain any $\alpha\beta^*$, where α and β are two premises (and then, according to our notations, two edges) of a same link.

In the whole paper, we will simply write "path" always meaning "straight path". We will often speak of "a path with starting link n and a terminal link m", meaning that the first edge of the path is the one which comes "after" n (w.r.t. the orientation of this path) and the last edge of the path is the one which comes "before" m (w.r.t. the orientation of this path). We will also speak of "a path with starting link n and a terminal edge a", meaning that the first edge of the path is the one which comes "after" n (w.r.t. the orientation of this path) and the last edge of the path is a. If Φ and Ψ are two paths of a proof-net R, we will denote by $\Phi \cap \Psi$ the set of all the links and the (oriented) edges that belong to Φ and to Ψ.

3.2. DEFINITION. Let R be a proof-net, and let Φ be a path of R with starting link n and terminal link m (resp. terminal edge a). We will say that Φ is a **direct path** iff the two following conditions hold:
(i) Φ crosses no cut link (except perhaps m), and no axiom link (except perhaps n)
(ii) every edge of Φ is oriented downwardly (i.e. it is an edge -and not a reverted edge- of R).

Intuitively, a path changes direction when it crosses an axiom or a cut link, and it can contain some edges oriented downwardly and some others oriented upwardly. This is not the case for a direct path: it cannot change direction (condition (i)) and all its edges are oriented downwardly (condition (ii)).

3.3. REMARK. Let R be a proof-net and m be a link of R. One and only one of the two following conditions holds:

(1) there exists a cut link c of R and a (unique) direct path Φ_m with starting link m and terminal link c
(2) there exists a conclusion a of R and a (unique) direct path Φ_m with starting link m and terminal edge a.

We need the following lemma to define cut-elimination for proof-nets, and the notion of substitution of a subproof-net by a proof-net. Having in mind the explanation following definition 2.12, one feels that the lemma is true. We nevertheless give a detailed proof.

3.4. LEMMA. *Let B be an additive box of the proof-net (R, j_R) and let t be a door of B.*
There exists an integer $h \geq 2$ and a set $Ax_t = \{z_1, \ldots, z_h\}$ of axiom links of B s.t. the following conditions hold:
(i) if $z \in Ax_t$ is a link of an additive box B' contained in B (one might have $B' = B$), there exists $z' \in Ax_t$ s.t. B' separates z and z'
(ii) if $z_i, z_j \in \{z_1, \ldots, z_h\}$ and $z_i \neq z_j$, there exists an addititve box B_{ij} contained in B (one might have $B_{ij} = B$) which separates z_i and z_j
(iii) $\forall i \in \{1, \ldots, h\}$ there exists a direct path Ψ_t with terminal link t s.t. one of the following statements holds for z_i:

- (α) *Ψ_t has z_i as starting link*
- (β) *there exists a weakening link m_i of B s.t. $z_i \in j_R(m_i)$ and Ψ_t has m_i as starting link.*

proof: Let L be the set of the axiom or weakening links l of B s.t. there exists a (unique) direct path Φ_l of T having l as starting link and t as terminal link. We denote by $\sharp\Phi_l$ the number of doors of additive boxes (including t) crossed by Φ_l. Clearly $L \neq \emptyset$.
We proceed by induction on $p = max\{\sharp\Phi_l : l \in L\}$.
If $p = 1$, then for $i \in \{1, 2\}$ there exists $y_i \in \gamma_i(B)$ s.t. the unique door of an additive box crossed by Φ_{y_i} is t (in general there are of course many of these links and we choose one of them in each of the two components of B). The set $Ax_t = Y_1 \cup Y_2$ will do, where for $i \in \{1, 2\}$ $Y_i = \{y_i\}$ if y_i is an axiom link, and $Y_i = j_R(y_i)$ if y_i is a weakening link. Observe that $\Psi_t = \Phi_{y_i}$, and if y_i is an axiom link then y_i satisfies condition (α), while if y_i is a weakening link then every link $z \in j_R(y_i)$ satisfies condition (β).
If $p > 1$ there exists a link $u \in L$ s.t. $\sharp\Phi_u > 1$. Suppose for example that $u \in \gamma_1(B)$. Let B_1 be the biggest additive box of $\gamma_1(B)$ containing u and let t_1 be the door of B_1 crossed by Φ_u. Let L_1 be the set of the axiom or

weakening links l of B_1 s.t. there exists a (unique) direct path Φ_l^1 of T having l as starting link and t_1 as terminal link. Every link l belonging to L_1 is a link of L, Φ_l^1 is a subpath of Φ_l and $\sharp\Phi_l^1 < \sharp\Phi_l$. Then $p_1 = max\{\sharp\Phi_l^1 : l \in L_1\} < p$ and we can apply the induction hypothesis to B_1: there exists an integer $h_1 \geq 2$ and a set $Ax_{t_1} = \{z_1^1, \ldots, z_{h_1}^1\}$ of axiom links of B_1 satisfying the conditions of our lemma.

If there exists also a link $u' \in \gamma_2(B)$ s.t. $\sharp\Phi_{u'} > 1$, then we call B_2 the biggest additive box of $\gamma_2(B)$ containing u', t_2 the door of B_2 crossed by $\Phi_{u'}$ and (like before) we apply the induction hypothesis to B_2: there exists an integer $h_2 \geq 2$ and a set $Y_2 = Ax_{t_2} = \{z_1^2, \ldots, z_{h_2}^2\}$ of axiom links of B_2 satisfying the conditions of our lemma. If such a link u' does not exist, then there exists $y_2 \in \gamma_2(B) \cap L$ s.t. the only door of an additive box crossed by Φ_{y_2} is t. In this case, if y_2 is an axiom link, we define $Y_2 = \{y_2\}$, while if y_2 is a weakening link we define $Y_2 = j_R(y_2)$. We are going to show that, in any case, the set $Ax_t = Ax_{t_1} \cup Y_2$ satisfies the conditions of our lemma.

Condition (i): it is satisfied by induction hypothesis for the additive boxes contained in B_1 (including B_1) and then by every additive box of $\gamma_1(B)$. If there exists $u' \in \gamma_2(B)$ s.t. $\sharp\Phi_{u'} > 1$, then condition (i) is also satisfied by the additive boxes contained in B_2 (including B_2) and then by every additive box of $\gamma_2(B)$. If such an axiom link u' does not exist, then y_2 has additive depth zero in $\gamma_2(B)$, and either $Y_2 = \{y_2\}$ (in which case (i) is vacuously satisfied by every additive box of $\gamma_2(B)$), or $Y_2 = j_R(y_2)$ and then by (1) of definition 2.12 condition (i) is satisfied by every additive box containing a link of Y_2 (and then by every additive box of $\gamma_2(B)$). The fact that condition (i) holds for the additive box B is immediate from the definition of Ax_t.

Condition (ii): let $z, z' \in Ax_t$. If $z, z' \in Ax_{t_1}$ or if $z, z' \in Y_2$, then (ii) is satisfied by induction hypothesis (and by (2) of definition 2.12). Otherwise suppose $z \in Ax_{t_1}$ and $z' \in Y_2$: B is then the additive box separating z and z'.

Condition (iii): first remember that Φ_u has $u \in L$ as starting link, the door t of B as terminal link, and that Φ_u crosses t_1. Let then $\Phi_{t_1 t}$ be the subpath of Φ_u having t_1 as starting link and t as terminal link. If $z \in Ax_{t_1}$, then by induction hypothesis there exists a direct path Ψ_{t_1} with terminal link t_1 s.t. Ψ_{t_1} and z satisfy (α) or (β). The direct path Ψ_t obtained by connecting Ψ_{t_1} and $\Phi_{t_1 t}$ has t as terminal link and Ψ_t and z satisfy (α) or (β). If $z \in Y_2 = Ax_{t_2}$, then one argues exactly like for $z \in Ax_{t_1}$. If $z \in Y_2 = \{y_2\}$, then y_2 is an axiom link and clearly y_2 and Φ_{y_2} satisfy (α). If $z \in Y_2 = j_R(y_2)$, then y_2 is a weakening link and every link of $j_R(y_2)$ and Φ_{y_2} satisfy (β). $\square$

Let α be a subproof-net of the proof-net T, and let β be a proof-net with the same conclusions as α. Let G be the graph obtained from T by substituting β to α. Up to a renamimg of some variables of G, the thus obtained graph satisfies the conditions of definitions 2.2 and 2.4. Let's call R this proof-structure.

3.5. DEFINITION. Let α be a subproof-net of the proof-net T, and let β be a proof-net with the same conclusions as α. One can associate with the proof-structure R previuosly defined a proof-net by defining the function j_R. In general, this can be done in several (different) ways. We denote by $T[\beta/\alpha]$ any of these proof-nets s.t. the function $j_{T[\beta/\alpha]}$ satisfies the three following conditions. Let n be a weakening link of R. If n is a link of α, then (see definition 2.19) all the links of $j_T(n)$ are links of α. We require that:
(1) if n is a link of β, then $j_{T[\beta/\alpha]}(n) = j_\beta(n)$
(2) if n is a link of T but not of α and if every link of $j_T(n)$ is not a link of α, then $j_{T[\beta/\alpha]}(n) = j_T(n)$
(3) if n is a link of T but not of α, let $j_T(n) = \{y_1, \dots, y_k\}$ and suppose that $y_1, \dots, y_l$ $(1 \leq l \leq k)$ are links of α while $y_{l+1}, \dots, y_k$ aren't. We then require that one (and only one) of the two following conditions holds:

- there exists an axiom link y of β with additive depth zero in β and $j_{T[\beta/\alpha]}(n) = \{y\} \cup \{y_{l+1}, \dots, y_k\}$
- there exists an additive box B with additive depth zero in β and a set $Ax = \{z_1, \dots, z_h\}$ of axiom links of B satisfying conditions (i) and (ii) of lemma 3.4, and $j_{T[\beta/\alpha]}(n) = Ax \cup \{y_{l+1}, \dots, y_k\}$.

