Additives of linear logic and normalization -Part II: the additive standardization theorem

Lorenzo Tortora de Falco*
Équipe de Logique Mathématique, Université Paris VII

September 21, 2000

Abstract

We prove the additive standardization theorem for full linear logic (without constants).

1 Introduction

Proof-nets of linear logic (LL) are proofs represented as graphs: they were introduced in [Girard 87], to give a more geometric account of proofs. The syntax of proof-nets has outstanding properties for the multiplicative fragment of LL, but its expressive power is dramatically poor. If one moves to the second order multiplicative and exponential fragment (the most studied) the situation is definitely better: proof-nets are not as simple as for the multiplicative fragment, but "good enough", and the expressive power of such a system is great. However, the additive connectives of LL are the natural candidates to type functions like "predecessor" and "if...then...else..." (see [Girard 95b] and [Girard 87]).
Proof-nets for the additives have never been completely convincing, despite years of efforts. In the original syntax of [Girard 87], the additive connective & is handled by means of a box, and normalization is defined but the Church-Rosser property does not hold (see [Tortora 00b]). In [Girard 95a], additive boxes are removed, but the sequentialization theorem is difficult to prove, and some steps of normalization are not defined. In [Tortora 00b], we

*tortora@logique.jussieu.fr; this research was partially supported by a fellowship of the Consiglio Nazionale delle Ricerche (CNR).

introduce the notion of "additive multibox" (a generalization of the standard notion of additive box) which yields a more satisfactory notion of normal form, but only a "weak Church-Rosser property" is proven.
Recently, it appeared that the sequentialization proof of [Girard 95a] could be simplified for a significant fragment of LL (containing of course the additives): the polarized fragment (see [Laurent 99]). The notion of "positive tree" of [Laurent 99] allows to give in [LQT 00] a very simple proof of strong normalization and confluence for polarized proof-nets.

This paper is a contribution to the study of proof-net normalization of full LL (without constants), in the traditional syntax of [Girard 87].
Despite the lack of confluence, we have proven in [Tortora 00a] that (roughly speaking) a proof-net which does not contain the connective & in its conclusions has a unique normal form.
But the main existing result on the subject is certainly the strong normalization theorem for second order linear logic proven in [Girard 87]. The proof makes an essential use of a so called "standardization theorem". A proof of this theorem is only quickly sketched in [Girard 87]. In his thesis, V. Danos called "striction" a different theorem which is enough to prove strong normalization. He gives a detailed proof of this result for the multiplicatives and the exponentials (see [Danos 90]), which relies on the confluence of this fragment.
The achievement of the present work is the proof of a (generalized) additive standardization theorem (theorem 4.2).
The proof that we give here is completely new. To be precise, the result that we prove (theorem 4.2) is more in the spirit of Danos' striction than of Girard' standardization. But the real point is that our result is enough to prove strong normalization (as explained in section 4).
Provided the technical point evoked by remark 7.2 has been carefully checked, we obtain that our variant of PN_2 (Girard's system of second order proof-nets, defined in [Girard 87]) is strongly normalizing.

The problem in the study of cut-elimination in presence of the additives, is how to deal with the commutative elementary reduction step (e.r.s.). Following the original definition of cut-elimination (see [Girard 87]), during this step an additive box swallows and duplicates a particular subproof-net having an occurrence of the cut formula among its conclusions (called the empire of this occurrence of formula). We propose a generalization of this e.r.s.: we allow *any* subproof-net having this occurrence of formula among its conclusions to be swallowed and duplicated. This is very natural: the embedding in LL of the system for classical logic LK^{tq} introduced in [DJS 97] requires, to be "homoreductive" (that is to be a morphism for cut-elimination steps, see [DJS 97] and [JST 00]), a variant of [Girard 87] 's commutative e.r.s. for the additives which is a particular case of the one considered in this paper. Our generalization of the commutative e.r.s. for the additives requires a suitable (and general enough) notion of *subproof-net*, and the presence of weakenings (which "disconnect" the graph) makes this notion delicate to define (see [Tortora 00a]).

The reader acquainted with proof-nets has two ways of reading the present paper: either to refer to its own (more or less precise) notion of proof-net and to follow the general structure of the paper, or first to read [Tortora 00a] and then to understand the present work in all details.
The paper is organized as follows. In section 2, we recall the main notions, notations and results of [Tortora 00a] that will be used in the sequel. Section 3 answers the following question: suppose that Q and S are two proof-nets plugged together by means of a cut link c; what happens during normalization if we "do not (really) reduce" c (nor any of its residues)? The answer is crucial in section 6. Section 4 basically recalls the main steps of Girard's proof of strong normalization and states the main theorem of the paper. In sections 5 and 6 we define a size on the proof-nets, and show that (under certain conditions) it cannot increase and it "often decreases" during normalization. Finally, in the last section we prove the theorem, showing that a certain size shrinks whatever normalization step is performed.

2 Preliminaries

The main result of this paper (theorem 4.2) relies on the notion of proof-net for second order LL. This notion was first given in [Girard 87]. Since then several improvements for fragments of LL have been proposed, but never generalized to full LL. This is not trivial in presence of the additive connectives. In [Tortora 00a] (see also [Tortora 00b] for a more detailed version), we introduced a notion of jump which allows to define proof-nets for second order LL: following the notations of [Girard 87], we denoted by PN_2 the set of these proof-nets (called in fact "proof-nets with jumps").
Recall that a proof-net with jumps is a couple (R, j_R) (in the sequel we will often simply write "a proof-net R") where R is a graph and j_R is a jump function from the set of the weakeming links of R to $\mathcal{P}(Ax)$, the power set of Ax (Ax being the set of all the axiom links of R). If n is a weakening link of R, then the axiom links of $j_R(n)$ are called the **jumps** of n. Of course

R and j_R have to satisfy certain conditions. Intuitively, the jump function allows to "connect" each weakening link to the rest of every "correctness graph" of the proof-net.
If B is an additive box, we denote by $\gamma_1(B)$ (resp. $\gamma_2(B)$) the first (resp. the second) **component** of B. We write $\gamma_i(B)$ to indicate one of the two components $\gamma_1(B)$ and $\gamma_2(B)$ of B. If n_1 (resp. n_2) is a link of $\gamma_1(B)$ (resp. $\gamma_2(B)$), then we say that B **separates** n_1 and n_2, or simply that n_1 and n_2 are separated. An additive box can also separate two boxes (additive or exponential), or two subproof-nets.

Cut-elimination for proof-nets (definition 3.14 of [Tortora 00a]) associates with a proof-net (R, j_R) and a given elementary reduction step (e.r.s.) $[x]$, a (in fact several) proof-net(s) $(R', j_{R'})$: the graph R' is uniquely determined by (R, j_R) and $[x]$, but in general the function $j_{R'}$ isn't. The jump function j_R is used to define the graph R' only when $[x]$ is an additive commutative e.r.s. (these steps are denoted by $[ccad]$), because j_R allows to define the crucial notion of **subproof-net** (definition 2.19 of [Tortora 00a]).
Let R' be a reduct of R. It is possible to define precisely the **lift** of the links, boxes, subproof-nets of R' in R. Conversely, one can define the **residues** of the links, boxes, subproof-nets of R in R' (see definitions 3.12 and 3.19 [Tortora 00a]).

Suppose now that $R \overset{[ccad]}{\rightarrow} R'$ (which means that R' is obtained by performing an additive commutative e.r.s. from R). Let S be the subproof-net of R duplicated by the $[ccad]$ e.r.s. If B is the additive box of R that swallows (and duplicates) S, we will say that this e.r.s. is **B-attractive**.

For us, the notion of **path** in a proof-net is the notion of "straight path" introduced in [DanReg 95] (see definition 3.1 of [Tortora 00a]): it is an oriented path changing direction only in axioms and cuts. A bit roughly speaking, a **direct path** (see definition 3.2 of [Tortora 00a]) is a path moving from a subformula to a conclusion or a cut.

We now recall some remarks and results proven in [Tortora 00a].

2.1. Remark. Let $R \xrightarrow{[x]} R'$, where (R, j_R) is a proof-net. Let m' be a weakening link of R' and m its lift in R.

If $[x]$ is not an axiom e.r.s. nor a weakening e.r.s., then $j_{R'}(m')$ contains only residues of links of $j_R(m)$.

If $[x]$ is not a contraction e.r.s., then $j_{R'}(m')$ contains all the residues of the links of $j_R(m)$.

2.2. LEMMA. **(substitution lemma)** *Let T, T' be two proof-nets, s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$. Let α be a subproof-net of T, and β any proof-net with the same conclusions as α.*
If, for every $[x] \in \sigma$ s.t. $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$, and for every residue γ_1 of α in T_1, the following conditions are satisfied:

i) *if $[x] = [ax]$ then x is not contained in γ_1 and the link ax erased by $[x]$ is not contained in γ_1*

ii) *if $[x] \neq [ax], [ccad]$ then none of the premises of x is an edge of γ_1 (in particular, any conclusion of γ_1 is not a premise of x)*

iii) *if $[x] = [ccad]$ then x is not a link of γ_1, $[x]$ is not attractive for a box of γ_1 and if $[x]$ duplicates a subproof-net of γ_1 then $[x]$ duplicates the whole γ_1*

then, if $\alpha'_1, \ldots, \alpha'_n$ are the residues of α in T', we have $T[\beta/\alpha] \overset{\sigma}{\twoheadrightarrow} T'[\beta/\alpha'_1, \ldots, \beta/\alpha'_n]$ (up to substitutions). If $[\forall/\exists] \in \sigma$, then we have, more precisely, $T[\beta/\alpha] \overset{\sigma}{\twoheadrightarrow} T'[\beta_1/\alpha'_1, \ldots, \beta_n/\alpha'_n]$, where $\beta_i = \beta$ in which we performed the substitution(s) required by the e.r.s. $[\forall/\exists]$.

2.3. PROPOSITION. **(separation property).** *Let T, T' be two proof-nets s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$, and let $\mathbb{B}$ be an additive box with exponential depth 0 in T. If $\mathbb{B}'$ and $\mathbb{B}''$ are two (different) residues of $\mathbb{B}$ in T', then $\mathbb{B}'$ and $\mathbb{B}''$ are separated.*

2.4. LEMMA. *Let T be a proof-net s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$, let c be a cut link of T, and suppose that if R is a proof-net appearing in σ, then every residue of c in R has exponential depth zero in R.*
If c' and c'' are two (different) residues of c in T', then c' and c'' are separated.

2.5. REMARK. Let R be a proof-net and let R' be any reduct of R. If l is a logical link, an axiom link or a *logical* cut link, with exponential depth zero in R, then any residue of l in R' has exponential depth zero in R'. The same holds for a subproof-net α of R different from an exponential box, and for an additive box B of R: if α and B have exponential depth zero in R, then every residue of α in R' and every residue of B in R' have exponential depth zero in R'.

This is because the only possibility for a link (different from a commutative exponential cut) to enter an exponential box is to be already in *another* exponential box, active in an exponential commutative e.r.s..

2.6. REMARK. Suppose that T and T' are two proof-nets s.t. $T \overset{[x]}{\rightarrow} T'$ and that x is a contraction cut link. Let n be the of course link of T whose conclusion is a premise of x and let $B^!$ be the exponential box of T associated with n. If R is a subproof-net of T which has a residue in T' and if R contains $B^!$, then $R = B^!$.

2.7. REMARK. Let R and R' be two proof-nets s.t. $R \overset{[x]}{\rightarrow} R'$, and let n be an axiom, a logical, a weakening, or a cut link of R. If one of the two following conditions holds, then n has no residue in R' and we say that n belongs to the subproof-net of R erased by $[x]$:
(a) $[x]$ is a $[w]$ e.r.s. and n is a link of the exponential box erased by $[x]$
(b) x is a $\&/\oplus$ cut link out of the main door of the additive box B of R, and n is a link of the component of B erased by $[x]$.
Conversely, if n has no residue in R', then one has the following possibilities:
(i) If n is a logical link, then (a) or (b) holds, or x is a logical cut link and the conclusion of n is a premise of x.
(ii) If n is an axiom link, then (a) or (b) holds, or $[x]$ is an axiom e.r.s. and n is the axiom link erased by $[x]$.
(iii) If n is a weakening link, then (a) or (b) holds, or the conclusion of n is a premise of the weakening cut link x and the exponential box erased by $[x]$ has no pax door.
(iv) If n is a cut link, then (a) or (b) holds, or $n = x$ is a logical cut link, or $n = x$ is a weakening cut link, or $n = x$ and $[x]$ is an axiom e.r.s..
Let us now introduce a convention:

Convention: Suppose that $T \overset{\sigma}{\twoheadrightarrow} T'$ and that $\mathbb{B}$ is an additive box with exponential depth 0 in T. If $\mathbb{B}'$ is a residue of $\mathbb{B}$ in T' with additive depth $p > 0$, then there exist p additive boxes $B_1, \ldots, B_p$ of T' different from $\mathbb{B}'$ and s.t. $\mathbb{B}'$ is a subproof-net of B_i $\forall i \in \{1, \ldots, p\}$. In the sequel we will speak of the **main subproof-nets** of T' (that will also be denoted by β_1,..., β_p) in the canonical representation of T' with respect to $\mathbb{B}'$, meaning the components of the additive boxes $B_1, ..., B_p$ which do not contain $\mathbb{B}'$.
The separation property states that all the residues of $\mathbb{B}$ different from $\mathbb{B}'$ are subproof-nets of a main subproof-net of T' in the canonical representation of T' w.r.t. $\mathbb{B}'$.