3.6. REMARK. (i) One can check that any proof-structure $T[\beta/\alpha]$ previously defined is a proof-structure with jumps which satisfies (inductively) condition (ACC) of definition 2.16, i.e. it is indeed a proof-net.
(ii) For every T, α, and β satisfying the hypothesis of the previous definition, there exists (by lemma 3.4) at least a proof-net $T[\beta/\alpha]$.
(iii) Let T, α, and β be like in the previous definition. Let δ be a subproof-net of T which is not a subproof-net of α, and let S_δ denote the proof-structure obtained from δ by substituting β to α (if α is not a subproof-net of δ, then $S_\delta = \delta$). For every proof-net $T[\beta/\alpha]$, we denote by $\delta[\beta/\alpha]$ the proof-net associated with S_δ s.t. the function $j_{\delta[\beta/\alpha]}$ is the restriction of the function $j_{T[\beta/\alpha]}$ to the weakening links of S_δ: $\delta[\beta/\alpha]$ is then a subproof-net of $T[\beta/\alpha]$.

We now define the e.r.s. associated with the cut links of PN_2. For every cut link x of a proof-net R, we first define the graph R' obtained from R by reducing the cut link x (definition 3.9), and we then define the function $j_{R'}$ obtained from j_R by reducing the cut link x (definition 3.14). Theorem 3.15 states that $(R', j_{R'})$ is a proof-net.

3.7. Definition. Let R be a proof-net and x a cut link of R.
x is called a **logical** cut link when the two premises of the link are conclusions of two (necessary dual) logical links: we also denote these cut links by $\wp/\otimes$, $!/?de$, $\&/\oplus_1$, $\&/\oplus_2$ (or simply $\&/\oplus$ when we do not want to specify which one of the two plus links occurs), $\forall/\exists$.
x is called a **contraction** (resp. a **weakening**) cut link when one premise of x is the conclusion of a contraction (resp. a weakening) link and the other one is the conclusion of an of course link: we also denote this cut link by *co* (resp. w).
x is called an **exponential commutative** cut link when one premise of x is the conclusion of a pax link and the other one is the conclusion of an of course link: we also denote this cut link by *cc*.
x is called an **axiom** cut link when one of the two premises of x is the conclusion of an axiom link.
x is called an **additive commutative** cut link when one of the two premises of x is the conclusion of a coad link: we also denote this cut link by *ccad*.

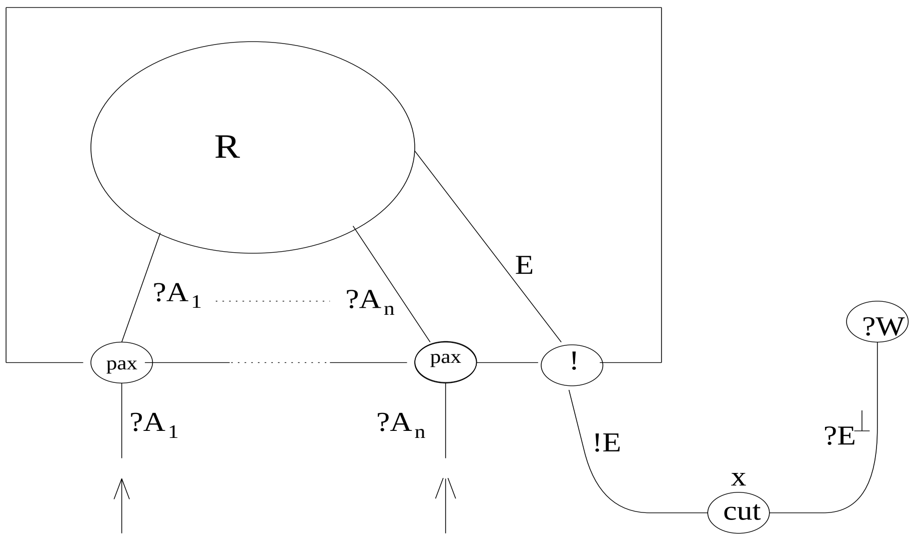
3.8. Remark. If x is a cut link of a proof-net, then x is of exactly one of the kinds mentioned in the previous definition, *except in one case*: if one of the premises of x is the conclusion of an axiom link and the other one is the conclusion of a coad link, then x is both an axiom and a *ccad* cut link.

We are now going to define the elementary reduction steps associated with the different kinds of cut links.

3.9. Definition. Let R be a proof-net and x a cut link of R.
If x is a logical (different from a $\forall/\exists$), a contraction, or a *cc* cut link, then we denote by $[x]$ the unique e.r.s. associated with x and defined in [Girard 87]. When x is a logical cut link, $[x]$ is called a **logical e.r.s.**.
If x is a $\forall/\exists$ cut link, then let $A[X]$ (resp. $A^{\perp}[T/X]$) be the label of the premise of the $\forall$ link (resp. the $\exists$ link) whose conclusion is a premise of x. The e.r.s. $[x]$ consits in replacing *every* occurrence of the (first/second order) variable X in R by the term/formula T, and by substituting the cut

link x by a cut link with premises labeled by $A[T/X]$ and $A^{\perp}[T/X]$ (this is nothing but a generalization to PN_2 of the e.r.s. defined in [Girard 91]).
If x is a weakening cut link, then we denote by $[x]$ the unique e.r.s. associated with x and defined in figure 3.
If x is an axiom cut link, then we denote by $[ax]$ the axiom e.r.s. associated with x defined in [Girard 87]. (In case both the premises of x are conclusions of axiom links, we have in fact *two* possible e.r.s.).
If x is a *ccad* cut link, let n be one of the coad links whose conclusion is a premise of x (there is at least one such link and there are at most two such links) and suppose that the conclusion of n is labeled by the formula A. Let now S be *any* subproof-net of R having the premise of x labeled by $A^{\perp}$ among its conclusions, and let's call B the additive box of R having the premise of x labeled by A among its conclusions. We then denote by [*ccad*] the e.r.s. defined in figure 4. This e.r.s. is said to be B-attractive (or attractive for the box B) because B is the additive box of R that swallows (and duplicates) the subproof-net S.
In the sequel, we will denote by $[x]$ any (given) e.r.s. associated with the cut link x.

The configuration



is replaced by the configuration

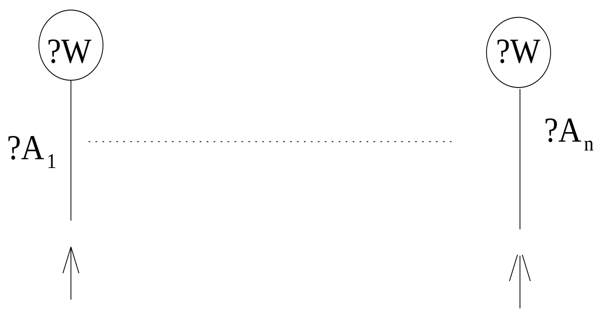
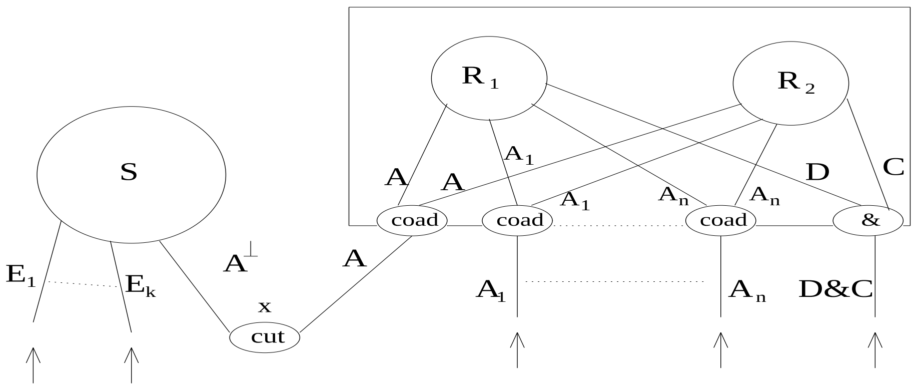


Figure 3. The $[w]$ elementary reduction step.

The configuration



is replaced by the configuration

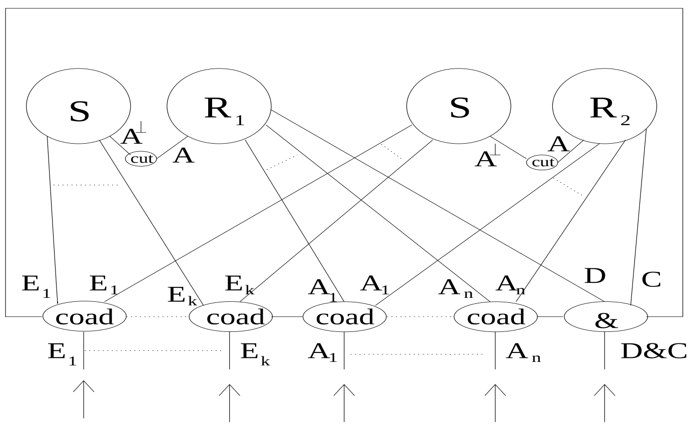


Figure 4. The [*ccad*] elementary reduction step.

3.10. Definition. Let R be a proof-net. We say that R' (which for the moment is simply a graph) is a one step reduct of R, if there exists a cut link x of R s.t. R' is obtained from R by applying the e.r.s. $[x]$: we write in that case $R \overset{[x]}{\to} R'$.

3.11. Remark. Let x be a cut link of the proof-net R, and suppose $R \overset{[x]}{\to} R'$. Then:
(i) if $[x] \neq [ccad]$, then the graph R' is uniquely determined by the cut link x. In this case, by combining a result of Danos' thesis ([Danos 90]) and a result of [Girard 91], one can show that if R is a proof-net in the sense of the preliminary definition (definition 2.8), then R' is also a proof-net in the sense of the preliminary definition. The proof can be found in [Tortora 00].
(ii) if $[x] = [ccad]$, then to determine the graph R' we still need to know which is the box B s.t. $[x]$ is B-attractive, and which is the subproof-net S of R swallowed by B when performing $[x]$. This is the only case in which the jump function is used to define the graph R' (remember that by definition 2.19 the notion of subproof-net is given with the help of the jump function).

3.12. Definition. Let R' be a one step reduct of the proof-net R.
Any logical or axiom link n of R' comes from a unique occurrence of ("the same") logical or axiom link $\overleftarrow{n}$ in R. We call $\overleftarrow{n}$ the **lift of the link** n in R. Conversely, we define the set of **residues of the** logical (resp. axiom) **link** l in R, as the set of all the occurrences l' of logical (resp. axiom) links in R' such that $\overleftarrow{l'} = l$.
In the particular case where the logical link considered in R' is a *with* or an *of course* link, it is actually an additive or an exponential box B, which comes from a unique occurrence of "the same" additive or exponential box $\overleftarrow{B}$ in R (observe, however, that the links contained in the box are not necessarily the same in R as in R'). We call $\overleftarrow{B}$ the **lift of the box** B in R. Conversely, we define the set of **residues of the box** B_1 of R as the set of all the occurrences B_1' of boxes in R' such that $\overleftarrow{B_1'} = B_1$.
Similarly, any occurrence of a cut link c in R' comes from a unique occurrence $\overleftarrow{c}$ of a cut link with the same active formulas (up to substitutions if the reduced cut is $\forall/\exists$) in R, except for those cut links which are created by a logical e.r.s.. In this last case, we shall say that the created links have no lift in R and that the logical cut link reduced has no residue in R'.

It is also possible to define the notions of lift and residue for weakening links.

3.13. Definition. Let R be a proof-net and $R \stackrel{[x]}{\to} R'$.
If $[x] \neq [w]$, then any weakening link n of R' stems from a unique occurrence of ("the same") weakening link $\overleftarrow{n}$ of R, that will be called (as usual) **the lift** of n in R. Conversely, we define the set of **residues of the** weakening link l of R, as the set of all the occurrences l' of weakening links in R' such that $\overleftarrow{l'} = l$.
If $[x] = [w]$, then any weakening link n of R' stems from a unique occurrence of ("the same") weakening link $\overleftarrow{n}$ of R, except for the k weakening links $n_1, \ldots, n_k$ which are created by the $[w]$ e.r.s. ($k \geq 0$). In this case, if m is the weakening link of R whose conclusion is active in x, we say that m is the lift of n_i ($i \in \{1, \ldots, k\}$) and that $\{n_1, \ldots, n_k\}$ is the set of residues of m in R'.

Let R' be a one step reduct of R. The following definition associates to R' *several* functions $j_{R'}$.

3.14. Definition. Let (R, j_R) be a proof-net and suppose that $R \xrightarrow{[x]} R'$. We distinguish all the possible cases for $[x]$ and we define $j_{R'}^{[x]}$ (that will simply be denoted by $j_{R'}$).
Let m' be a weakening link of R', m its lift in R and let $j_R(m) = \{y_1, \ldots, y_k\}$ be the set of the jumps of m. We have to define $j_{R'}(m')$.
Case 1: $[x]$ is an axiom e.r.s.. Let n be the axiom link erased by $[x]$. $\forall i \in \{1, \ldots, k\}$, if $y_i \neq n$ then y_i has a unique residue y'_i in T'.
If $\forall i \in \{1, \ldots, k\}$ $y_i \neq n$, then we define $j_{R'}(m') = \{y'_1, \ldots, y'_k\}$. Otherwise suppose that $n = y_1$. Let B_1 be the smallest additive box containing x and $n = y_1$ if it exists. We call α the component of B_1 containing x and y_1 if B_1 exists, and we call α the proof-net R otherwise. y_1 is a link with additive depth zero in α, so that (by condition (2) of definition 2.12) $\forall i \in \{2, \ldots, k\}, y_i$ is not a link of α.
There exists an axiom or a weakening link l and a direct path Φ_l of R with starting link l and terminal link x which does not cross y_1. (Observe that l and Φ_l are not uniquely determined). Either l is contained in an additive box of R which does not contain x (first case), or l is contained in all the additive boxes of R containing x (second case).
In the second case, if l is an axiom link then we call l' its unique residue in R' and we define $j_{R'}(m') = \{l', y'_2, \ldots, y'_k\}$. If l is a weakening link, then

every jump of l is a link of α, it is different from y_1 (otherwise a correctness graph of R would contain a cycle, thus contradicting definition 2.16), and it has a unique residue in R'. We call Z' the set of all these residues and we define $j_{R'}(m') = Z' \cup \{y'_2, \dots, y'_k\}$.
In the first case, let B be the biggest among these boxes and let t be the door of B crossed by Φ_l. Let $Ax_t = \{z_1, \dots, z_h\}$ $(h \geq 2)$ be the set of axiom links of R given by lemma 3.4. $\forall j \in \{1, \dots, h\}$ z_j has a unique residue z'_j in R'. Let Ax'_t be the set of the h residues of the links of Ax_t. We define $j_{R'}(m') = Ax'_t \cup \{y'_2, \dots, y'_k\}$.
Case 2: x is a $\wp/\otimes$, a $\forall/\exists$ or a $!/?de$ cut link. Then $\forall i \in \{1, \dots, k\}$, y_i has a unique residue y'_i and we define $j_{R'}(m') = \{y'_1, \dots, y'_k\}$.
Case 3: x is a $\&/\oplus$ cut link. Then we define $j_{R'}(m')$ as the set of the residues of the links of $j_R(m)$ (in this case some links of $j_R(m)$ might have no residue in R'). Condition (1) of definition 2.12 is here crucial: it implies that $j_{R'}(m') \neq \emptyset$.
Case 4: x is a weakening cut link. Let then n be the of course link whose conclusion is a premise of x, let $B^!$ be the exponential box associated with n and let z be the weakening link whose conclusion is a premise of x. Every link of $j_R(z)$ is not a link of $B^!$ (by definition 2.16), and has then a unique residue in R'. Let Z' be the set of these residues. It is important to stress the fact that in any case $Z' \neq \emptyset$. Let J' be the set of the residues of the links of $j_R(m)$. We distinguish 3 cases:

(4.1) $m \neq z$ and every jump of m is not a link of $B^!$: in this case $j_{R'}(m')$ is simply the set J' of the residues of the links of $j_R(m)$ (which is not empty)

(4.2) $m \neq z$ and there is a jump of m in $B^!$: one then defines $j_{R'}(m') = J' \cup Z'$

(4.3) $m = z$: then m' is one of the weakening links created by $[x]$, and one defines $j_{R'}(m') = Z'$.

Case 5: $[x]$ is a contraction e.r.s.. Let then n be the of course link whose conclusion is a premise of x, let $B^!$ be the exponential box associated with n and let $B'^!_1$ and $B'^!_2$ be its two residues in R'. Let Y be the subset of $j_R(m)$ containing all the links of $j_R(m)$ which are contained in $B^!$, and let Y'_1 (resp. Y'_2) be the set of the residues of the links of Y which are contained in $B'^!_1$ (resp. $B'^!_2$). Let Z' be the set of the residues in R' of the links of $j_R(m)$ which are not contained in Y.

If m is a link of $B^!$, then ($Y = j_R(m)$ and) there exists $i \in \{1,2\}$ s.t. m' is a link of $B_i'^!$. We then define $j_{R'}(m') = Y_i'$.
If m is not a link of $B^!$, then we distinguish the case $Y = \emptyset$ from the case $Y \neq \emptyset$. If $Y = \emptyset$, then we define $j_{R'}(m') = Z' = \{y_1', \dots, y_k'\}$. If $Y \neq \emptyset$, then we define $j_{R'}(m') = Z' \cup Y_i'$ where we have chosen the integer $i \in \{1,2\}$ arbitrarily.
Case 6: x is a cc cut link. In this case $\forall i \in \{1, \dots, k\}$ y_i has a unique residue y_i', and we simply define $j_{R'}(m') = \{y_1', \dots, y_k'\}$.
Case 7: $[x]$ is a $[ccad]$ e.r.s.. Let B be the additive box of R s.t. $[x]$ is B-attractive and let (S, j_S) be the subproof-net of (R, j_R) duplicated by $[x]$. Let B' be the residue of B in R'. Let Y be the subset of $j_R(m)$ containing the links of $j_R(m)$ which are links of S, and let Y_1' (resp. Y_2') be the set of the residues of the links of Y which are contained in the first (resp. the second) component of B'.
If m is a link of S, then (by definition 2.19 of subproof-net) $j_R(m) = Y$ and there exists $i \in \{1,2\}$ s.t. m' is a link of $\gamma_i(B')$. We then define $j_{R'}(m') = Y_i'$.
If m is not a link of S, then we simply define $j_{R'}(m')$ as the set of all the residues of the links of $j_R(m)$.

3.15. THEOREM. *If (R, j_R) is a proof-net, x is a cut link of R, $R \xrightarrow{[x]} R'$, and $j_{R'}$ is one of the functions associated with j_R by definition 3.14, then $(R', j_{R'})$ is a proof-net.*

proof: See [Tortora 00]. □

3.16. REMARK. Let $R \xrightarrow{[x]} R'$, where (R, j_R) is a proof-net. Let m' be a weakening link of R' and m its lift in R.
If $[x]$ is not an axiom e.r.s. nor a weakening e.r.s., then $j_{R'}(m')$ contains only residues of links of $j_R(m)$.
If $[x]$ is not a contraction e.r.s., then $j_{R'}(m')$ contains all the residues of the links of $j_R(m)$.

3.17. DEFINITION. Let R' be a one step reduct of the proof-net R. There exists an obvious bijection $i_{R,R'}$ from the conclusions of R to the ones of R'. If $i_{R,R'}(a) = a'$, we will say that the edge a (conclusion of R) is **the lift** of the edge a' (conclusion of R') and that a' is **the residue** of a in R'.

The notion of lift and residue is slightly more delicate for subproof-nets than for links. Let $R \overset{[x]}{\to} R'$. Because during cut-elimination a subproof-net can be erased (by a $[\&/\oplus]$ or a $[w]$ e.r.s.) or modified (by an e.r.s. $[x]$ s.t. x is a cut link belonging to the subproof-net), a subproof-net α of R' comes from *at most* one occurrence of ("the same" up to substitutions) subproof-net $\overleftarrow{\alpha}$ of R. One would like to say that such a subproof-net of R (if it exists) is the lift of α in R. But this is still ambiguous: let $B^!$ be an exponential box of R', $\overleftarrow{B^!}$ its lift in R, suppose that x is a contraction cut link and that the conclusion of the of course door of $\overleftarrow{B^!}$ is a premise of x. Suppose that the subgraph α of R' consisting in the box $B^!$ and in the weakening link n (which does not belong to $B^!$) and its conclusion is a subproof-net of R' (we are simply saying that $j_{R'}(n)$ contains only links of $B^!$). Are we going to consider the subproof-net of R consisting in $\overleftarrow{B^!}$ and in the lift $\overleftarrow{n}$ of n (and its conclusion) as the lift of α? (Observe that $\overleftarrow{B^!}$ has two residues while $\overleftarrow{n}$ has only one residue in R'). The answer is "no", as stated by the following definition.