We now prove the following useful lemma

2.8. LEMMA. *Let T, T' be two proof-nets such that $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}$ be an additive box with exponential depth zero in T and $\mathbb{B}'$ a residue of $\mathbb{B}$ in T'. Given a proof-net R appearing in σ, we will refer to the canonical representation of R with respect to the (unique) lift of $\mathbb{B}'$ in R.*
Let T_1 and T_2 be two proof-nets in the following decomposition of σ: $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{\sigma_2}{\twoheadrightarrow} T_2 \overset{\sigma_3}{\twoheadrightarrow} T'$. If β is a main subproof-net of T_1, then all its residues in T_2 are subproof-nets of main subproof-nets of T_2.

proof: By induction on σ_2.
If $T_1 = T_2$, then the lemma holds.
Otherwise, $T_1 \overset{\sigma_2'}{\twoheadrightarrow} T_2' \overset{[x]}{\rightarrow} T_2$. If β has no residue in T_2 then the lemma vacuously holds. Otherwise, let β be a main subproof-net of T_1, β'' a residue of β in T_2 and β' its lift in T_2'.
By induction hypothesis, β' is a subproof-net of a main subproof-net γ of T_2'.
If γ has no residue in T_2, then $\gamma \overset{[x]}{\rightarrow} \delta$ (γ cannot disappear in a $[\&/\oplus]$ e.r.s., because in this case β' would disappear too) where δ is a main subproof-net of T_2, and β'' is a subproof-net of δ.
Otherwise, β'' is a subproof-net of a residue of γ in T_2. We are now going to show that for every possible $[x]$, any residue of γ in T_2 is a subproof-net of a main subproof-net of T_2.
Let $\mathbb{B}_2'$ (resp. $\mathbb{B}_2$) be the lift of $\mathbb{B}'$ in T_2' (resp. T_2). Let B' be the additive box of T_2' s.t. γ is a component of B'. By definition of main subproof-net, $\mathbb{B}_2'$ is a subproof-net of the component of B' different from γ.
Observe that B' has one or two residues in T_2 (otherwise γ or $\mathbb{B}_2'$ would have no residue in T_2). Let's now consider these two possible cases.
If B' has one residue B in T_2, then $\mathbb{B}_2$ (the unique residue of $\mathbb{B}_2'$ in T_2) and the unique residue of γ in T_2 will (still) be subproof-nets of the two different components of B. To conclude, remember that the component of B which does not contain $\mathbb{B}_2$, i.e. the one containing γ, is a main subproof-net of T_2.
If B' has two residues B_1 and B_2 in T_2, then $[x] = [ccad]$ and B' is a subproof-net of the subproof-net of T_2' duplicated by $[x]$. This means that there are two residues of γ: the component γ_1 of B_1 and the component γ_2 of B_2. Let B'' be the box of T_2' s.t. $[x]$ is B''-attractive, and let's call B_3 the unique residue of B'' in T_2. $\mathbb{B}_2$ is either a subproof-net of B_1 or a subproof-net of B_2. Suppose for example that $\mathbb{B}_2$ is a subproof-net of B_1: then γ_1 is the component of B_1 which does not contain $\mathbb{B}_2$ (i.e. γ_1 is a main

subproof-net of T_2), and γ_2 (is a subproof-net of B_2 which) is a subproof-net of the component of B_3 which does not contain $\mathbb{B}_2$ (and this component of B_3 is a main subproof-net of T_2). □

2.9. REMARK. Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}$ be an additive box with exponential depth zero in T, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let β be a main subproof-net of T' in its canonical representation w.r.t. $\mathbb{B}'$. If there is a residue of $\mathbb{B}$ in β, then there exists an additive box B' of β containing all the residues of $\mathbb{B}$ that are subproof-nets of β. Indeed, were $\mathbb{B}'_1$ and $\mathbb{B}'_2$ two residues of $\mathbb{B}$ in β not contained in any additive box of β, then $\mathbb{B}'_1$ and $\mathbb{B}'_2$ would not be separated, thus contradicting the separation property.

3 Normalization without interaction

Let Q be a proof-net having the edge a labeled by A among its conclusions, let S be a proof-net having the edge $a^\perp$ labeled by $A^\perp$ among its conclusions, and let T be the proof-net obtained from Q and S by connecting the edges a and $a^\perp$ with a cut link c. In the whole section, Q, S, T will denote proof-nets of the previous form, and c will denote the cut link between Q and S.
Suppose that $T \overset{\sigma}{\twoheadrightarrow} T'$ and that for every $[x] \in \sigma$, $[x]$ is not a [*ccad*] e.r.s. and x is not a residue of c. Then the reader will easily convince herself/himself that there exists a subproof-net Q' of T' and a subproof-net S' of T', s.t. $Q \twoheadrightarrow Q'$, $S \twoheadrightarrow S'$, and if we connect Q' and S' by means of a cut link with premises the residue of a in Q' and the residue of $a^\perp$ in S', then we obtain T'. Intuitively, this means that until we don't reduce a residue of c, Q and S will never interact. Because of the "global move" involved in a [*ccad*] e.r.s., our previous statement is not true if one allows this kind of step: just reduce a *ccad* cut link with additive and exponential depth 0 in Q and duplicate a subproof-net of T containing some links of S...
The main point of this section (proposition 3.9) is to show that even though the previous statement is wrong if we allow [*ccad*] e.r.s., its intuitive meaning is still true. This result, as well as the five lemmas closing the section, will be crucial in section 6.

3.1. REMARK. Up to the renaming of bound variables, the label A (resp. $A^\perp$) of the conclusion a (resp. $a^\perp$) of Q (resp. S) does not contain any occurrence of a free variable X s.t. X is bound by a forall link somewhere in T. This implies that if $T \overset{\sigma}{\twoheadrightarrow} T'$, then the two premises of every residue of c in T' are still labeled by A and $A^\perp$.

Let us recall some more terminology of [Tortora 00a].
Intuitively, any link of a proof-net R "comes from" a logical, an axiom or a weakening link, as stated more precisely in the following lemma. We will then say that the link n of the proof-net R is a **main link** of R if n is a logical, an axiom or a weakening link.

3.2. LEMMA. *Let T be a proof-net. For every link n of T, there exists a main link l and a direct path Φ of T having l as starting link and n as terminal link.*

3.3. REMARK. Let R be a proof-net and m be a link of R. One and only one of the two following conditions holds:
(1) there exists a cut link c of R and a (unique) direct path Φ_m with starting link m and terminal link c
(2) there exists a conclusion a of R and a (unique) direct path Φ_m with starting link m and terminal edge a.

The following lemma says (in particular) that under certain circumstances the conditions (1) and (2) of remark 3.3 are stable w.r.t. cut elimination.

3.4. LEMMA. *Let R, R' be two proof-nets such that $R \overset{[x]}{\to} R'$. Let n be a main link of R and Φ a direct path of R with starting link n.*
(i) Suppose that Φ has the cut link c_1 as terminal link, and suppose that c_1 has a residue in R'. For every residue n' of n in R', there exists a residue c' of c in R', and a direct path Φ' having n' as starting link and c' as terminal link.
(ii) Suppose that Φ has the conclusion a of R as terminal edge. For every residue n' of n in R', there exists a direct path Φ' having n' as starting link and the residue a' of a in R' as terminal edge.

proof: Let n' be a residue of n in R' (if there is no such link, then the lemma vacuously holds). We proceed by inspection of all the possible cases for $[x]$.
If x is a logical cut link: then n' is the unique residue of n in T'.
- in case (i): $x \neq c_1$ and c_1 has a unique residue c' in R'. The direct path Φ' of R' is then the same as the path Φ of R, except when x is a $\&/\oplus$ or a $!/?de$ cut link and n belongs to the box opened by $[x]$ while c_1 doesn't: in this case Φ' crosses a coad or a pax link less than Φ.

- in case (ii): the direct path Φ' of R' is the same as the path Φ of R, except when x is a $\&/\oplus$ or a $!/?de$ cut link and n belongs to the box opened by $[x]$: in this case Φ' crosses a coad or a pax link less than Φ.
If $[x]$ is an axiom e.r.s.: then n' is the unique residue of n in T'.
- in case (i): $x \neq c_1$ and c_1 has a unique residue c' in R'. The direct path Φ' of R' is then the same as the path Φ of R.
- in case (ii): the direct path Φ' of R' is the same as the path Φ of R.
If $[x]$ is a contraction or a $[ccad]$ e.r.s.:
- in case (i), we still have to distinguish four cases:

(i.1) c_1 is duplicated by $[x]$ (which means that $[x] = [c_1]$ or that c_1 belongs to the subproof-net of R duplicated by $[x]$) and n is also duplicated by $[x]$ (which means that n belongs to the subproof-net of R duplicated by $[x]$): then whatever of the two residues of n one chose for n', the direct path Φ' of R' is the same as the path Φ of R and c' is the appropriate residue of c_1 in R'.

(i.2) c_1 is duplicated by $[x]$ and n isn't: then $[x] = [c_1]$. If $[x]$ is a contraction e.r.s., then c' is the appropriate residue of c_1 in R' and Φ' crosses one contraction link less than Φ. If $[x]$ is a $[ccad]$ e.r.s., then c' is the appropriate residue of c_1 in R' and Φ' crosses one coad link less than Φ.

(i.3) n is duplicated by $[x]$ and c_1 isn't: then $x \neq c_1$ and c_1 has a unique residue c' in R'. If $[x]$ is a contraction e.r.s., then Φ' crosses one contraction link more than Φ. If $[x]$ is a $[ccad]$ e.r.s., then Φ' crosses one coad link more than Φ.

(i.4) Neither n nor c_1 is duplicated by $[x]$: then n and c_1 both have a unique residue in R' and Φ' is the same as Φ.

- in case (ii) we still have to distinguish two cases:

(ii.1) n is duplicated by $[x]$ (which means that n belongs to the subproof-net of R duplicated by $[x]$): then Φ' crosses one contraction (resp. one coad) link more than Φ if $[x]$ is a contraction (resp. a $[ccad]$) e.r.s..

(ii.2) n is not duplicated by $[x]$: then Φ' is the same as Φ.