Intuitively, any link of a proof-net R "comes from" a logical, an axiom or a weakening link, as stated more precisely in the following lemma. We will then say that the link n of the proof-net R is a **main link** of R if n is a logical, an axiom or a weakening link.

3.18. LEMMA. *Let T be a proof-net. For every link n of T, there exists a main link l and a direct path Φ of T having l as starting link and n as terminal link.*

proof: Straightforward. $\square$

3.19. DEFINITION. Suppose that R and R' are two proof-nets s.t. $R \overset{[x]}{\to} R'$, and let α be a subproof-net of R'. α comes from at most one occurrence of "the same" (up to substitutions if $[x] = [\forall/\exists]$) subproof-net $\overleftarrow{\alpha}$ of R. If such a subproof-net $\overleftarrow{\alpha}$ exists, we shall say that it is the **lift of the subproof-net** α when one (and only one) of the following conditions holds:
- every main link of α has a unique residue in R'
- every main link of α has two residues in R' (i.e. it comes from a duplication).
Conversely, we define the set of **residues of the subproof-net** β of R, as the set of all the occurrences β' of subproof-nets of R' such that β is the lift of β'.

What motivated the previous definition is in fact the following remark.

3.20. REMARK. Suppose that T and T' are two proof-nets s.t. $T \overset{[x]}{\to} T'$ and that x is a contraction cut link. Let n be the of course link of T whose conclusion is a premise of x and let $B^!$ be the exponential box of T associated with n. If R is a subproof-net of T which has a residue in T' and if R contains $B^!$, then $R = B^!$.

3.21. REMARK. Let R and R' be two proof-nets s.t. $R \overset{[x]}{\to} R'$, and let n be an axiom, a logical, a weakening, or a cut link of R. If one of the two following conditions holds, then n has no residue in R' and we say that n belongs to the subproof-net of R erased by $[x]$:
(a) $[x]$ is a $[w]$ e.r.s. and n is a link of the exponential box erased by $[x]$
(b) x is a $\&/\oplus$ cut link out of the main door of the additive box B of R, and n is a link of the component of B erased by $[x]$.
Conversely, if n has no residue in R', then one has the following possibilities:
(i) If n is a logical link, then (a) or (b) holds, or x is a logical cut link and the conclusion of n is a premise of x.
(ii) If n is an axiom link, then (a) or (b) holds, or $[x]$ is an axiom e.r.s. and n is the axiom link erased by $[x]$.
(iii) If n is a weakening link, then (a) or (b) holds, or the conclusion of n is a premise of the weakening cut link x and the exponential box erased by $[x]$ has no pax door.
(iv) If n is a cut link, then (a) or (b) holds, or $n = x$ is a logical cut link, or $n = x$ is a weakening cut link, or $n = x$ and $[x]$ is an axiom e.r.s..

3.22. REMARK. Let R and R' be two proof-nets s.t. $R \overset{[x]}{\to} R'$, and consider the particular case of the previous remark in which n is a *with* link (i.e. an additive box) or an *of course* link (i.e. an exponential box).
(i) If B is an additive box of R and if there is no residue of B in R', then there are exactly three possibilities:
- $[x] = [w]$ is a weakening e.r.s. which erases an exponential box containing the additive box B
- $[x] = [\&/\oplus]$ is a logical e.r.s. which erases a subproof-net of R containing the additive box B
- $[x] = [\&/\oplus]$, where $x = \&/\oplus$ is a logical cut out of the main door of B.
In the first two cases we will say that B is erased by $[x]$, while in the third we will say that B is opened by $[x]$.
(ii) If $B^!$ is an exponential box of R and if there is no residue of $B^!$ in R', then there are exactly three possibilities:

- $[x] = [w]$ is a weakening e.r.s. and the box $B_1^!$ of R s.t. the conclusion of the of course link of $B_1^!$ is a premise of x contains $B^!$ (one may have $B_1^! = B^!$)
- $[x] = [\&/\oplus]$ is a logical e.r.s. which erases a subproof-net of R containing the exponential box $B^!$
- x is a $!/?de$ cut link out of the *of course* door of $B^!$.
In the first two cases we will say that B is erased by $[x]$, while in the third we will say that B is opened by $[x]$.

3.23. Remark. (i) If σ is a sequence of e.r.s. starting from (R, j_R), then we can define the notion of σ-reduct (or simply reduct) of R. If $(R', j_{R'})$ is a σ-reduct of R, then we write $(R, j_R) \overset{\sigma}{\twoheadrightarrow} (R', j_{R'})$. We will also write $R \overset{\sigma}{\twoheadrightarrow} R'$, or simply $R \twoheadrightarrow R'$.
(ii) The notions of lift and of residue of a logical, axiom, cut, weakening link and those of a box, a subproof-net, a conclusion, can obviously be generalized so as to be relative to a reduction sequence.
(iii) Let us stress again the fact that $(R', j_{R'})$ is not uniquely determined by (R, j_R) and σ; the notation $(R, j_R) \overset{\sigma}{\twoheadrightarrow} (R', j_{R'})$ simply means that *there exists* a way to obtain $(R', j_{R'})$ from (R, j_R) by applying to the e.r.s. of σ definition 3.14.
In particular, when we will make use of the notion of substitution of a proof-net to a subproof-net in a given proof-net (defined in 3.5), we will sometimes write that (under certain hypothesis, of course) if $T \overset{\sigma}{\twoheadrightarrow} T'$, then $T[\beta/\alpha] \overset{\sigma}{\twoheadrightarrow} T'[\beta/\alpha'_1, \ldots, \beta/\alpha'_n]$, where α is a subproof-net of T, β a proof-net with the same conclusions as α, and $\alpha'_1, \ldots, \alpha'_n$ are the residues of α in T' (see definition 3.19). What we mean is that there exists a jump function $j_{T'[\beta/\alpha'_1,\ldots,\beta/\alpha'_n]}$, and a way to obtain $(T'[\beta/\alpha'_1, \ldots, \beta/\alpha'_n], j_{T'[\beta/\alpha'_1,\ldots,\beta/\alpha'_n]})$ from $(T[\beta/\alpha], j_{T[\beta/\alpha]})$ by applying to the e.e.r. of σ definition 3.14.

The following useful lemma is constantly used in the paper.

3.24. Lemma. **(substitution lemma)** Let T, T' be two proof-nets, s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$. Let α be a subproof-net of T, and β any proof-net with the same conclusions as α.
If, for every $[x] \in \sigma$ s.t. $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$, and for every residue γ_1 of α in T_1, the following conditions are satisfied:

i) if $[x] = [ax]$ then x is not contained in γ_1 and the link ax erased by $[x]$ is not contained in γ_1

ii) if $[x] \neq [ax], [ccad]$ then none of the premises of x is an edge of γ_1 (in particular, any conclusion of γ_1 is not a premise of x)

iii) if $[x] = [ccad]$ then x is not a link of γ_1, $[x]$ is not attractive for a box of γ_1 and if $[x]$ duplicates a subproof-net of γ_1 then $[x]$ duplicates the whole γ_1

then, if $\alpha'_1, \ldots, \alpha'_n$ are the residues of α in T', we have $T[\beta/\alpha] \overset{\sigma}{\twoheadrightarrow} T'[\beta/\alpha'_1, \ldots, \beta/\alpha'_n]$ (up to substitutions). If $[\forall/\exists] \in \sigma$, then we have, more precisely, $T[\beta/\alpha] \overset{\sigma}{\twoheadrightarrow} T'[\beta_1/\alpha'_1, \ldots, \beta_n/\alpha'_n]$, where $\beta_i = \beta$ in which we performed the substitution(s) required by the e.r.s. $[\forall/\exists]$.

proof: Intuitively, the conditions $(i), (ii), (iii)$ guarantee that $[x]$ does not modify anything in any residue of α: it can only duplicate or erase residues. More formally, we proceed by induction on σ. If σ is empty, then $T = T'$ and the result is obvious.
Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. By induction hypothesis, if $\alpha_1^1 \ldots \alpha_1^m$ are the residues of α in T_1, we have $T[\beta/\alpha] \overset{\sigma_1}{\twoheadrightarrow} T_1[\beta/\alpha_1^1, \ldots, \beta/\alpha_1^m]$ (up to substitutions). To achieve the result, it is enough to check that for any residue α_1^i of α in T_1 and for all the residues α' of α_1^i in T', we have that $T_1[\beta/\alpha_1^i] \overset{[x]}{\rightarrow} T'[\beta/\alpha']$ (up to substitutions if $[x] = [\forall/\exists]$), where $[x]$ is any e.r.s. satisfying the conditions $i)$, $ii)$ and $iii)$. In case $[x]$ is a $[ccad]$ e.r.s. which duplicates the subproof-net δ of T_1, it is important to observe that (thanks to (iii) of remark 3.6) $\delta[\beta/\alpha_1^1, \ldots, \beta/\alpha_1^m]$ is a subproof-net of $T_1[\beta/\alpha_1^1, \ldots, \beta/\alpha_1^m]$. $\square$

4 The separation property

In this short section, we prove a property of normalization (that we call "separation property") which is tightly linked to the presence of the connective &, as we show in remark 4.3 (see also figure 5). This property plays a central role in the standardization theorem of [Tortora 00b].
Following the intuition (corroborated by the coherent denotational semantics) that an additive box is nothing but a syntactical way to represent the superimposition of two computational processes, the separation property can be (informally) stated as: $\ll$normalization preserves superimposition *at exponential depth zero*$\gg$.

The following (seemingly obvious) remark is fundamental.

4.1. Remark. Let R be a proof-net and let R' be any reduct of R. If l is a logical link, an axiom link or a *logical* cut link, with exponential depth zero in R, then any residue of l in R' has exponential depth zero in R'. The same holds for a subproof-net α of R different from an exponential box, and for an additive box B of R: if α and B have exponential depth zero in R, then every residue of α in R' and every residue of B in R' have exponential depth zero in R'.
This is because the only possibility for a link (different from a commutative exponential cut) to enter an exponential box is to be already in *another* exponential box, active in an exponential commutative e.r.s..

We are now going to prove the separation property.

4.2. Proposition. **(separation property).** Let T, T' be two proof-nets s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$, and let $\mathbb{B}$ be an additive box with exponential depth 0 in T.
If $\mathbb{B}'$ and $\mathbb{B}''$ are two (different) residues of $\mathbb{B}$ in T', then $\mathbb{B}'$ and $\mathbb{B}''$ are separated.

proof: We have to show the existence of an additive box B' of T' which separates $\mathbb{B}'$ and $\mathbb{B}''$.
We proceed by induction on σ. If $T' = T$, then the result is obvious.
Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$.
Let $\mathbb{B}'_1$ (resp. $\mathbb{B}''_1$) be the lift of $\mathbb{B}'$ (resp. $\mathbb{B}''$) in T_1.
In case $\mathbb{B}'_1 = \mathbb{B}''_1 = \mathbb{B}_1$, then (because $\mathbb{B}'' \neq \mathbb{B}'$) $\mathbb{B}_1$ is duplicated during the e.r.s. $[x]$. But as $\mathbb{B}_1$ has exponential depth zero in T_1 (from remark 4.1), the only possibility is that $[x] = [ccad]$. Let then B_1 be the additive box of T_1 s.t. $[x]$ is B_1-attractive, and let S be the subproof-net of T_1 swallowed by B_1 when performing $[x]$. $\mathbb{B}_1$ is (obviously) a subproof-net of S. The box B' of T' that we are looking for is the unique residue of B_1 in T'.
Suppose now that $\mathbb{B}'_1 \neq \mathbb{B}''_1$. By induction hypothesis, there exists an additive box B_1 of T_1 which separates $\mathbb{B}'_1$ and $\mathbb{B}''_1$.
Observe that the box B_1 of T_1 has one or two residue(s) in T': if B_1 would have no residue in T', then (see also (ii) of remark 3.22) at least one of the two components of B_1 would also disappear, so that one among $\mathbb{B}'_1$ and $\mathbb{B}''_1$ would have no residue in T', which is of course impossible.
Let's then consider the two possibilities:

(i) B_1 has one residue in T': in this case the box B' of T' that we are looking for is the unique residue of B_1 in T'.

(ii) B_1 has two residues in T': let B'_{11} and B'_{12} be these two boxes of T'. Again because B_1 has exponential depth zero, $[x] = [ccad]$ and B_1 is contained in the subproof-net S of T_1 duplicated by the e.r.s. $[x]$. Let's call B_2 the additive box of T_1 s.t. $[x]$ is B_2-attractive, and let's call B'_2 the (unique) residue of B_2 in T'.

As subproof-nets of B_1 (then of S), $\mathbb{B}'_1$ and $\mathbb{B}''_1$ have both two residues in T'. More precisely, one residue of $\mathbb{B}'_1$ (resp. of $\mathbb{B}''_1$) is a subproof-net of B'_{11} and the other one is a subproof-net of B'_{12}. Let's call $\mathbb{B}'_{11}$ (resp. $\mathbb{B}''_{11}$) the residue of $\mathbb{B}'_1$ (resp. of $\mathbb{B}''_1$) contained in B'_{11} and $\mathbb{B}'_{12}$ (resp. $\mathbb{B}''_{12}$) the one contained in B'_{12}. We have that $\mathbb{B}' = \mathbb{B}'_{1i}$ and $\mathbb{B}'' = \mathbb{B}''_{1j}$, where $i, j \in \{1, 2\}$.

If $i = j = 1$ (resp. $i = j = 2$), then the box B'_{11} (resp. B'_{12}) is the additive box of T' that we are looking for. If $i \neq j$, then the box B'_2 is the additive box of T' that we are looking for. □

4.3. Remark. (i) Like every box which separates two links or boxes, the additive box B' given by the separation property is unique.
(ii) The separation property is wrong if we omit the hypothesis "$\mathbb{B}$ has exponential depth zero in T". Suppose for example that $T \overset{[co]}{\to} T'$ and that $\mathbb{B}$ is inside the exponential box duplicated by the $[co]$ step. The two residues $\mathbb{B}'$ and $\mathbb{B}''$ of $\mathbb{B}$ in T' are not subproof-nets of two different components of an additive box of T'. This contradicts the conclusion of the lemma, and also (what is even worse) its intuitive meaning: $\mathbb{B}'$ and $\mathbb{B}''$ are not superimposed in T', so that the $[co]$ step of normalization did not preserve superimposition. Let us just mention the fact that this phenomenon seems to be related to the well-known (crucial) denotational isomorphism between the coherent spaces $!(A\&B)$ and $!A\otimes!B$: $\ll$a with link which has exponential depth greater than zero is not a true with link$\gg$.
(iii) The separation property says (in particular) that, in any reduct of a proof-net T, two different residues of an additive box of exponential depth zero in T *cannot interact* (they will never belong to two different subproof-nets connected by a $\otimes$-link, for example). Observe that this is not at all the case for the residues of an exponential box. Intuitively, this means that the duplication involved in an additive commutative e.r.s. is of a completely different nature from the one involved in a contraction e.r.s..
(ii) From the separation property, it follows that a residue $\mathbb{B}'$ of an additive box $\mathbb{B}$ of exponential depth 0 in T cannot be a subproof-net of a residue $\mathbb{B}''$ of $\mathbb{B}$ different from $\mathbb{B}'$. Let us stress the fact that this is a very strong

property, which allows for example to bound *a priori* the maximal additive depth of the reducts of a proof-net with no exponential boxes.
This is of course wrong in general (if $\mathbb{B}$ has not exponential depth zero): like the exponential boxes, the additive ones can (in some sense) "swallow themselves", and this has the notable consequence that one cannot bound *a priori* the maximal additive depth (like the maximal exponential depth) of the reducts a proof-net. This phenomenon does not desappear in the systems with "light" complexity such as ELL and LLL: even though the exponential depth cannot increase in those systems, the contents of two different exponential boxes can interract, so that the phenomenon of an additive box "swallowing itself" can still occur. In [Girard 95b], this is carefuly avoided: one does not perform the e.r.s. [*ccad*], and we will see in next section, that this is usually very reasonable...

Figure 5 gives an example of an exponential box "swallowing itself". The example is a bit more complicated than what would be strictly necessary, so that the reader acquainted with ELL can see that, despite the fact that the exponential depth is constant in ELL, an *additive* box contained in the exponential box B of figure 5 can swallow one of its residues.
Let's observe also that the proof-net we started from (in figure 5) corresponds to the λ-term $(\lambda f(f)(f)x)\lambda yt$: the phenomenon we are analyzing appears then in full light.

The proof of the following lemma is very much like the proof of the separation property.

4.4. LEMMA. *Let* T *be a proof-net s.t.* $T \overset{\sigma}{\twoheadrightarrow} T'$, *let* c *be a cut link of* T, *and suppose that if* R *is a proof-net appearing in* σ, *then every residue of* c *in* R *has exponential depth zero in* R.
If c' *and* c'' *are two (different) residues of* c *in* T', *then* c' *and* c'' *are separated.*

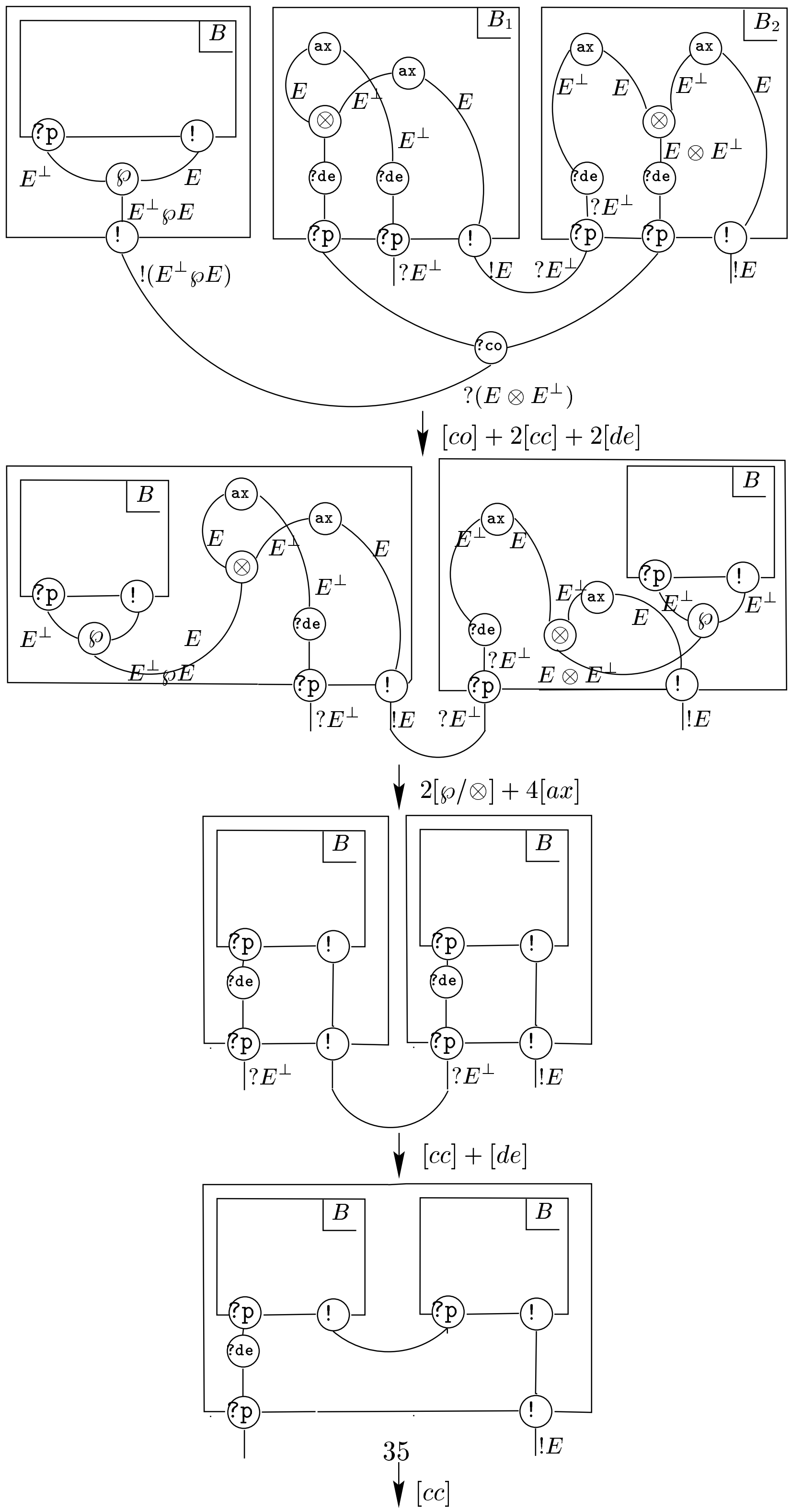


Figure 5. A residue of B swallows another residue of B

5 LL and the Church-Rosser property

As stated in the introduction, full linear logic does not enjoy the confluence property. This can be easily seen for example by eliminating the cut between the two canonical proofs of $\vdash A^{\perp}\&A^{\perp}, A$ and $\vdash A\&A, A^{\perp}$. The point is that both the premises of the cut link are conclusions of an auxiliary door of an additive box: one has to choose which box swallows and duplicates the other one (observe that this conflict occurs also with the original definition of [*ccad*] given in [Girard 87]).
Nevertheless, the separation property seems to indicate that this lack of confluence is not, so to speak, very damaging. The aim of this section is precisely to discuss a mathematical version of this informal claim. The result will be the following: if among the conclusions of the proof-net R there are no occurrences of the connective & nor of the quantifier $\exists^2$ (existential quantifier of the second order), then the normalization procedure defined in section 3 associates to R a unique cut-free proof-net (theorem 5.12). Let us stress again the fact that this case is very important to "program" with proof-nets (following the approach described in [KriPar 90]). Data types are actually formulas without & nor $\exists^2$, even though these connectives appear when performing cut-elimination (see also remark 5.18, [Girard 95b] and [DJ 99]).