If x is a weakening cut link: then n cannot belong to the exponential box erased by $[x]$ (we are supposing that n has a residue n' in R'). Let m be the weakening link of R whose conclusion is one of the premises of x. If

$n \neq m$, then in both cases ((i) and (ii)) Φ' is the same as Φ. If $m = n$, then we should be in case (i) and we should have $c_1 = x$, but this is impossible because c_1 would have no residue in R'.
If x is a cc cut link: then n has a unique residue in R' and (in case (i)) c_1 has also a unique residue c' in R'. Let $B_1^!$ be the exponential box of R swallowed by the exponential box $B_2^!$ of R when performing $[x]$. If n does not belong to $B_1^!$ nor $B_2^!$, then Φ' is the same as Φ. Otherwise:
- in case (i): if n belongs to $B_1^!$ and $c_1 = x$ or if n and c_1 belong to $B_1^!$, then Φ' is the same as Φ. If n belongs to $B_1^!$ and $c_1 \neq x$ doesn't, then Φ' crosses one pax link more than Φ. If n and c_1 belong to $B_2^!$ or if n belongs to $B_2^!$ and $c_1 \neq x$ doesn't, then Φ' is the same as Φ. If n belongs to $B_2^!$ and $c_1 = x$, then Φ' crosses one pax link less than Φ.
- in case (ii): if n belongs to $B_1^!$ then Φ' crosses one pax link more than Φ, and if n does not belong to $B_1^!$ then Φ' is the same as Φ. □

3.5. Lemma. *Let R, R' be two proof-nets s.t. $R \overset{[x]}{\to} R'$. Let n_1, n_2 be two main links of R s.t. $n_1 \neq n_2$, and let c_3 be a cut link of R s.t. if $x = c_3$ then $[x] = [ccad]$. Let n_1' (resp. n_2') be a residue of n_1 (resp. n_2) in R' and let c' be a residue of c_3 in R'.
Suppose that there exists a couple (Φ_1, Φ_2) of direct paths of R s.t. Φ_i has n_i as starting link and c_3 as terminal link. Suppose also that there exists a couple (Φ_1', Φ_2') of direct paths of R' s.t. Φ_i' has n_i' as starting link and c' as terminal link. If $\Phi_1 \cap \Phi_2 = \{c_3\}$, then $\Phi_1' \cap \Phi_2' = \{c'\}$.*

proof: Clearly $c' \in \Phi_1' \cap \Phi_2'$. Φ_1' and Φ_2' are nothing but the direct paths associated with Φ_1 and Φ_2 given by (i) of the previous lemma. We know that $\Phi_1 \cap \Phi_2 = \{c_3\}$ and we have to show that the e.r.s. $[x]$ does not change this situation. We just explain why the lemma holds: from the proof of (i) of the previous lemma, one can see that for $i \in \{1, 2\}$ either $\Phi_i' = \Phi_i$, or Φ_i' crosses one contraction, coad or pax link more or less than Φ_i. It is then easy to see that, in any case, none of the links created by $[x]$ can be crossed by both Φ_1' and Φ_2' (the hypothesis $n_1 \neq n_2$ is of course crucial).
A detailed proof can be given by inspecting (again!) all the possible cases for $[x]$. □

Let $T \overset{\sigma}{\twoheadrightarrow} T'$. Every main link of T' is a residue of a link of Q or of a link of S. The following definition introduces a special terminology for these links, and gives also a similar "status" to the links of T' which are not main links.

3.6. Definition. Suppose that $T \overset{\sigma}{\twoheadrightarrow} T'$. Let n' be a link of T', and let m' be a main link of T' s.t. there exists a direct path with starting link m' and terminal link n'. The existence of m' is given by lemma 3.2 (if n' is a main link, then $m' = n'$). Let m be the lift of m' in T.
We say that m' is a **main Q-link** (resp. a **main S-link**) iff m is a link of the subproof-net Q (resp. S) of T. We say that n' is a **Q-link** (resp. an **S-link**) iff m' is a main Q-link (resp. a main S-link). The conclusion of an S-link (resp. a Q-link) is called an **S-edge** (resp. a **Q-edge**). An additive box B is called a **Q-box** (resp. an **S-box**) iff its front door is a main Q-link (resp. a main S-link).
We say that n' is a **QS-link** (or an **SQ-link**) if n' is both an S-link and a Q-link. We use the same terminology for edges.

3.7. Remark. Let T' be a reduct of T.
(i) Any main link of T' is a main Q-link or a main S-link and cannot be both.
(ii) Let Φ and Ψ be two direct paths of T' s.t. the starting link of Φ (resp. Ψ) is a Q-link (resp. an S-link). If m is a link of T' s.t. $m \in \Phi \cap \Psi$, then m is a QS-link.
(iii) The unique link of T which is a QS-link is the cut link c.

The following lemma is obviously wrong if we substitute to $B^!$ an additive box (suppose for example that $[c]$ is a $[ccad]$ e.r.s. and that $T \overset{[c]}{\to} T'$).

3.8. Lemma. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$ and let $B^!$ be an exponential box of T'. If for every $[x] \in \sigma$ s.t. $[x] \neq [ccad]$ x is not a QS-link, then one (and only one) of the two following conditions holds:*
(1) every link of $B^!$ is a Q-link which is not an S-link
(2) every link of $B^!$ is an S-link which is not a Q-link.

proof: By induction on σ. If $T' = T$ the result is obvious. Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\to} T'$. By induction hypothesis, the result holds for the exponential boxes of T_1. Suppose now that $B^!$ does not satisfy (1) nor (2) of our lemma, let n' be the of course door of $B^!$ and let n_1 be its lift in T_1.
If n' is a main Q-link, then n_1 is a main Q-link. By induction hypothesis the exponential box $B_1^!$ of T_1 associated with n_1 contains then only Q-links that are not S-links. The only possibility to have a different situation for $B^!$ in T' is that some S-links enter the box $B_1^!$ when performing $[x]$. Then $[x]$ must be a $[cc]$ e.r.s. s.t. an exponential box $B_2^!$ of T_1 (containing some

S-links) is swallowed by $B_1^!$. But if $B_2^!$ contains an S-link, then (by induction hypothesis) every link of $B_2^!$ is an S-link. On the other hand the pax doors of $B_1^!$ are Q-links, so that x is a QS-link of T_1, thus contradicting the hypothesis on σ.
If n' is a main S-link the proof is obviously the same. □

We have seen that the unique link of T which is a QS-link is the cut link c. Roughly speaking, the following proposition states that until one doesn't reduce any residue of c, this is still the case for any reduct of T. We give a detailed proof of this result.

3.9. PROPOSITION. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s.*
If a link n of T' is a QS-link, then it is a residue of c in T'. (In particular, there are no QS-edges in T').

proof: By induction on σ. If $T = T'$, then the result is obvious. Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. There is no link of T' which is a main Q-link and a main S-link, and this means that there exist a main Q-link n_1' of T' (resp. a main S-link n_2' of T') and a direct path Ψ_1' (resp. Ψ_2') with starting link n_1' (resp. n_2') and terminal link n. $n_1' \neq n_2'$. Let then n_1 (resp. n_2) be the lift of n_1' (resp. n_2') in T_1: by induction hypothesis n_1 is a main Q-link which is not an S-link, and n_2 is a main S-link which is not a Q-link. In particular $n_1 \neq n_2$.

Part I: We first show that (2) of remark 3.3 cannot hold for n.

If (2) holds for n, then (2) holds for n_1' and n_2'. Let a' and Φ' be the conclusion and the direct path of T' associated with n given by remark 3.3. Let a_1 be the lift of a' in T_1.
If (2) of remark 3.3 holds for n_1 and for n_2, then from lemma 3.4, we deduce the existence of a direct path Φ_1 (resp. Φ_2) with starting link n_1 (resp. n_2) and terminal edge a_1. Then a_1 is a QS-edge, thus contradicting the induction hypothesis.
We can then suppose (for example) that (1) of remark 3.3 holds for n_1. Let c_1 and Ψ_1 be the cut link and the direct path of T_1 associated with n_1 given by remark 3.3. By lemma 3.4 c_1 has no residue in T'. Because n_1' is a residue of n_1 in T', c_1 cannot belong to a subproof-net of T_1 erased by $[x]$. Then (from remark 2.7) $x = c_1$ is a logical cut link (case (i)) or $x = c_1$ is a weakening cut link (case (ii)), or $[x] = [c_1]$ is an axiom e.r.s. (case (iii)).

In case (i), clearly (1) of remark 3.3 would still hold for n'_1, and this cannot be the case.
Case (ii) is represented in figure 5. n_1 is necessarily the weakening link whose conclusion is a premise of x. n_1 is a main Q-link and c_1 is then a Q-link. Because $[c_1] \in \sigma$ and $[c_1] \neq [ccad]$, c_1 is not a residue of c in T_1, and then (by induction hypothesis) c_1 is not an S-link. Let $B^!_1$ be the exponential box erased by $[c_1]$. The induction hypothesis applied to all the proof-nets appearing in σ_1 allows to deduce that for every $[y] \in \sigma_1$ s.t. $[y] \neq [ccad]$, y is not a QS-link: otherwise y would be a residue of c and we would have $[y] \in \sigma$, thus contradicting the hypothesis on the reduction sequence σ. We can then apply lemma 3.8 to $B^!_1$: the conclusion of the main door of $B^!_1$ is a Q-edge which is not an S-edge (because c_1 is not an S-link) and then every link of $B^!_1$ is a Q-link which is not an S-link.
The S-link n_2 cannot be a link of $B^!_1$, and then (2) of remark 3.3 holds for n_2: if (1) would hold for n_2, then the cut link c_2 of T_1 given by remark 3.3 would necessarily have a residue in T' and (from lemma 3.4) (1) would hold for n'_2. Lemma 3.4 gives the existence of a direct path Ψ_2 (represented in figure 5) of T_1 with starting link n_2 and terminal edge a_1. The fact that (2) of remark 3.3 holds for n'_1 implies the existence of the path Ψ (represented in figure 5) with starting link the pax door m_1 of $B^!_1$ and terminal edge a_1. m_1 is a Q-link while n_2 is a (main) S-link, so that figure 5 clearly shows that a_1 is a QS-edge, which contradicts the induction hypothesis.

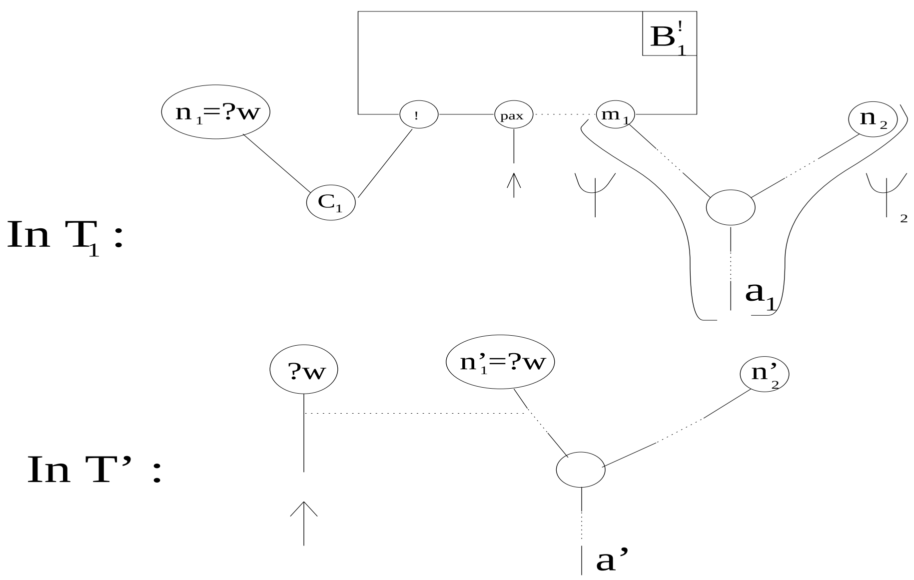

Figure 5.

Case (iii) is represented in figure 6. Let m_1 be the axiom link erased by $[x] = [c_1]$. The existence of the direct paths Ψ_1' and Φ' of T' gives the existence of a direct path Ψ_1 of T_1 with starting link m_1 and terminal edge a_1. By hypothesis on σ c_1 is not a residue of c, and then by induction hypothesis c_1 is not a QS-link of T_1. Because n_1 is a Q-link, c_1 is also a Q-link, so that m_1 is a (main) Q-link which is not an S-link. n_2 is a main S-link which is not a Q-link, and then no direct path with starting link m_1 can cross n_2. On the other hand, because (2) of remark 3.3 holds for n_2', it also holds for n_2 and (from lemma 3.4) there exists a direct path Ψ_2 of T_1 with starting link n_2 and terminal edge a_1. But in this case a_1 is a QS-edge of T_1, thus contradicting the induction hypothesis.

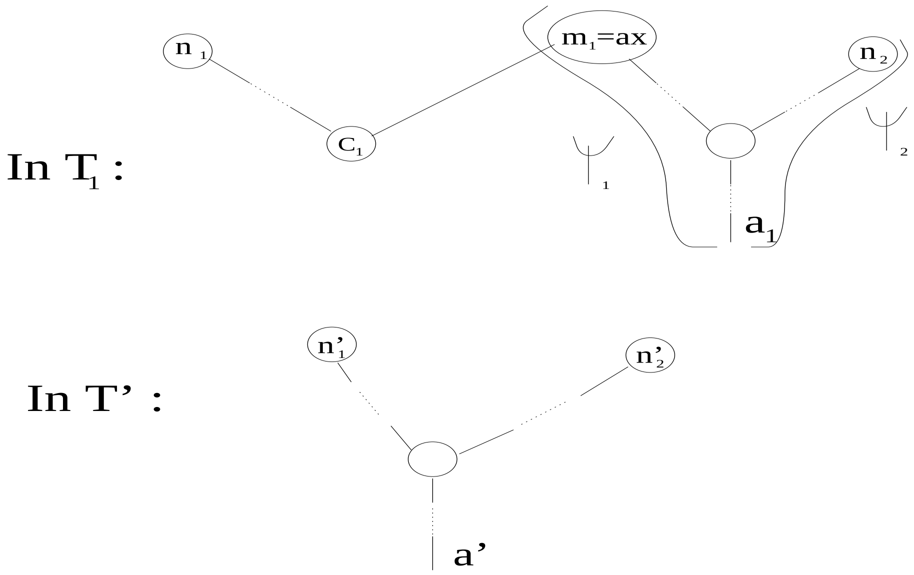

Figure 6.