In this section, we will consider the following sets of proof-nets:
- PN_2: the set of proof-nets with jumps defined in 2.16, whose elements will be called proof-nets with jumps
- PN_2^+: the set of proof-nets (without jumps) defined in 2.8, whose elements will be called proof-nets
- PN_2^-: the subset of PN_2^+ whose elements are the proof-nets which do not contain any link & nor any link $\oplus$.
With every proof-net with jumps one can obviously associate a proof-net (simply forget the jumps), so that each proof-net is actually an equivalence class of proof-nets with jumps.
Except for the [*ccad*] e.r.s., cut-elimination for proof-nets is perfectly well-defined by definition 3.9 (see also remark 3.11), so that the rewriting rule $\overset{\neg[ccad]}{\longrightarrow}$ obtained from the usual one by forbidding [*ccad*] e.r.s., is well-defined on proof-nets. We'll denote by $\overset{\neg[ccad]}{\twoheadrightarrow}$ its reflexive and transitive closure. We will also say that σ is a $\neg[ccad]$ reduction sequence when $\forall[x] \in \sigma$ $[x] \neq [ccad]$.
Let (R, j_R) and $(R', j_{R'})$ be two proof-nets with jumps s.t. $(R, j_R) \overset{\sigma}{\twoheadrightarrow} (R', j_{R'})$,

where σ is a $\neg[ccad]$ reduction sequence. Then one has for the proof-nets R, R': $R \overset{\sigma}{\twoheadrightarrow} R'$. Because in the absence of $[ccad]$ e.r.s. the notion of jump does not interfere with cut-elimination (as pointed out by remark 3.11), the converse is also true: if (R, j_R) is a proof-net with jumps and R' is a proof-net s.t. $R \overset{\sigma}{\twoheadrightarrow} R'$ (where σ is a $\neg[ccad]$ reduction sequence), then there exists a function $j_{R'}$ s.t. $(R', j_{R'})$ is a proof-net with jumps and $(R, j_R) \overset{\sigma}{\twoheadrightarrow} (R', j_{R'})$ (just follow definition 3.14, and use theorem 3.15).

We will apply very often the substitution lemma (lemma 3.24): if any e.r.s. $[\forall/\exists]$ is involved in one of these applications, then certain proof-nets have to be considered up to substitutions.
We will denote by $\emptyset$ an empty reduction sequence (i.e. an empty *ordered* set of e.r.s.).

We are going to use the following results:

5.1. THEOREM. **([Girard 87], [Danos 90]).** Every proof-net of PN_2^- is strongly normalizable (i.e. every reduction sequence starting from such a proof-net is finite).

proof: It is, essentially, Girard's strong normalization proof for system F, adapted to proof-nets (see [Girard 87]). However, the syntax of LL requires to achieve strong normalization one more result, known as "standardization theorem" or "propriété de striction". For the fragment PN_2^-, this has been proven by Vincent Danos (see [Danos 90]). $\square$

5.2. THEOREM. **([Bechet 94]).** There exists an embedding of PN_2^+ in PN_2^-, denoted by $[\cdot]$, and s.t.: if R is a proof-net of PN_2^+ and $R \overset{[x]}{\to} R'$ with $[x] \neq [ccad]$, then one has in PN_2^- $[R] \overset{n}{\twoheadrightarrow} [R']$, where $n > 0$.

5.3. COROLLARY. *Every proof-net of PN_2^+ is strongly normalizable, with respect to the rewriting rule $\overset{\neg[ccad]}{\longrightarrow}$.*

The rewriting rule $\neg[ccad]$

Bechet's result relies on his discovery of the following coding of the additives in PN_2^-:
$U \oplus' V = \forall X.?(X \otimes U) \wp ?(X \otimes V) \wp X^\perp$
$U \&' V = \exists X.!(X^\perp \wp U) \otimes !(X^\perp \wp V) \otimes X$.

Theorem 5.2 has been proven in [Bechet 94] in the framework of sequent calculus. The precise statement needed here is nothing but a reformulation of Bechet's proof in terms of proof-nets, and is therefore omitted (see [Tortora 00] for a detailed proof).

5.4. Remark. Let R be a proof-net and B an additive box of R. Suppose that $R \overset{\sigma}{\twoheadrightarrow} R'$, where σ is a $\neg[ccad]$ reduction sequence. Then, no residue B' of B in R' can contain as subproof-net a residue B'' of B different from B': this is because there is simply no way to enter an additive box.

5.5. Proposition. *In PN_2^+, the rewriting rule $\overset{\neg[ccad]}{\twoheadrightarrow}$ enjoys the Church-Rosser property.*

proof: This is a consequence of strong normalization and local confluence of $\xrightarrow{\neg[ccad]}$, where local confluence is the following property of proof-nets: if R is a proof-net s.t. $R \xrightarrow{[x_1]} R_1$ and $R \xrightarrow{[x_2]} R_2$, then there exist two reduction sequences σ_1 and σ_2 and a proof-net R' s.t. $R_1 \overset{\sigma_1}{\twoheadrightarrow} R'$ and $R_2 \overset{\sigma_2}{\twoheadrightarrow} R'$. This can be proven by a simple inspection of cases. $\square$

5.6. Remark. Let (R, j_R) be a proof-net with jumps and suppose that $R \xrightarrow{[x]} R'$, where $[x] \neq [ccad]$. Due to the several possible ways to define the function $j_{R'}$ associated with the e.r.s. $[x]$ (see definition 3.14), the previuous proposition is clearly wrong in PN_2.

The following result is the (obvious) generalization of the same result proven in [Girard 95a] for $MALL_2$.

5.7. Proposition. *Let R be a proof-net with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions.*
If R is normal w.r.t. $\xrightarrow{\neg[ccad]}$, (i.e. there are only cut links of type ccad in R), then R is cut-free.

proof: This is more or less obvious: R cannot contain only cut links of type *ccad*. The idea is that, if c_1 is a cut out of the auxiliary door of an additive box B_1, then there is a (straight) path Φ_1 starting from the main door of B_1 which has to cross a cut link c_2 (because there are no occurrences of the connective & *nor of the quantifier* $\exists^2$ in the conclusions of R). But then, c_2 is a cut out of the auxiliary door of the box B_2, and there exists a path Φ_2 starting from the main door of B_2 which has to cross a cut link c_3 ... Of course this cannot go on forever (because of the acyclicity of every correctness graph, see definition 2.8). $\square$

The (restricted) Church-Rosser Property for PN_2

5.8. LEMMA. *Let (R, j_R) be a proof-net with jumps s.t. $(R, j_R) \xrightarrow{[ccad]} (R_1, j_{R_1})$. Let $\mathbb{B}$ be the additive box of R attractive for the e.r.s. $[ccad]$, and (S, j_S) be the subproof-net of (R, j_R) swallowed by $\mathbb{B}$. Let's call (α, j_α) the subproof-net of (R, j_R) obtained by connecting $\mathbb{B}$ and S through the cut link reduced by $[ccad]$, and B the unique residue of $\mathbb{B}$ in (R_1, j_{R_1}). We obviously have $R = R_1[\alpha/B]$.*
Suppose that $R_1 \overset{\sigma}{\twoheadrightarrow} T$, where σ is a $\neg[ccad]$ reduction sequence which does not contain any e.r.s. reducing a cut inside a residue of B. If we call $B_1, \ldots, B_n$ the n residues of B in T, then $R \overset{\sigma}{\twoheadrightarrow} T[\alpha/B_1, \ldots, \alpha/B_n]$.

proof: Basically, this lemma holds because during such a $\neg[ccad]$ reduction sequence, a residue of B can only be duplicated (by a $[co]$-step), erased (by a $[w]$-step or by a $[\&/\oplus]$-step) or opened.
Observe also, that the only reason why we cannot apply directly the substitution lemma (lemma 3.24) is that among the e.r.s. of σ there might be $[x] = [\&/\oplus]$ which opens a residue of B. What we are actually going to show, is precisely that this is not damaging because $[x]$ has the same effect when we substitute α to this residue of B.
We prove the lemma by induction on σ.
If $\sigma = \emptyset$, then the result is obvious.
Otherwise, suppose that $R_1 \overset{\sigma'}{\twoheadrightarrow} T' \xrightarrow{[x]} T$, and let $B_1, \ldots, B_n$ be the residues of B in T'. By induction hypothesis $R \overset{\sigma'}{\twoheadrightarrow} T'[\alpha/B_1, \ldots, \alpha/B_n]$.
We have to show that $\forall i \in \{1, \ldots, n\}$ $T'[\alpha/B_i] \xrightarrow{[x]} T[\alpha/B_i^1, \ldots, \alpha/B_i^{h_i}]$, where $B_i^1, \ldots, B_i^{h_i}$ are the h_i residues of B_i in T ($h_i \in \{0, 1, 2\}$).
If x is not a logical cut $\&/\oplus$ out of the front door of a residue of B, then we simply apply the substitution lemma to each B_i.
Let x be a logical cut $\&/\oplus$ out of the front door of B_k. First observe that we can still apply the substitution lemma to all the B_j, with $j \neq k$ (because none among B_j 's conclusions is the conclusion of a link $\oplus$, and of course also because of remark 5.4). Then, simply check that $T'[\alpha/B_k] \xrightarrow{[\&/\oplus]} T$. Because there is no residue of B_k in T, this ends the proof. $\square$

The following lemma is actually the induction step of the next one. We suggest to the reader to read first the next lemma and its proof.

5.9. LEMMA. *Let B be an additive box of the proof-net R. Suppose that $R \overset{\sigma_1}{\twoheadrightarrow} R_1 \xrightarrow{[x]} R_2 \overset{\sigma_2}{\twoheadrightarrow} R_0$ is a $\neg[ccad]$ reduction sequence, R_0 does not contain any residue of B, and σ_2 does not contain any e.r.s. reducing a cut inside a residue of B.*
Then, there exists a $\neg[ccad]$ reduction sequence σ s.t. $R \overset{\sigma_1}{\twoheadrightarrow} R_1 \overset{\sigma}{\twoheadrightarrow} R_0$, and σ does not contain any e.r.s. reducing a cut inside a residue of B.

proof: There is something to prove only when the cut link x belongs to the residue B_1 of B in R_1. Let B_2 be the unique residue of B_1 in R_2.
The idea is that, because eventually all the residues of B_2 will disappear, we have for any of these residues one of the two following cases:

- the residue is erased by an e.r.s. of σ_2, in which case $[x]$ is useless (for this residue)
- the residue is opened by an e.r.s. $[\&/\oplus]$, in which case, either $[x]$ is again useless, or "it is the same" to apply $[x]$ at the beginning of the reduction sequence or immediately after the e.r.s. $[\&/\oplus]$.

More formally, we prove that for every decomposition of σ_2
$R_2 \overset{\tau}{\twoheadrightarrow} T \overset{\sigma_2\backslash\tau}{\twoheadrightarrow} R_0$, there exists a $\neg[ccad]$ reduction sequence θ s.t.
$R_1 = R_2[B_1/B_2] \overset{\theta}{\twoheadrightarrow} T[B_1/B_2^1, \dots, B_1/B_2^n]$, and $\forall[z] \in \theta$, z does not belong to any residue of B (we are of course speaking of any residue of the box B of R), where $B_2^1, \dots, B_2^n$ are the n residues of B_2 in T (w.r.t. the reduction sequence τ). The conclusion will then follow immediately from the case $\tau = \sigma_2$.
We proceed by induction on τ.
If $\tau = \emptyset$, then $T = R_2$ and the result is straightforward with $\theta = \tau = \emptyset$.
Otherwise, $R_2 \overset{\tau_1}{\twoheadrightarrow} T_1 \xrightarrow{[y]} T$. Let $B_2^1, \dots, B_2^m$ be the residues of B_2 in T_1.
The induction hypothesis gives $R_1 = R_2[B_1/B_2] \overset{\theta_1}{\twoheadrightarrow} T_1[B_1/B_2^1, \dots, B_1/B_2^m]$ where $\forall[z] \in \theta_1$, z does not belong to any residue of B.
$\forall i \in \{1, \dots, m\}$, let $B_2^{1,i}, \dots, B_2^{h_i,i}$ be the h_i residues of B_2^i in T ($h_i \in \{0, 1, 2\}$). If $[y]$ does not open any of the boxes $B_2^1, \dots, B_2^m$, then the substitution lemma gives, $\forall i \in \{1, \dots, m\}$, $T_1[B_1/B_2^i] \xrightarrow{[y]} T[B_1/B_2^{1,i}, \dots, B_1/B_2^{h_i,i}]$, and we can take $\theta = [y] o \theta_1$.

Suppose now that $[y] = [\&/\oplus]$ opens the box B_2^k. Then $\forall i \in \{1, \ldots, m\}$, $i \neq k$, $h_i = 1$; and there is no residue of B_2^k in T. For $i \neq k$ the substitution lemma still holds, and we have $T_1[B_1/B_2^i] \xrightarrow{[y]} T[B_1/B_2^i]$. For $i = k$, remember that in σ_2 there are no e.r.s. reducing a cut in a component of any residue of B_2, so that from $B_1 \xrightarrow{[x]} B_2$, we deduce $B_1 \xrightarrow{[x]} B_2^k$. Let T_2 be s.t. $T_1[B_1/B_2^k] \xrightarrow{[y]} T_2$. We have the two following possibilities:

- if x belongs to the component of B_1 erased by $[y] = [\&/\oplus]$, then $T_2 = T$
- if x belongs to the component of B_1 which is not erased by $[y] = [\&/\oplus]$, then $T_2 \xrightarrow{[x]} T$. (Observe that in this case x does not belong to any residue of B in T_2, because otherwise in T_1 a residue of B would be contained in B_2^k, thus contradicting remark 5.4).

In the first case we will take $\theta = [y] o \theta_1$, while in the second we will take $\theta = [x] o [y] o \theta_1$.
To be more precise, we should also check that y does not belong to any residue of B *w.r.t.* $\theta_1 o \sigma_1$: this is clear from the construction of θ_1. $\square$

5.10. LEMMA. *Let R be a proof-net and B an additive box of R. If $R \overset{\sigma'}{\twoheadrightarrow} R_0$, where σ' is a $\neg[ccad]$ reduction sequence and there is no residue of B in R_0, then there exists a $\neg[ccad]$ reduction sequence σ s.t. $R \overset{\sigma}{\twoheadrightarrow} R_0$ and σ does not contain any e.r.s. reducing a cut inside a residue of B.*

proof: Let $n(\sigma')$ be the number of e.r.s. $[y]$ of σ' s.t. y belongs to a residue of B. We prove the lemma by induction on $n(\sigma')$.
If $n(\sigma') = 0$, then $\sigma = \sigma'$ will do.
Otherwise, let $[x]$ be the last e.r.s. of σ' s.t. x belongs to a residue of B. We have for σ' the following decomposition: $R \overset{\sigma_1}{\twoheadrightarrow} R_1 \xrightarrow{[x]} R_2 \overset{\sigma_2}{\twoheadrightarrow} R_0$, where σ_2 does not contain any e.r.s. reducing a cut inside a residue of B. From the previous lemma, we get a $\neg[ccad]$ reduction sequence τ s.t. $R \overset{\sigma_1}{\twoheadrightarrow} R_1 \overset{\tau}{\twoheadrightarrow} R_0$ and τ does not contain any e.r.s. reducing a cut inside a residue of B. Then, if we define $\theta := \tau o \sigma_1$, we obtain $R \overset{\theta}{\twoheadrightarrow} R_0$ and $n(\theta) < n(\sigma')$. We conclude by applying the induction hypothesis to θ. $\square$

5.11. LEMMA. *Let R be a proof-net with no occurrences of the connective $\&$ nor of the quantifier $\exists^2$ in its conclusions, and let (R, j_R) be a proof-net with jumps.*

If $(R, j_R) \stackrel{[ccad]}{\longrightarrow} (R_1, j_{R_1})$, then there exists a $\neg[ccad]$ reduction sequence σ, and a cut-free proof-net R_0 s.t. $R \stackrel{\sigma}{\twoheadrightarrow} R_0$ and $R_1 \stackrel{\sigma}{\twoheadrightarrow} R_0$.

proof: Let $\mathbb{B}$ be the additive box of R attractive for the e.r.s. $[ccad]$, and (S, j_S) be the subproof-net (with jumps) of (R, j_R) swallowed by $\mathbb{B}$. Let's call (α, j_α) the subproof-net (with jumps) of R obtained by connecting $\mathbb{B}$ and S through the cut link reduced by $[ccad]$, and B the unique residue of $\mathbb{B}$ in (R_1, j_{R_1}). We obviously have that $R_1[\alpha/B] = R$.
Let σ' be any $\neg[ccad]$ reduction sequence starting from R_1 and leading to the unique $\neg[ccad]$ normal form R_0 of R_1. From proposition 5.7 we know that R_0 is cut-free. By the subformula property, we deduce that there is no residue of B (w.r.t. σ') in R_0. Then, lemma 5.10 applies and there exists a $\neg[ccad]$ reduction sequence σ s.t. $R_1 \stackrel{\sigma}{\twoheadrightarrow} R_0$ and σ does not contain any e.r.s. reducing a cut inside a residue of B. But now we can apply lemma 5.8 to σ and deduce that $R \stackrel{\sigma}{\twoheadrightarrow} R_0$ (remember that there is no residue of B in R_0). $\square$

We are going to prove the

5.12. THEOREM. *Let* R be a proof-net with no occurrences of the connective & nor of the quantifier $\exists^2$ (i.e. existential quantifier of the second order) in its conclusions.
Let j_1 and j_2 be two functions s.t. (R, j_1) and (R, j_2) are two proof-nets with jumps.
If $(R, j_1) \stackrel{\sigma_1}{\twoheadrightarrow} (R_0, j_1^0)$ and $(R, j_2) \stackrel{\sigma_2}{\twoheadrightarrow} (T_0, j_2^0)$, where R_0 and T_0 are cut-free proof-nets, then $R_0 = T_0$.
In particular (take $j_1 = j_2$), if $(R, j_1) \stackrel{\sigma_1}{\twoheadrightarrow} (R_0, j_1^0)$ and $(R, j_1) \stackrel{\sigma_2}{\twoheadrightarrow} (T_0, j_2^0)$, then $R_0 = T_0$.