Part II: We now know that (1) of remark 3.3 holds for the link n of T' and we prove the proposition in this case.

Let then c' and Ψ' be the cut link and the direct path associated with n given by remark 3.3. We have two possibilities:
(1.1) c' has no lift in T_1: then x is a logical cut link and c' is created by the e.r.s. $[x]$. Let a'_1 and a'_2 be the two premises of c' in T', and let l and $l^\perp$ be the dual logical links whose conclusions are the premises of x in T_1. Because n is a QS-link of T', either a'_i is a QS-edge (for some $i \in \{1, 2\}$) or $n = c'$ and (for example) a'_1 is a Q-edge while a'_2 is an S-edge. In the first case one among l and $l^\perp$ is a QS-link of T_1, thus contradicting the induction hypothesis. In the second case x is a QS-link, and then (by induction hypothesis) x is a residue of c in T_1, thus contradicting the hypothesis on the reduction sequence σ. This means that the case (1.1) cannot occur.
(1.2) c' has the lift c_3 in T_1: let Φ'_1 (resp. Φ'_2) be the direct path of T' with starting link n'_1 (resp. n'_2) and terminal link c'. For $j = 1, 2$, Φ'_j is obtained by connecting Ψ'_j with Ψ'. Obviously, $n \in \Phi'_1 \cap \Phi'_2$.
Let $j \in \{1, 2\}$. (1) of remark 3.3 holds for n_j in T_1: if (2) holds for n_j, then (by lemma 3.4) (2) holds for n'_j, which contradicts the fact that (1) holds for n. Let's then denote by c_j and Φ_j the cut link and the direct path of

T_1 associated with n_j given by remark 3.3. Observe that if $c_1 = c_2$, then c_j is a QS-link and (by induction hypothesis) c_j is a residue of c in T_1. This implies that $x \neq c_j$. This means that if $x = c_j$ then $c_1 \neq c_2$.
If c_j has a residue in T', then (by lemma 3.4) c' is one of the residues of c_j (i.e. $c_j = c_3$).
If c_j has no residue in T', then c_j cannot belong to a subproof-net of T_1 erased by $[x]$ (otherwise n_j would have no residue in T'). Then (from remark 2.7) $x = c_j$ is a logical cut link (case (i)) or $x = c_j$ is a weakening cut link (case (ii)), or $[x] = [c_j]$ is an axiom e.r.s. (case (iii)).
Case (i) has to be excluded: if $x = c_j$ is a logical cut, then n'_j is the unique residue of n_j and c' is necessarily created by $[x]$. But we are supposing that c' has a lift in T_1, so that this is impossible.
If case (ii) occurs (for example) for c_1, then n_1 is the weakening link whose conclusion is a premise of $c_1 = x$, and $c_2 \neq c_1$ has a residue in T'. If case (iii) occurs (for example) for c_1, then $c_1 \neq c_2$.
This means that we are done, once we have proven the proposition in the three following situations:

(α) c_1 and c_2 have both c' among their residues: then $c_1 = c_2 = c_3$

(β) $[x] = [c_1]$ is an axiom e.r.s.: $c_2 \neq c_1$ has a residue in T', so that $c_2 = c_3$

(γ) $[x] = [c_1]$ is a $[w]$ e.r.s., and n_1 is the weakening link whose conclusion is a premise of c_1: then $c_2 = c_3$.

In case (α), $c_3 \in \Phi_1 \cap \Phi_2$ and every link of $\Phi_1 \cap \Phi_2$ is a QS-link, so that by induction hypothesis $\{c_3\} = \Phi_1 \cap \Phi_2$ and c_3 is a residue of c in T_1. If $[x] \neq [ccad]$, then (by hypothesis on σ) $c_3 \neq x$. This means that we can apply lemma 3.5 to T_1 and T': $\{c'\} = \Phi'_1 \cap \Phi'_2$. Then $n = c'$ is a residue of c in T'.
Case (β) is represented in figure 7. Let m_1 be the axiom link erased by $[x] = [c_1]$. Because in T' $c' \in \Phi'_1 \cap \Phi'_2$, there exists a direct path Ψ_1 of T_1 with starting link m_1 and terminal link c_3. By hypothesis on σ, c_1 is not a residue of c, and then by induction hypothesis c_1 is not a QS-link of T_1. Because n_1 is a Q-link, c_1 is also a Q-link, so that m_1 is a (main) Q-link which is not an S-link. $c_3 \in \Psi_1 \cap \Phi_2$ and every link of $\Psi_1 \cap \Phi_2$ is a QS-link, so that by induction hypothesis $\{c_3\} = \Psi_1 \cap \Phi_2$ and c_3 is a residue of c in T_1. This clearly implies that $\{c'\} = \Phi'_1 \cap \Phi'_2$, and then $n = c'$ is a residue of c in T'.

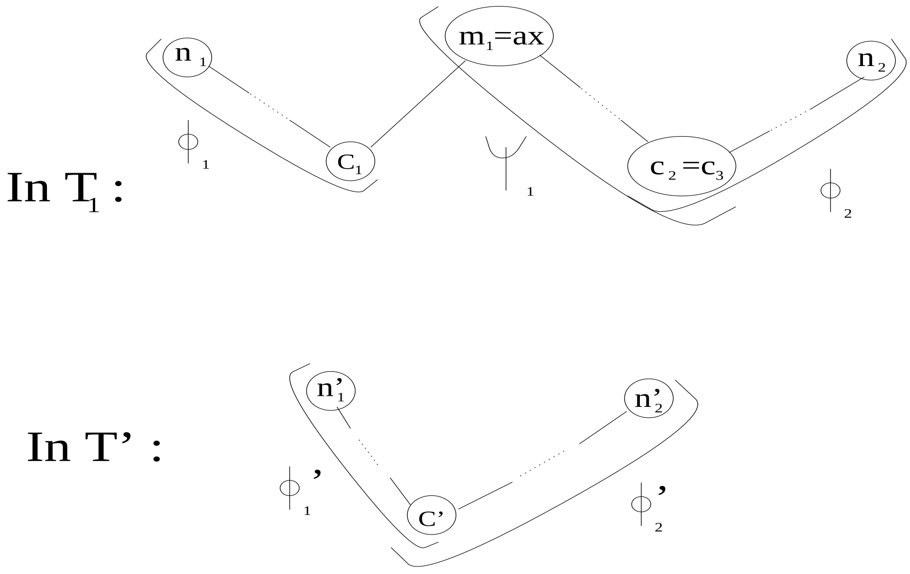

Figure 7.

Case (γ) is represented in figure 8. We argue like in case (ii) of part I. n_1 is a main Q-link and c_1 is then a Q-link. Because $[c_1] \in \sigma$ and $[c_1] \neq [ccad]$, c_1 is not a residue of c in T_1, and then (by induction hypothesis) c_1 is not an S-link. Let $B_1^!$ be the exponential box erased by $[c_1]$. By induction hypothesis we can apply lemma 3.8 to $B_1^!$: the conclusion of the main door of $B_1^!$ is a Q-edge (because c_1 is not an S-link) and then every link of $B_1^!$ is a Q-link which is not an S-link.
The existence of Φ_1' in T' implies the existence of a direct path Ψ in T_1 (represented in figure 8) with starting link the pax door m_1 of $B_1^!$ and terminal link c_2. m_1 is a Q-link while n_2 is a main S-link, so that c_2 is a QS-link and then by induction hypothesis $\{c_2\} = \Psi \cap \Phi_2$ and c_2 is a residue of c in T_1. Otherwise said the situation is the one in figure 8, and this clearly shows that $\{c'\} = \Phi_1' \cap \Phi_2'$. Then $n = c'$ is a residue of c in T'.

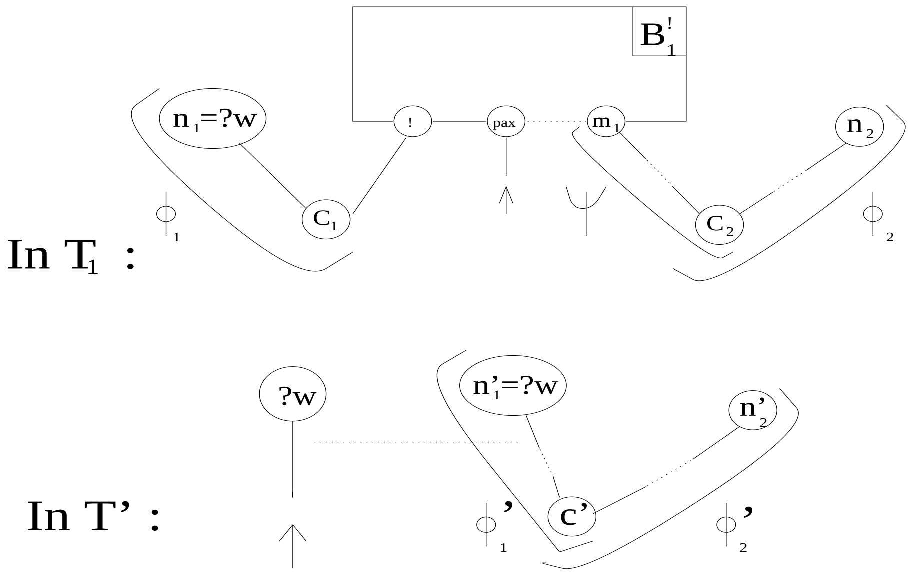

Figure 8.

□

The previous proposition allows to formulate lemma 3.8 in the following way:

3.10. Corollary. *Let* $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s.
If $B^!$ is an exponential box of T', then one (and only one) of the two following conditions holds:
(1) every link of $B^!$ is a Q-link which is not an S-link
(2) every link of $B^!$ is an S-link which is not a Q-link.

3.11. Remark. Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s. Every residue of c in T' has exponential depth 0 in T'.
This is because c has exponential depth 0 in T, and if $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{\sigma_2}{\twoheadrightarrow} T'$ is a decomposition of σ and the residue c_1 of c in T_1 is a cc cut link, then $[c_1] \notin \sigma_2$.

3.12. REMARK. Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s. Then:
(i) If Φ' is a direct path of T' crossing a Q-link (resp. an S-link) and s.t. the terminal link of Φ' is not a residue of c in T', then every link of Φ' is a Q-link (resp. an S-link) which is not an S-link (resp. a Q-link). This is an immediate consequence of proposition 3.9.
(ii) By remark 3.11, the hypothesis of lemma 2.4 are satisfied by the residues of c when $T \overset{\sigma}{\twoheadrightarrow} T'$, and for every $[x] \in \sigma$ s.t. $[x] \neq [ccad]$ x is not a residue of c.

We are now going to prove some lemmas which will be used in section 6.

3.13. LEMMA. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s. Let B' be an additive box of T' and let $i, j \in \{1, 2\}$, $i \neq j$.*
If there is a residue c'_i of c in $\gamma_i(B')$, then there is a residue c'_j of c in $\gamma_j(B')$.

proof: By induction on σ. If $T' = T$, then the result is obvious.
Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. Let B_1 be the lift of B' in T_1, and let c_i be the lift of c'_i in T_1.
If c_i is a link of B_1, then it is a link of $\gamma_i(B_1)$, and by induction hypothesis there exists a residue c_j of c in T_1 which is a link of $\gamma_j(B_1)$. If c_j has a residue in T', then one of these residues is the cut link we are looking for. If c_j has no residue in T', then $[x] \neq [ccad]$ so that (by hypothesis on σ) $c_j \neq x$. From remark 2.7, we deduce that c_j must then belong to a subproof-net of T_1 erased by $[x]$. From remark 3.11 we deduce that x is a logical $\&/\oplus$ cut link out of the front door of the additive box B_2 of T_1, and c_j belongs to the component $\gamma_k(B_2)$ of B_2 erased by $[x]$. B_2 is necessarily a box contained in the component j of B_1 (B_2 cannot contain B_1, otherwise c_i would have no residue in T_1). By induction hypothesis, there exists a residue c_h of c in the component of B_2 different from $\gamma_k(B_2)$. The residue of c_h in T' is the cut link we are looking for.
If c_i is not a link of B_1, then $[x] = [ccad]$ and, either $[c_i] = [x]$ or c_i is a cut link contained in the subproof-net of T_1 swallowed by B_1 when performing $[x]$. In both cases c_i has two residues: the cut link we are looking for is the residue of c_i different from c'_i. $\square$

3.14. LEMMA. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s.*

There exists at least a residue of c in T'.

proof: By induction on σ. If $T' = T$, then the result is obvious.
Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. By induction hypothesis, let c_1 be a residue of c in T_1. If c_1 has a residue in T', then the result is obvious. Otherwise $[x] \neq [ccad]$, then $x \neq c_1$, so that (again by remark 2.7) c_1 belongs to a subproof-net of T_1 erased by $[x]$. From remark 3.11 we deduce that x is a logical $\&/\oplus$ cut link out of the front door of the additive box B_1 of T_1, and c_1 belongs to the component of B_1 which is erased by $[x]$. But from lemma 3.13, the component of B_1 which is not erased by $[x]$ contains a residue c_2 of c in T_1. The residue of c_2 in T' is a residue of c in T'. $\square$

The notion of jump is crucial to prove the last three lemmas of this section.

3.15. LEMMA. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s.*
Let $j_{T'}$ be one of the functions associated with σ and given by definition 3.14 of [Tortora 00a]. If m' is a weakening link of $(T', j_{T'})$ which is a (main) Q-link (resp. S-link) then its jumps are all (main) Q-links (resp. S-links).

proof: By induction on σ. If $T' = T$, then the result is obvious.
Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. Let m' be a weakening link of T' which is a (main) Q-link. Let m_1 be the lift of m' in T_1: m_1 is a (main) Q-link and (by induction hypothesis) its jumps are all Q-links.
If $[x]$ is not an axiom nor a weakening e.r.s., then by (ii) of remark 2.1 $j_{T'}(m')$ contains only residues of links of $j_{T_1}(m_1)$. By induction hypothesis the links of $j_{T_1}(m_1)$ are (main) Q-links, and then the links of $j_{T'}(m')$ are all (main) Q-links.
If $[x]$ is a weakening e.r.s., let then n be the of course link whose conclusion is a premise of x, let $B^!$ be the exponential box associated with n and let z_1 be the weakening link whose conclusion is a premise of x. By definition 3.14 of [Tortora 00a], if for every $z \in j_{T_1}(m_1)$ z is not a link of $B^!$, then $j_{T'}(m')$ is the set of the residues of the links of $j_{T_1}(m_1)$, which are all Q-links. Otherwise by proposition 3.9 and corollary 3.10, $B^!$ contains only Q-links that are not S-links, and x, z_1 are Q-links that are not S-links. In this case $j_{T'}(m')$ contains only residues of links of $j_{T_1}(m_1)$ and of $j_{T_1}(z_1)$, then only Q-links.
If $[x]$ is an axiom e.r.s., let then n be the axiom link erased by $[x]$. If $n \notin j_{T_1}(m_1)$, then (by definition 3.14 of [Tortora 00a]) $j_{T'}(m')$ is the set of

the residues of the links of $j_{T_1}(m_1)$. As usual in this case we conclude by applying the induction hypothesis. If $n \in j_{T_1}(m_1)$, then we have to prove that every link of $j_{T'}(m')$ which is not the residue of a link of $j_{T_1}(m_1)$ is a main Q-link. Definition 3.14 of [Tortora 00a] shows that the lift z_1 in T_1 of every such link z' of T' satisfies one of the two following conditions:
(α) there exists a direct path Ψ_1 of T_1 with starting link z_1 and terminal link x
(β) there exists a weakening link u_1 of T_1 s.t. $z_1 \in j_{T_1}(u_1)$ and there exists a direct path Ψ_1 of T_1 with starting link u_1 and terminal link x.
Observe that $n \in j_{T_1}(m_1)$ is a Q-link (by induction hypothesis), and then x is a Q-link which is not an S-link (by hypothesis on σ). In case (α) (resp. (β)), by remark 3.12 z_1 (resp. u_1) is a main Q-link. In case (α), this implies that z' is a Q-link and we are done. In case (β), by induction hypothesis we deduce that z_1 is a main Q-link, and then z' is a Q-link. □
The following lemma is a consequence of definitions 2.16 of proof-nets and 2.19 of subproof-net of [Tortora 00a].

3.16. LEMMA. *Let (T, j_T) be a proof-net. Let's call T_j the graph obtained from T by drawing the edges connecting the weakening links of T with their jumps. If (R, j_R) is a subproof-net of (T, j_T), and if m and n are two links of R, then there exists an oriented path of T_j with starting link m and terminal link n which crosses only links of R.*

3.17. LEMMA. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s.*
Let $j_{T'}$ be one of the functions associated with σ and given by definition 3.14 of [Tortora 00a], and let $(R', j_{R'})$ be a subproof-net of $(T', j_{T'})$. If n' (resp. m') is a Q-link (resp. an S-link) of T' which is a link of R', then there is a residue of c in R'.

proof: Let T'_j be the graph obtained from T' by drawing the edges connecting the weakening links of T' and their jumps. By lemma 3.16, there exists an oriented path Φ' of T'_j with starting link n', terminal link m' and crossing only links of R'. Let z' be the first link of Φ' which is an S-link.
If $z' = n'$, n' is a QS-link and then (by proposition 3.9) a residue of c, so that we are done.
If $z' \neq n'$, let u' be the last link of Φ' which is a Q-link and not an S-link. Let's denote by $\overrightarrow{u'z'}$ the (oriented) edge of Φ' connecting u' and z'. z' is not a jump of u': if it were the case, by lemma 3.15 z' would be a Q-link, then

a QS-link and by proposition 3.9 z' would then be a residue of c in T', thus contradicting the fact that a jump is an axiom link. For the same reasons, u' is not a jump of z'. This means that there exists an oriented edge α of the proof-net T connecting u' and z'. u' is the source and z' is the goal of α: otherwise u' would be an S-link, thus contradicting the fact that z' is the first S-link crossed by Φ'. Then z' is also a Q-link, which implies (still by proposition 3.9) that z' is a residue of c in T'. $\square$

3.18. LEMMA. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and suppose that for every $[x] \in \sigma$ if x is a residue of c, then $[x]$ is a $[ccad]$ e.r.s.*
If B' is an additive box of T' containing a Q-link and an S-link, then B' contains a residue of c in T'.

proof: By induction on σ. If $T' = T$, then the result is obvious (no additive box of T contains a Q-link and an S-link). Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. Let B_1 be the lift of B' in T_1.
If B_1 contains an S-link and a Q-link, then by induction hypothesis B_1 contains a residue c_1 of c in T_1. Suppose for example that c_1 is a link of $\gamma_1(B_1)$. By lemma 3.13 there exists a residue c_2 of c in T_1 which is a link of $\gamma_2(B_1)$. B_1 has at least a residue in T' (it is B'), so that at least one among c_1 and c_2 has a residue in T' which is a link of B'.
If B_1 contains only (say) Q-links, then one has necessarily $[x] = [ccad]$ and the subproof-net R_1 duplicated by $[x]$ contains some S-links. We have then two possibilities:
- R_1 contains only S-links that are not Q-links: then x is a QS-link of T_1, that is (by proposition 3.9) a residue of c in T_1. In that case any of the two residues of x in T' will do
- R_1 contains S-links and Q-links: then by lemma 3.17 R_1 contains a residue c_1 of c in T_1. Any of the two residues of c_1 in T' will then do. $\square$

4 The additive standardization theorem

The following is theorem 4.26.9 p.74 of [Girard 87]:

4.1. THEOREM. *(**Girard 87**) All proof-nets of PN_2 are reducible.*

If R is any proof-net with conclusions $E_1, \ldots, E_k$ and α_i ($i \in \{1, \ldots, k\}$) is a *candidat de réductibilité* (see [Girard 87] for the definition) having $E_i^\perp$ among its conclusions, then Girard shows, by induction on the proof-net R, that the proof-net T obtained by cuting R with the α_i is strongly normalizable. As we already wrote, the notion of proof-net and the definition of cut-elimination of [Girard 87] are not exactly the ones that we have given in sections 2 and 3 of [Tortora 00a]. However, the method of the *candidats de réductibilité* is known to yield a very general proof of strong normalization, and there is not the slightest problem to apply it to the system PN_2 that we have previously defined.

In case R is an additive box $\mathbb{B}$ with front door $D\&C$ and auxiliary doors $A_1, ..., A_n$, let's call Q_i $(i \in \{1, ..., n\})$ the *candidat de réductibilité* having $A_i^\perp$ among its conclusions, and S the *candidat de réductibilité* having $D^\perp \oplus C^\perp$ among its conclusions. Girard shows that for S it is enough to take a *candidat de réductibilité* having among its last links a $\oplus$ link with conclusion the edge labeled by $D^\perp \oplus C^\perp$ (which will be one of the two premises of the cut link of T between $\mathbb{B}$ and S). In this paper we will assume that this $\oplus$ link is a $\oplus_1$ link (in case it is a $\oplus_2$ link the proof is of course the same).
Let R_1 (resp. R_2) be the first (resp. the second) component of $\mathbb{B}$, and S_1 be the subproof-net of S obtained by removing the (terminal) $\oplus_1$ link. We call $C_1(T)$ the proof-net obtained by cuting R_1 with the Q_i $(i \in \{1, ..., n\})$ and with S_1, and we call $C_2(T)$ the proof-net obtained by cuting R_2 with the Q_i $(i \in \{1, ..., n\})$.
In Girard's proof of strong normalization, by applying the induction hypothesis one obtains that $C_1(T)$ and $C_2(T)$ are strongly normalizable. What is missing to conclude is then the following

4.2. Theorem. (**Additive standardization**) If $C_1(T)$ and $C_2(T)$ are strongly normalizable, then so is T.

From now on, T, $C_1(T)$, $C_2(T)$, S and $\mathbb{B}$ will denote proof-nets of the previous form. If $T \overset{\sigma}{\twoheadrightarrow} T'$, we shall call **special cut** (resp. **special link** $\oplus$) any residue of the logical cut between $\mathbb{B}$ and S (resp. any residue of the link $\oplus_1$ of S) in T'.
To prove the theorem, we define a size of every reduct of T, and show that this size shrinks when performing any e.r.s..

4.3. Remark. For the same reasons evoked in remark 3.1, when $T \overset{\sigma}{\twoheadrightarrow} T'$, the label of the front door of any residue of $\mathbb{B}$ in T' is $D\&C$, the label of the conclusions of the special $\oplus$ links of T' is $D^\perp \oplus C^\perp$, and the labels of the premises of the special cuts of T' are $D\&C$ and $D^\perp \oplus C^\perp$.

The following useful lemma basically holds because all the residues of $\mathbb{B}$, all the special $\oplus_1$ links and all the special cuts, have exponential depth 0 in T'. The proof of this lemma requires some technicalities, and we suggest to the reader who is not interested in the proof to go directly to definition 5.1.

4.4. LEMMA. *Suppose that $T \overset{\sigma}{\twoheadrightarrow} T'$. Then:*

1) *Let & (resp. $\oplus_1$) be the front door of a residue of $\mathbb{B}$ (resp. a special $\oplus_1$ link) in T'. There exists a path Φ with starting link & (resp. $\oplus_1$) and terminal link a special cut, s.t. every link of Φ different from its starting link and from its terminal link, is a coad link.*

2) *Let c be a special cut in T', and let Φ be a path having c as starting link and crossing at least one link different from c and from a coad link (such a path obviously exists for every special cut c).*

   If n is the first link crossed by Φ such that $n \neq c$ and $n \neq coad$, then n is either a special $\oplus_1$ link either the front door of a residue of $\mathbb{B}$.

Let $T \overset{\sigma}{\twoheadrightarrow} T'$. Observe that $\mathbb{B}$ and $\oplus_1$ have exponential depth zero in T, so that remark 2.5, the separation property and lemma 2.8 apply. The previous lemma shows that every special cut of T' has exponential depth zero in T', so that also lemma 2.4 applies.

4.5. DEFINITION. Let Φ be a path in a proof-net, with starting link n and terminal link m. We will say that Φ is a **lucid** path if every link l crossed by Φ and s.t. $l \neq m, n$ is a coad link. We will say that Φ is a **maximal lucid** path if Φ is lucid and $n, m \neq coad$.