proof: We prove that if (R_0, j_{R_0}) is any normal form of (R, j_R), then there exists a $\neg[ccad]$ reduction sequence τ, s.t. $R \stackrel{\tau}{\twoheadrightarrow} R_0$. Confluence of $\stackrel{\neg[ccad]}{\twoheadrightarrow}$ (proposition 5.5) will then be enough to conclude: if $(R, j_1) \stackrel{\sigma_1}{\twoheadrightarrow} (R_0, j_1^0)$ and $(R, j_2) \stackrel{\sigma_2}{\twoheadrightarrow} (T_0, j_2^0)$, then we will have $R \stackrel{\tau_1}{\twoheadrightarrow} R_0$ and $R \stackrel{\tau_2}{\twoheadrightarrow} T_0$ (where τ_1 and τ_2 are $\neg[ccad]$ reduction sequences), so that $R_0 = T_0$.
Let τ' be any reduction sequence s.t. $(R, j_R) \stackrel{\tau'}{\twoheadrightarrow} (R_0, j_{R_0})$. We prove the existence of τ by induction on $l(\tau') :=$ number of e.r.s. of τ'.
If $l(\tau') = 0$, then $\tau = \tau' = \emptyset$ will do.
Otherwise $(R, j_R) \stackrel{[x]}{\longrightarrow} (R_1, j_{R_1}) \stackrel{\tau_1'}{\twoheadrightarrow} (R_0, j_{R_0})$. By induction hypothesis (of course (R_1, j_{R_1}) is a proof-net with jumps with no occurrences of the con-

nective & nor of the quantifier $\exists^2$ in its conclusions) there exists a $\neg[ccad]$ reduction sequence τ_1 s.t. $R_1 \overset{\tau_1}{\twoheadrightarrow} R_0$. If $[x] \neq [ccad]$, then we are done. If $[x] = [ccad]$, then we apply lemma 5.11: there exists a $\neg[ccad]$ reduction sequence τ and a cut-free proof-net T_0 s.t. $R \overset{\tau}{\twoheadrightarrow} T_0$ and $R_1 \overset{\tau}{\twoheadrightarrow} T_0$. From confluence of $\overset{\neg[ccad]}{\twoheadrightarrow}$, we deduce that $R_0 = T_0$ (remember that R_0 and T_0 are both cut-free). $\square$

5.13. Corollary. **(Confluence)** Let (R, j) be a proof-net with jumps, with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions.
If $(R, j) \overset{\sigma_1}{\twoheadrightarrow} (R_1, j_1)$ and $(R, j) \overset{\sigma_2}{\twoheadrightarrow} (R_2, j_2)$, then there exists a cut-free proof-net R_0, two jump functions j_1^0 and j_2^0, and two reduction sequences τ_1 and τ_2 s.t. $(R_1, j_1) \overset{\tau_1}{\twoheadrightarrow} (R_0, j_1^0)$ and $(R_2, j_2) \overset{\tau_2}{\twoheadrightarrow} (R_0, j_2^0)$.

proof: Strong normalization of $\neg[ccad]$ and proposition 5.7 guarantee that by applying the necessary number of e.r.s. to (R_1, j_1) and to (R_2, j_2), we will obtain two $\neg[ccad]$ reduction sequences τ_1 and τ_2 and two cut-free proof-nets with jumps (T_1^0, j_1^0), (T_2^0, j_2^0) s.t. $(R_1, j_1) \overset{\tau_1}{\twoheadrightarrow} (T_1^0, j_1^0)$ and $(R_2, j_2) \overset{\tau_2}{\twoheadrightarrow} (T_2^0, j_2^0)$. The previous theorem states $T_1^0 = T_2^0$. $\square$

5.14. Remark. (i) Of course, the reader noticed that one has to forbid to the second order existential quantifier to occur in the conclusions of the proof-net, simply because it can "hide" occurrences of the connective &.
(ii) The previous theorem basically states that any "correct" way to perform cut-elimination starting from a proof-net R (with the suitable conclusions) yields the same normal form. More precisely, the following sequence of operations always gives the same result (the same proof-net):
- take a jump function j s.t. (R, j) is a proof-net with jumps
- perform cut-elimination until you reach a normal form
- forget the jump function.
(iii) The theorem seems to be the best result one can obtain, *unless* one decides to change the syntax of the additives. In [Girard 95a], boxes for the additives are removed thanks to a weight function on the edges of the proof-net (see also [Laurent 99] for a more elegant version in the restricted polarized case), but the problem of the additive commutative e.r.s. is not solved. In chapter 5 of [Tortora 00], we introduce the notion of "multiboîte" (an additive box with several front doors, with the terminology of definition 2.2) which allows to define a confluent procedure of cut-elimination, in the restricted case of multiplicative and additive linear logic ($MALL$).

5.15. REMARK. There was also a semantical approach to the proof of theorem 5.12: it would have been enough to prove that two different normal proof-nets of PN_2 without occurrences of & and of $\exists^2$ are interpreted by two different cliques (in the usual or even in the multiset semantics). A first attempt was to prove that this holds for two different normal proof-nets in the multiplicative and exponential fragment. But (quite surprisingly) this turned out to be ... wrong! Several counterexamples, as well as some positive results on these topics, can be found in [Tortora 00].

The lazy cut-elimination procedure

We are now going to make some remarks on the *lazy* cut-elimination procedure, due to its importance in the recent works on the logical systems with "light" complexity ELL and LLL. The procedure that we are going to define is the obvious generalization to PN_2^+ of the one defined in [Girard 95a].

5.16. DEFINITION. Let x be a cut link of the proof-net R. We will say that x is a **lazy cut link** when x has additive depth zero in R, and x is not *only* of type *ccad* (remember remark 3.8).
Similarly, if x is a lazy cut link, then we will say that $[x]$ is a **lazy e.r.s.**, *when* $[x] \neq [ccad]$ (this has to be stated, still because of remark 3.8). We will say that a reduction sequence σ is a **lazy reduction sequence** when every e.r.s. of σ is lazy. We will denote $\longrightarrow_l$ this rewriting rule and, as usual, $\twoheadrightarrow_l$ its reflexive and transitive closure.

5.17. PROPOSITION. *Let R be a proof-net with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions. Then:*

(1) *Every sequence of lazy e.r.s. starting from R is finite.*

(2) *Every lazy normal form of R is cut free: if a proof-net T is normal w.r.t. $\longrightarrow_l$, (i.e. T contains only cut links of type ccad or with additive depth greater than zero), then T is cut free.*

(3) *There exists a unique lazy normal form of R (which is cut free).*

proof: (1): Because every lazy e.r.s. is also a $\neg[ccad]$ e.r.s., the rewriting rule $\longrightarrow_l$ is srtongly normalizing (as a consequence of proposition 5.3).
(2): the same as the proof of 5.7.

(3): Let R_0 and R_1 be two lazy normal forms of R. (2) implies that R_0 and R_1 are cut free. Because every sequence of lazy reductions is also a sequence of $\neg[ccad]$ reductions, R_0 and R_1 are also normal forms w.r.t. the rewriting rule $\neg[ccad]$, which is confluent. Then $R_0 = R_1$. $\square$

5.18. REMARK. The lazy procedure of cut-elimination is "almost" the one defined in [Girard 95b] to perform cut-elimination in the system LLL, and it precisely coincides with that procedure for the system $MALL_2$ (the fragment of PN_2 containing only the additive connectives and the second order quantifiers). In [Girard 95a], it is shown that this cut-elimination procedure can be performed in linear time in $MALL_2$.

The "default" of Girard's procedure is that it does not always terminate on a cut free proof-net. But in case the conclusions of the proof-net that we normalize do not contain any occurrence of the connective & nor of the quantifier $\exists^2$, (2) of proposition 5.17 tells us that the lazy procedure terminates on a cut free proof-net. We already mentioned the importance of this case.

In the following remark, we use the previous proposition and confluence of $\stackrel{\neg[ccad]}{\longrightarrow}$ (proposition 5.5).

5.19. REMARK. Let R be a proof-net with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions. The unique $\neg[ccad]$ normal form of R can be reached in a lazy way.
What we have shown (theorem 5.12) is that the unique $\neg[ccad]$ normal form of R is actually also the unique normal form of the proof-net with jumps (R, j) (where j is *any* jump function), and (by what we just wrote) it can be reached in a lazy way. Our result yields then a complete justification of the lazy procedure.

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References

[Bechet 94] Bechet D., Second Order Connectives and Proof Transformations in Linear Logic, presented at the colloquium “Preuves, Réseaux et Types” held in Marseille, France, October 1994, and to appear as a prépublication Université Paris 13, 1999

[Danos 90] Danos V., La logique linéaire appliquée à divers processus de normalisation (principalement du lambda-calcul). Thèse de doctorat, Université de Paris 7, 1990

[DJ 99] Danos V., Joinet J.-B., Elementary Recursive Functions in Linear Logic. A paraître dans les actes du colloque “International Workshop on Implicit Computational Complexity” (ICC'99), 1999

[DanReg 89] Danos V., Regnier L., The structure of multiplicatives. Archives for Mathematical Logic, 28, pp. 181-203, 1989

[DanReg 95] Danos V., Regnier L., Proof-nets and the Hilbert space. In: J.Y. Girard, Y. Lafont and L. Regnier editors, Advances in Linear Logic, pp. 307-328. Cambridge Univerity Press, 1995. London Mathematical Society Lecture Note serie 222, Proceedings of the 1993 workshop on Linear Logic, Cornell University, Ithaca.

[Girard 87] Girard J.Y., Linear logic. Theoretical Computer Science, 50:1-101, 1987.

[Girard 91] Girard J.Y., Quantifiers in linear logic II. In: Corsi, G. and Sambin, G. editors, Nuovi problemi della logica e della filosofia della scienza, Volume II. Proceedings of the conference with the same name, Viareggio, 8-13 gennaio 1990. CLUEB, Bologna (Italy).

[Girard 95a] Girard J.Y., Proof-nets: the parallel syntax for proof-theory, In Ursini and Agliano editors, Logic and Algebra, New-York, 1995. Marcel Dekker.

[Girard 95b] Girard J.Y., Light Linear Logic, Information and Computation, vol. 143, n. 2, p.175-204, 1998

[KriPar 90] Krivine J.-L., Parigot M., Programming with proofs. Journal of Information Processing and Cybernetics EIK (formerly Elektronische Informationsverarbeitung und Kybernetik) 26, 1990

[Laurent 99] Laurent Olivier, Polarized Proof-Nets: Proof-Nets for LC (Extended Abstract). In Jean-Yves Girard, editor, Typed Lambda Calculi and Applications '99, volume 1581 of LNCS, pages 213-227. Springer-Verlag

[Tortora 00] Tortora de Falco L., "Réseaux, cohérence et expériences obsessionnelles", Thèse de doctorat, Université de Paris 7, January 2000, available at the address: http://www.logique.jussieu.fr/www.tortora/index.html

[Tortora 00b] Tortora de Falco L., "Additives of Linear Logic and Normalization Part II: the additive standardization theorem", submitted to Theoretical Computer Science