4.6. REMARK. (i) If Φ is a lucid path of a proof-net, then either Φ is itself a direct path, or the path $\Phi^\star$ obtained from Φ by reverting all its edges is a direct path.
(ii) If n is a cut link in a proof-net, then there always exists a maximal lucid path with starting link n.

Lemma 4.4 is an immediate consequence of the following lemma (just take $R = R_1 = T$, $R' = R_2 = T'$, $\sigma_2 = \sigma$, l the front door of $\mathbb{B}$, $l^\perp$ the unique special $\oplus_1$ link of T and c the unique special cut of T).

4.7. LEMMA. *Let R be a proof-net s.t. $R \overset{\sigma_1}{\twoheadrightarrow} R_1 \overset{\sigma_2}{\twoheadrightarrow} R_2 \overset{\sigma_3}{\twoheadrightarrow} R'$. Let l and $l^\perp$ be two (dual) propositional logical links of R s.t. the two conclusions of l and $l^\perp$ are the two premises of the logical cut link c, and suppose that l, $l^\perp$ and c have exponential depth zero in R.*

If:

(i.1) For every residue l_1 (resp. $l_1^\perp$) of l (resp. of $l^\perp$) in R_1, there exists a residue c_1 of c in R_1 and a lucid path Φ_1 with starting link l_1 (resp. $l_1^\perp$) and terminal link c_1

(ii.1) Every maximal lucid path with starting link a residue c_1 of c in R_1, has a residue of l or a residue of $l^\perp$ as terminal link

then:

(i.2) For every residue l_2 (resp. $l_2^\perp$) of l (resp. of $l^\perp$) in R_2, there exists a residue c_2 of c in R_2 and a lucid path Φ_2 with starting link l_2 (resp. $l_2^\perp$) and terminal link c_2

(ii.2) Every maximal lucid path with starting link a residue c_2 of c in R_2, has a residue of l or a residue of $l^\perp$ as terminal link.

proof: By induction on σ_2. If $R_1 = R_2$, the result is obvious.
Otherwise $R_1 \overset{\sigma_2'}{\twoheadrightarrow} R_2' \overset{[x]}{\rightarrow} R_2$. In this proof we will constantly use remark 2.5: every residue of l and every residue of c has exponential depth zero.
Let's first prove (i.2). Let l_2 be a residue of l in R_2 and let l_2' be its lift in R_2'. By induction hypothesis ((i.2) applied to l_2'), there exists a residue c_2' of c in R_2' and a lucid path Φ_2' with starting link l_2' and terminal link c_2'. By induction hypothesis ((ii.2) applied to c_2'), if c_2' is not a logical cut link, then $[c_2'] = [ccad]$. By remark 2.7, the only possibility for c_2' to have no residue in R_2 would be to disappear in a $[\&/\oplus]$ e.r.s.. But any component of an additive box B of R_2' containing c_2' will necessarily contain l_2', and c_2' has a residue in R_2. The lucid path Φ_2' of R_2' is a direct path, so that we can apply lemma 3.4: there exists a direct path Φ_2 of R_2 with starting link l_2 and terminal link a residue of c_2' in R_2. The proof of lemma 3.4 shows that if $\Phi_2 \neq \Phi_2'$, then Φ_2 crosses one coad link more or less than Φ_2': Φ_2 is lucid.
Let's prove now (ii.2). Let c_2 be a residue of c in R_2 and let Φ_2 be a maximal lucid path of R_2 with starting link c_2. Let's call l_2 the terminal link of Φ_2. We have to show that l_2 is a residue of l or a residue of $l^\perp$ in R_2.

Let m_2 be a main link of R_2 s.t. there exists a direct path Ψ_2 with starting link m_2, terminal link c_2, which crosses l_2 and which does not cross any main link of R_2 different from m_2 (we will show that, actually, $m_2 = l_2$). Let m'_2 and c'_2 be, respectively, the lifts in R'_2 of m_2 and c_2. m'_2 satisfies condition (1) of remark 3.3: otherwise by lemma 3.4 m_2 would satisfy condition (2) of remark 3.3, and we know that this is not the case. Let then Ψ'_2 and c'_3 be the path and the cut link of R'_2 given by remark 3.3. We are going to show that $c'_2 = c'_3$. If $c'_2 \neq c'_3$, then by lemma 3.4 c'_3 has no residue in R_2. This implies (by remark 2.7) that $c'_3 = x$ is a logical cut link (case (i)), or $c'_3 = x$ is a weakening cut link (case (ii)), or $c'_3 = x$ and $[x]$ is an axiom e.r.s. (case (iii)).
In case (i), c_2 is a cut link of R_2 created by $[x]$, with no lift in R'_2, and this is impossible.
In case (ii), m'_2 is the weakening link whose conclusion is a premise of c'_3 and there exists a direct path Φ'_2 with starting link an auxiliary door of the biggest exponential box erased by $[x]$ and terminal link c'_2. By definition of Ψ_2, Φ'_2 crosses no main link, and this contradicts the induction hypothesis (ii.2) applied to the cut link c'_2 of R'_2.
In case (iii), there exists a direct path Φ'_2 with starting link the axiom link t'_2 of R'_2 erased by $[x]$ and terminal link c'_2. By definition of Ψ_2, t'_2 is the unique main link of Φ'_2, and this contradicts the induction hypothesis (ii.2) applied to the cut link c'_2 of R'_2.
We now know that $c'_2 = c'_3$. The induction hypothesis (ii.2) applied to the cut link c'_2 of R'_2, gives the existence of a maximal lucid path with starting link a residue l'_2 of l or of $l^\perp$ in R'_2 and terminal link c'_2, which is a subpath of Ψ'_2. The proof of lemma 3.4 shows that if $\Psi_2 \neq \Psi'_2$, then Ψ_2 crosses one coad link more or less than Ψ'_2. This fact, and the fact that Ψ_2 crosses no main link different from m_2, implies that $l'_2 = m'_2$ and $l_2 = m_2$. Then l_2 is a residue of l or of $l^\perp$ in R_2. □

5 The first projection of the reducts of T

The following is the definition of $C_1(T')$, the analogue of $C_1(T)$ for any reduct T' of T: every residue of $\mathbb{B}$ in T' is replaced by its first component and we erase every special $\oplus_1$ link (the residues of the $\oplus$ link of S whose conclusion is a premise of c).

5.1. DEFINITION. **(first projection)** Let T' be such that $T \overset{\sigma}{\twoheadrightarrow} T'$.

For any residue $\mathbb{B}_{T'}$ of $\mathbb{B}$ in T' with first component $\gamma_1(\mathbb{B}_{T'})$, and for any subproof-net γ of T' with a special link $\oplus_1$ among its last links, we define

$$C_1(T') := T'\left[\gamma_1(\mathbb{B}_{T'})\Big/_{\mathbb{B}_{T'}}, \gamma\backslash\{\oplus_1\}\Big/_{\gamma}\right]$$

where $\gamma\backslash\{\oplus_1\}$ is obtained from γ by eliminating the special link $\oplus_1$ (then $\gamma\backslash\{\oplus_1\}$ will have $D^\perp$ instead of $D^\perp \oplus C^\perp$ among its conclusions).
As $C_1(T')$ is not exactly defined by substitution (following definition 3.5 of [Tortora 00a]), we still have to define the function $j_{C_1(T')}$. If n is a weakening link of $C_1(T')$, we simply define $j_{C_1(T')}(n)$ as the set of links of $j_{T'}(n)$ which are links of $C_1(T')$ (we only keep the jumps that are still defined).

5.2. Remark. (i) $C_1(T')$ is well-defined thanks to the separation property and to lemma 4.4: special cuts are replaced by cuts with active formula D. The fact that $(T', j_{T'})$ is a proof-net implies that $(C_1(T'), j_{C_1(T')})$ is a proof-net (the "cognoscendi" might notice that condition (1) of definition 2.6 of [Tortora 00a] for $(T', j_{T'})$ plays an essential role).

(ii) For $T' = T$, $C_1(T)$ is the proof-net previously defined.

(iii) If in T' there is no residue of $\mathbb{B}$, then from lemma 4.4 $C_1(T') = T'$

(iv) Let x be a cut link in the reduct T' of T. If x is not a *logical* special cut, and if x does not appear in the second component of any residue of $\mathbb{B}$ in T', then we have "the same" cut link x in $C_1(T')$.

The following proposition states that C_1 "projects" a sequence of reductions onto a sequence of reductions/equalities.

5.3. Proposition. *Let T_1 and T_2 be two proof-nets such that $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2$.*
One has $C_1(T_1) = C_1(T_2)$ when:

- $[x] = [ccad]$ attractive for a residue of $\mathbb{B}$ in T_1
- x is a logical special cut
- x is a cut link in the second component of a residue of $\mathbb{B}$ in T_1.

Otherwise $C_1(T_1) \overset{[x]}{\rightarrow} C_1(T_2)$.
In particular, if T' is a proof-net such that $T \twoheadrightarrow T'$ and if $C_1(T)$ is strongly normalizable, then $C_1(T')$ is also strongly normalizable.

proof: If $[x] = [ccad]$ attractive for a residue of $\mathbb{B}$ in T_1, x is a logical special cut, or x is a cut link in the second component of a residue of $\mathbb{B}$ in T_1, then we say that we are in situation (a). Otherwise we say that we are in situation (b).
If there is no residue of $\mathbb{B}$ in T_1, then there is no residue of $\mathbb{B}$ in T_2; we have that $C_1(T_1) = T_1$ and $C_1(T_2) = T_2$, and from lemma 4.4 there is no special cut in T_1: we are in situation (b) and we have that $C_1(T_1) \overset{[x]}{\rightarrow} C_1(T_2)$.
If x is a logical special cut, then from the definition of C_1, one clearly deduce $C_1(T_1) = C_1(T_2)$.
If x is a cut link in the first (resp. the second) component of a residue of $\mathbb{B}$ in T_1, then we are in situation (b) (resp. in situation (a)) and $C_1(T_1) \overset{[x]}{\rightarrow} C_1(T_2)$ (resp. $C_1(T_1) = C_1(T_2)$).
If $[x] = [ccad]$ attractive for a residue of $\mathbb{B}$ in T_1, then the only effect of $[x]$ is to increase the number of links (and maybe of jumps) contained in the residue (without duplicating any residue of $\mathbb{B}$, because of the separation property), but the definition of C_1 doesn't care: we are in situation (a) and $C_1(T_1) = C_1(T_2)$.
Otherwise, on the one hand from remark 5.2 x appears in $C_1(T_1)$, and on the other hand the hypothesis of the substitution lemma (lemma 2.2) are satisfied: x is neither logical special neither contained in a residue of $\mathbb{B}$ in T_1 and, if $[x] = [ccad]$, then $[x]$ is not attractive for any residue of $\mathbb{B}$ in T_1, and it is impossible for $[x]$ to duplicate a subproof-net of a residue $\mathbb{B}_1$ of $\mathbb{B}$ in T_1 without duplicating the whole $\mathbb{B}_1$. Then, if $\mathbb{B}_1^1, \ldots, \mathbb{B}_1^n$ are all the residues of $\mathbb{B}$ in T_1 and $\alpha_1, \ldots, \alpha_n$ are any proof-nets s.t. $\mathbb{B}_1^i$ and α_i have the same conclusions, we have, up to substitutions if $[x] = [\forall/\exists]$ (and again thanks to the separation property), that $T_1[\alpha_1/\mathbb{B}_1^1, \ldots, \alpha_n/\mathbb{B}_1^n] \xrightarrow{[x]} T_2[\alpha_1/\mathbb{B}_1^{1'}, \ldots, \alpha_n/\mathbb{B}_1^{n'}]$ where $\mathbb{B}_1^{i'}$ is any residue of $\mathbb{B}_1^i$ in T_2 (there are at most two for each i). In particular, thanks to lemma 4.4, we can substitute to any residue of $\mathbb{B}$ in T_1 its first component, provided we substitute any special cut by a cut with active formula D: we are in situation b) and $C_1(T_1) \overset{[x]}{\rightarrow} C_1(T_2)$. $\square$

Discussion: The first idea would now be to define in a similar way $C_2(T')$ (taking the second component of the residues of $\mathbb{B}$ and erasing the first) for any reduct T' of T, and then to prove the analogue of the previous proposition for C_2. But here is where the dissymmetry between the role played by the first and the second component of $\mathbb{B}$ appears (in theorem 4.2 we decided to choose a special link $\oplus_1$ and not $\oplus_2$): when we reduce a

logical special cut, it is always the *second* component of the residue of $\mathbb{B}$ which disappears.
Actually, consider for example the following reduction $T \overset{[ccad]}{\to} T_1 \overset{[\&/\oplus_1]}{\to} T_2$, where:

- T_1 is obtained from T applying an e.r.s. $[ccad]$ attractive for a box B contained in one of the Q_i, in such a way that the subproof-net swallowed by the box B contains $\mathbb{B}$ and the logical special cut,

- T_2 is obtained from T_1 by reducing precisely one of the two logical special cuts (that are residues in T_1 of T 's logical cut between $\mathbb{B}$ and S).

In T_1, we have two residues of $\mathbb{B}$ as subproof-nets of, respectively, the first and the second component of the residue B_1 of B in T_1; but in T_2, we have only one residue of $\mathbb{B}$, that is a subproof-net of (only) one of the two components of the residue B_2 of B in T_2.
In T_1, when we take the second component of the two residues of $\mathbb{B}$, we obtain two proof-nets with C among their conclusions, and we can connect the two edges by a link *coad* (which will be added to the box B_1) to define $C_2(T_1)$. But in T_2, we have only one residue of $\mathbb{B}$, so that when we take the second component of this residue of $\mathbb{B}$, we obtain a proof-net with C among its conclusions, and there is no way to obtain a proof-net with C among its conclusions from the component of B_2 which does not contain any residue of $\mathbb{B}$. This is because (in the canonical representation of T_1 with respect to the residue of $\mathbb{B}$ that has a residue in T_2) we have reduced a cut in a *main subproof-net* (defined by the convention of section 2) of T_1.
The reader is invited to draw a small picture which should clarify the whole thing.
Then, if $T \overset{\sigma}{\twoheadrightarrow} T'$, we will select a residue $\mathbb{B}'$ of $\mathbb{B}$ in T', and will distinguish the case in which (taking for any proof-net appearing in σ its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net) σ doesn't contain any e.r.s. reducing a cut inside a main subproof-net so that it will be possible to define $C_2^{\mathbb{B}'}(T')$, from the one in which σ contains an e.r.s. reducing a cut inside a main subproof-net, that will be handled in a different way (see proposition 7.1).

6 The second projection of the reducts of T

Let's call Q the subproof-net of T obtained from T by erasing S and the (unique) special cut link of T. We are now in the situation described in section 3, where the cut c of section 3 is here the special cut of T between Q and S. We can then use the terminology (and the results) of section 3.
Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let's consider for every proof-net appearing in the reduction sequence σ, its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net. The aim of the present section is to show that if σ does not contain any e.r.s. reducing a cut inside a main subproof-net, then one can define $C_2^{\mathbb{B}'}(T')$ by erasing (roughly speaking) all the S-links of T' (see definition 6.7), and one can prove that $C_2(T) \twoheadrightarrow C_2^{\mathbb{B}'}(T')$ (see proposition 6.9). This is possible, mainly thanks to proposition 3.9 and lemmas 3.15 and 3.18.

The following remark is an immediate consequence of lemma 4.4.

6.1. REMARK. Let $T \overset{\sigma}{\twoheadrightarrow} T'$. If c' is a special cut of T', then c' is a QS-link.

The following lemma allows to apply, in the sequel of this section, all the results of section 3.

6.2. LEMMA. *Let* $T \overset{\sigma}{\twoheadrightarrow} T'$, *let* $\mathbb{B}'$ *be a residue of* $\mathbb{B}$ *in* T', *and let's consider for any proof-net appearing in the reduction sequence* σ, *its canonical representation with respect to the lift of* $\mathbb{B}'$ *in this proof-net. Suppose that* σ *does not contain any e.r.s. reducing a cut inside a main subproof-net.*
If $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$ *is a decomposition of* σ *and if* x *is a special cut link, then* $[x]$ *is a* $[ccad]$ *e.r.s.*

proof: Let $\mathbb{B}_1$ be the lift of $\mathbb{B}'$ in T_1. By lemma 4.4, if x is a special cut link of T_1 and $[x] \neq [ccad]$ then x is a $\&/\oplus$ cut link out of the front door of a residue of $\mathbb{B}$ in T_1. But this is impossible: on the one hand x cannot be a cut link of a main subproof-net of T_1 (by hypothesis on σ), and on the other hand x cannot be a cut link out of the front door of $\mathbb{B}_1$ (which has a residue in T_2). $\square$

Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let's consider for every proof-net appearing in the reduction sequence σ, its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net. We are going to prove that in the absence of a reduction inside a main subproof-net, every additive box B of T' which is contained in a main subproof-net and which contains a special cut, "is not new": as a subproof-net of T', B is the residue of an additive box containing the lift of $\mathbb{B}'$. This result is an essential ingredient of section 6.7 (more precisely of case 2.3.2.2 of section 6.7): if B is an additive box of T' containing a special cut, we will have to define $C_2^{\mathbb{B}'}(B)$ only when B contains $\mathbb{B}'$.

When one writes "an additive box B of a proof-net R", one might refer to B as a box or as a subproof-net. In [Tortora 00a], we defined the notions of lift and residues both for a box and a subproof-net, and these two notions do not coincide: if $T \overset{\sigma}{\twoheadrightarrow} T'$, then every additive box of T' has a lift in T as a box, but it might not have a lift in T as a subproof-net. This is because the content of the box might "have changed" during cut-elimination.
In the following lemma, we will speak of "the lift and the residues of the subproof-net $\alpha = B$ (or simply of $\alpha = B$)" where B is an additive box, to indicate the fact that we think of B as a *subproof-net* rather than as a box: we then refer to the (stronger) notion of lift and residue of a subproof-net (definition 3.19 of [Tortora 00a]) and not to the (weaker) notion of lift and residue of an additive box (definition 3.12 of [Tortora 00a]).

6.3. Lemma. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let's consider for every proof-net appearing in the reduction sequence σ, its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net. Suppose that σ does not contain any e.r.s. reducing a cut inside a main subproof-net.*
Let B be an additive box of the main subproof-net β' of T', suppose that B contains all the special cut links of T' which are links of β', and that there exists at least one such link. Then, there exists a proof-net R and a decomposition of σ s.t.:
- $T \overset{\sigma'}{\twoheadrightarrow} R \overset{\sigma''}{\twoheadrightarrow} T'$, where σ'' is not empty
- there exists a lift $\alpha_R = B_R$ in R of the subproof-net $\alpha = B$ of T', s.t. $\alpha_R = B_R$ is an additive box containing the lift of $\mathbb{B}'$ in R.

proof: By induction on σ. If $T' = T$, then there is nothing to prove. Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$.
By hypothesis B contains a special cut link, so that by lemma 4.4 B contains a residue of $\mathbb{B}$ in T' and B is not itself a residue of $\mathbb{B}$ in T'. Let B_1 be the lift of B in T_1 and $\mathbb{B}_1$ be the lift of $\mathbb{B}'$ in T_1.

We first show that B_1 contains a residue of $\mathbb{B}$ in T_1. If no residue of $\mathbb{B}$ in T_1 is contained in B_1, then $[x]$ is a $[ccad]$ e.r.s. B_1-attractive, and the subproof-net S_1 of T_1 duplicated by $[x]$ contains a residue of $\mathbb{B}$ in T_1. By hypothesis on σ, x is not a cut link of one of the main subproof-nets of T_1, and then S_1 is not a subproof-net of one of the main subproof-nets of T_1. The only possibility is then that S_1 contains the lift $\mathbb{B}_1$ of $\mathbb{B}'$ in T_1. But in this case B contains $\mathbb{B}'$, thus contradicting the hypothesis of the lemma.
We now distinguish the two possible cases: B_1 contains or does not contain $\mathbb{B}_1$. First suppose that B_1 contains $\mathbb{B}_1$. Because a residue of B_1 is contained in a main subproof-net of T', $[x] = [ccad]$ and the subproof-net of T_1 duplicated by $[x]$ contains B_1. Let α_1 be the subproof-net B_1 of T_1. α_1 has two residues in T' and $\alpha = B$ is one of them. Then $R = T_1$ is the proof-net that we are looking for, where $\alpha_1 = \alpha_R = B_R = B_1$.
Suppose now that B_1 does not contain $\mathbb{B}_1$. Then B_1 is a box of a main subproof-net β_1 of T_1. By hypothesis on σ, x is not a link of β_1 and β_1 cannot be erased by $[x]$: β_1 has one or two residue(s) in T'. Let β_1' be the residue of β_1 containing the subproof-net $\alpha = B$ of T'. $\alpha = B$ has then the subproof-net $\alpha_1 = B_1$ as lift in T_1. We want to apply the induction hypothesis to the subproof-net $\alpha_1 = B_1$ of T_1. By lemma 2.8 β_1' is a subproof-net of the main subproof-net β' of T'. Then every special cut link c_1 of T_1 which is a link of β_1 has itself a residue c' in T'. Because (by hypothesis on B and β') c' is contained in B, c_1 has to be contained in B_1. And the lift in T_1 of any special cut of T' contained in B is a link of B_1. By induction hypothesis, there exists a proof-net R and a decomposition of σ_1 s.t.:
- $T \overset{\sigma_1'}{\twoheadrightarrow} R \overset{\sigma_1''}{\twoheadrightarrow} T_1$, where σ_1'' is not empty
- there exists a lift $\alpha_R = B_R$ in R of the subproof-net $\alpha_1 = B_1$ of T_1, s.t. $\alpha_R = B_R$ is an additive box containing the lift of $\mathbb{B}_1$ in R. Clearly the same proof-net R and the decomposition $T \overset{\sigma_1'}{\twoheadrightarrow} R \overset{\sigma_1''}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$ of σ will do for the subproof-net $\alpha = B$ of T'. □

The following lemma allows to define $C_2(B)$ when the additive box B of a reduct of T does not contain any S-link (see (ii) of remark 6.6).

6.4. Lemma. *Let $T \overset{\sigma}{\twoheadrightarrow} T'$ and let B be an additive box of T' containing a residue of $\mathbb{B}$ in T'. Suppose that B does not contain any special cut link and that B is different from a residue of $\mathbb{B}$ in T'.*
There exists a unique auxiliary door n of B s.t. the following condition is satisfied:

($\star$) for every residue $\mathbb{B}'$ of $\mathbb{B}$ contained in B, there exists a lucid path Φ' of T' with starting link the front door of $\mathbb{B}'$ and terminal link n.

proof: Let $p_{\mathbb{B}'}$ be the (additive) depth in $\gamma_i(B)$ ($i \in \{1, 2\}$) of the residue $\mathbb{B}'$ of $\mathbb{B}$ in T'. Let $p = max\{p_{\mathbb{B}'} : \mathbb{B}'$ residue of $\mathbb{B}$ in T' contained in $B\}$. We proceed by induction on p.
If $p = 0$, then let $\mathbb{B}_1$ be a residue of $\mathbb{B}$ contained in B. $\mathbb{B}_1$ has (additive) depth zero in (say) $\gamma_1(B)$. By lemma 4.4 the conclusion of the front door of $\mathbb{B}_1$ is the premise of a coad link n of B. And still by the same lemma, there exists a lucid path Φ_2 of T', having as starting link the front door of a residue $\mathbb{B}_2$ of $\mathbb{B}$ in T' which is contained in $\gamma_2(B)$, and as terminal link n. Because $p = 0$, the conclusion of the front door of $\mathbb{B}_2$ is actually the premise of n. The separation property implies that there is no residue of $\mathbb{B}$ in T' contained in B and different from $\mathbb{B}_1$ and $\mathbb{B}_2$.
If $p > 0$, there exists a residue $\mathbb{B}_1$ of $\mathbb{B}$ in T' with additive depth greater than zero in (say) $\gamma_1(B)$. Let B_1 be the biggest additive box of $\gamma_1(B)$ containing $\mathbb{B}_1$. $p > 0$ implies that B_1 is different from $\mathbb{B}_1$. The separation property implies that all the residues of $\mathbb{B}$ in T' contained $\gamma_1(B)$ are contained in B_1. By induction hypothesis there exists a unique auxiliary door n_1 of B_1 satifying condition ($\star$). By lemma 4.4 and because there is no special cut in B, the conclusion of n_1 is a premise of a coad link n of B. Still by lemma 4.4 there exists a lucid path Φ_2 having as starting link the front door of a residue $\mathbb{B}_2$ of $\mathbb{B}$ in T' which is contained in $\gamma_2(B)$, and as terminal link n. If $\mathbb{B}_2$ has (additive) depth zero in $\gamma_2(B)$, then the conclusion of its front door is a premise of n and (by the separation property) $\mathbb{B}_2$ is the unique residue of $\mathbb{B}$ in T' which is contained in $\gamma_2(B)$: in this case we are done. If $\mathbb{B}_2$ has (additive) depth greater than zero in $\gamma_2(B)$, then we call B_2 the additive box of $\gamma_2(B)$ with the same properties as B_1 in $\gamma_1(B)$. Let n_2 be the coad link of B_2 satisfying condition ($\star$) whose conclusion is a premise of a coad link n' of B. Lemma 4.4 implies that $n = n'$. $\square$

6.5. Definition. Let $T \overset{\sigma}{\twoheadrightarrow} T'$, and let B be an additive box of T' containing a residue of $\mathbb{B}$ in T', and s.t. B does not contain any S-link. We define $C_2(B) = B[\gamma_2(\mathbb{B}')/\mathbb{B}']$, where $\mathbb{B}'$ is any residue of $\mathbb{B}$ in T' which is contained in B. Like in the case of C_1 (definition 5.1), for every weakening link n of $C_2(B)$, we define $j_{C_2(B)}(n)$ as the set of links of $j_B(n)$ which are links of $C_2(B)$.
In particular, for every residue $\mathbb{B}'$ of $\mathbb{B}$ in T', $C_2(\mathbb{B}') = \gamma_2(\mathbb{B}')$.

6.6. Remark. Let $T \overset{\sigma}{\twoheadrightarrow} T'$.

(i) The separation property and lemma 4.4 imply that no special cut can be contained in a residue of $\mathbb{B}$ in T'. Lemma 3.18 allows then to claim that there is no S-link in any residue of $\mathbb{B}$ in T' (we use this property in the previous definition).
(ii) Let B be an additive box of T' containing a residue of $\mathbb{B}$ in T', and s.t. B does not contain any S-link. $C_2(B)$ is obviously a proof-net when B is itself a residue of $\mathbb{B}$ in T'. Otherwise, this is still the case because (by remark 6.1) B does not contain any special cut link, so that we can apply lemma 6.4 (and the separation property): we substitute the coad link n with premises and conclusion labeled by $D\&C$, by a coad link with premises and conclusion labeled by C. (Of course, like for C_1, we should mention the fact that $(C_2(B), j_{C_2(B)})$ is a proof-net because (B, j_B) is a proof-net).
The conclusions of $C_2(B)$ are then exactly the conclusions of B, except the edge labeled by $D\&C$ which is replaced by an edge labeled by C.

6.7 Second projection

Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let's consider for every proof-net appearing in the reduction sequence σ, its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net. Suppose that σ does not contain any e.r.s. reducing a cut inside a main subproof-net.
The present subsection is entirely devoted to define $C_2^{\mathbb{B}'}(T')$, the second projection of T', *with respect to the residue* $\mathbb{B}'$.
The work previously done (and specially sections 3 and 4) is far enough to give a reasonnably rigourous definition. However, to understand it in all details, requires from the reader a certain amount of patience.

We proceed by induction on σ. The reader is invited to notice that we will actually define $C_2^{\mathbb{B}'}(R')$ for every subproof-net R' of T' containing $\mathbb{B}'$.
If $T' = T$, then we define $C_2^{\mathbb{B}'}(T') = C_2(T)$. Otherwise we have to distinguish among different cases.
Case 1: If every additive box of T' containing $\mathbb{B}'$ does not contain any S-link, let B be the biggest among these boxes. From remark 6.1, we know that B does not contain any special cut link of T'. From lemma 4.4, we deduce that there exists a special cut link c' with additive depth zero in T'. But then, from lemma 2.4 c' is the unique special cut link of T'. From proposition 3.9, we know that c' is the unique QS-link of T'. For all these reasons, and because of lemma 6.4, T' is the proof-net of figure 9.1 (in case $B \neq \mathbb{B}'$), where q (resp. s) is a subgraph of T' containing only Q-links (resp.

S-links) that are not S-links (resp. Q-links). In case $B = \mathbb{B}'$ just replace the box B by the box $\mathbb{B}'$, and the coad link with conclusion $D\&C$ by a with link with the same conclusion. Lemma 3.15 guarantees that every jump of a weakening link of q is a Q-link, and belongs then to q or to B. The graph R obtained from T' by erasing s and c' is then a proof-net, and we can define $C_2^{\mathbb{B}'}(T') = R[C_2(B)/B]$. $C_2^{\mathbb{B}'}(T')$ is a proof-net whose conclusions are the Q-edges conclusions of T' and an edge labeled by C.

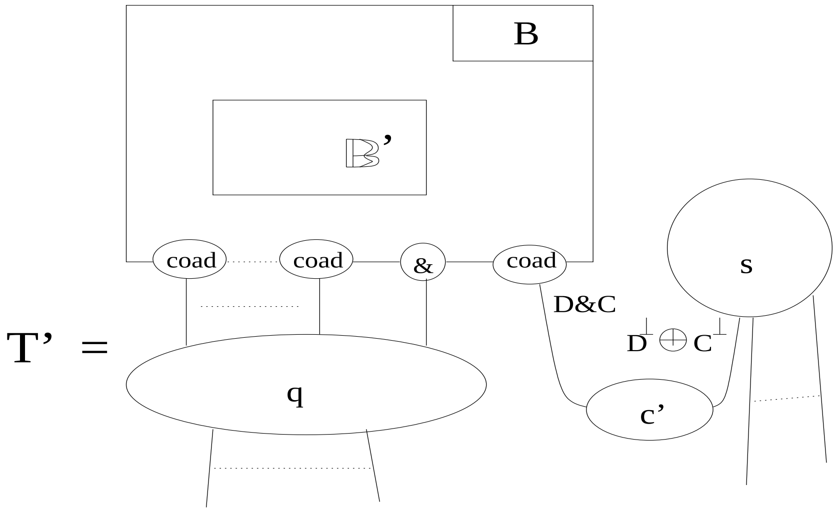

Figure 9.1

Case 2: If there exists an additive box of T' containing $\mathbb{B}'$ and some S-links, then let B be one of these boxes. Let $i \in \{1,2\}$ be s.t. $\mathbb{B}'$ is contained in $\gamma_i(B)$ and let $j \in \{1,2\}$ s.t. $i \neq j$. In the sequel of the definition, the natural

numbers i and j will always be s.t. $\mathbb{B}'$ is (resp. is not) a box of $\gamma_i(B)$ (resp. $\gamma_j(B)$). By lemma 3.18 and lemma 3.13, we know that there is a special cut in each of the two components of B. This also means (by lemma 4.4) that there is a residue of $\mathbb{B}$ in $\gamma_j(B)$.
We define $C_2^{\mathbb{B}'}(B)$ and $C_2^{\mathbb{B}'}(\gamma_i(B))$ by induction on (σ and on) the maximal (additive) depth d of the special cuts in $\gamma_i(B)$, and we show that the following condition holds:

($\bullet$) the conclusions of $C_2^{\mathbb{B}'}(\gamma_i(B))$ (resp. of $C_2^{\mathbb{B}'}(B)$) are the conclusions of $\gamma_i(B)$ (resp. of B) that are Q-edges and an edge labeled by C.

The reader should notice that we did not mention $C_2^{\mathbb{B}'}(\gamma_j(B))$: there is only one case (case 2.3.2 below) in which this definition is needed in order to define $C_2^{\mathbb{B}'}(B)$. In that case we will see that $C_2^{\mathbb{B}'}(\gamma_j(B))$ can be defined, either by means of $C_2(B_j)$ (of definition 6.5) for some additive box B_j (this is case 2.3.2.1), or (using lemma 6.3) by means of $C_2^{\mathbb{B}_R}(B_R)$ where B_R is an additive box containing the lift $\mathbb{B}_R$ of $\mathbb{B}'$ in R, and $R \neq T'$ appears in the reduction sequence σ (so that we can apply the induction hypothesis on σ: this is case 2.3.2.2).

We first define $C_2^{\mathbb{B}'}(\gamma_i(B))$ in case $d = 0$ (**step 2.1**). Then we define $C_2^{\mathbb{B}'}(\gamma_i(B))$ in case $d > 0$ assuming that we have defined $C_2^{\mathbb{B}'}(\gamma_i(B'))$ and $C_2^{\mathbb{B}'}(B')$ when the maximal additive depth d' of the special cuts contained in $\gamma_i(B')$ satisfies $d' < d$ (**step 2.2**). Finally, we define $C_2^{\mathbb{B}'}(B)$ in case $d = 0$ and in case $d > 0$ (the definition is the same in both cases), assuming that we have defined $C_2^{\mathbb{B}'}(\gamma_i(B))$ (**step 2.3**).

step 2.1: If $d = 0$, then by lemma 2.4 there is a unique special cut c_i in $\gamma_i(B)$ and we can argue for $\gamma_i(B)$ exactly like we did before (in case 1) for T'. $\gamma_i(B)$ is the proof-net of figure 9.2, where B_i is the biggest box of T' which does not contain any S-link and containing $\mathbb{B}'$, and q_i (resp. s_i) is a subgraph of T' containing only Q-links (resp. S-links) that are not S-links (resp. Q-links). Let's call R_i the subproof-net of T' obtained by erasing s_i and c_i. We define $C_2^{\mathbb{B}'}(\gamma_i(B)) = R_i[C_2(B_i)/B_i]$. $C_2^{\mathbb{B}'}(\gamma_i(B))$ satisfies ($\bullet$).

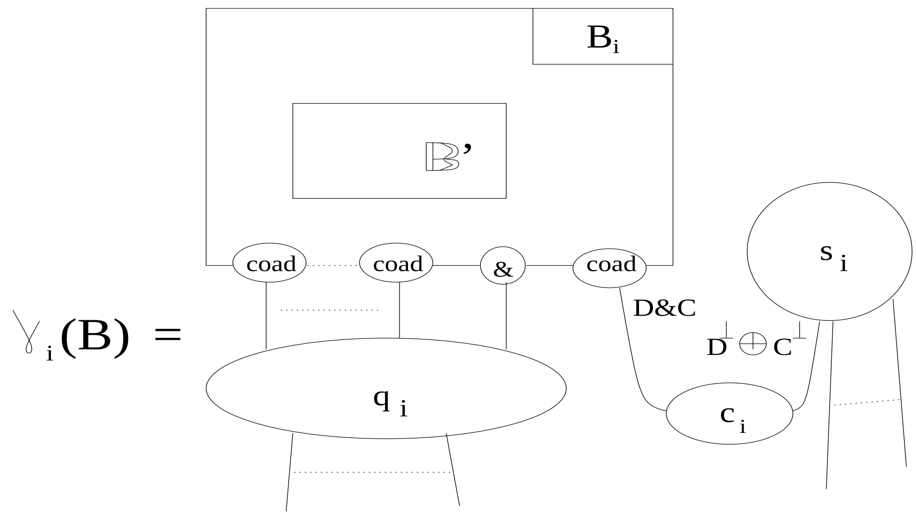

Figure 9.2

step 2.2: If $d > 0$, then let B_i be the biggest box of $\gamma_i(B)$ containing $\mathbb{B}'$. Let c be the special cut link crossed by the maximal lucid path starting from the front door of $\mathbb{B}'$ (the existence of c is given by lemma 4.4). By lemma 2.4, if c has additive depth 0 in $\gamma_i(B)$, then c is the unique special cut link of $\gamma_i(B)$, thus contradicting $d > 0$. This means that c is contained in B_i, and (again by lemma 2.4) B_i contains all the special cut links of $\gamma_i(B)$. By proposition 3.9, $\gamma_i(B)$ is the proof-net of figure 10.1, where the conclusion of the front door of B_i is an edge of q_i (resp. of s_i) when B_i is a Q-box (resp. an S-box), and q_i

(resp. s_i) is a subgraph of T' containing only Q-links (resp. S-links) that are not S-links (resp. Q-links). Let α be the component of B_i containing $\mathbb{B}'$. B_i is an additive box of T' containing $\mathbb{B}'$ and some S-links (because B_i contains some special cuts) and s.t. the maximal depth d' of the special cuts in α satisfies $d' < d$. We can then apply the induction hypothesis (on the depth) to B_i: $C_2^{\mathbb{B}'}(B_i)$ is a proof-net satisfying ($\bullet$). Lemma 3.15 guarantees that every jump of a weakening link of q_i is a Q-link, and belongs then to q_i or to B_i. The graph R_i obtained from $\gamma_i(B)$ by erasing s_i is then a proof-net, and we can define $C_2^{\mathbb{B}'}(\gamma_i(B)) = R_i[C_2^{\mathbb{B}'}(B_i)/B_i]$, which satisfies ($\bullet$).

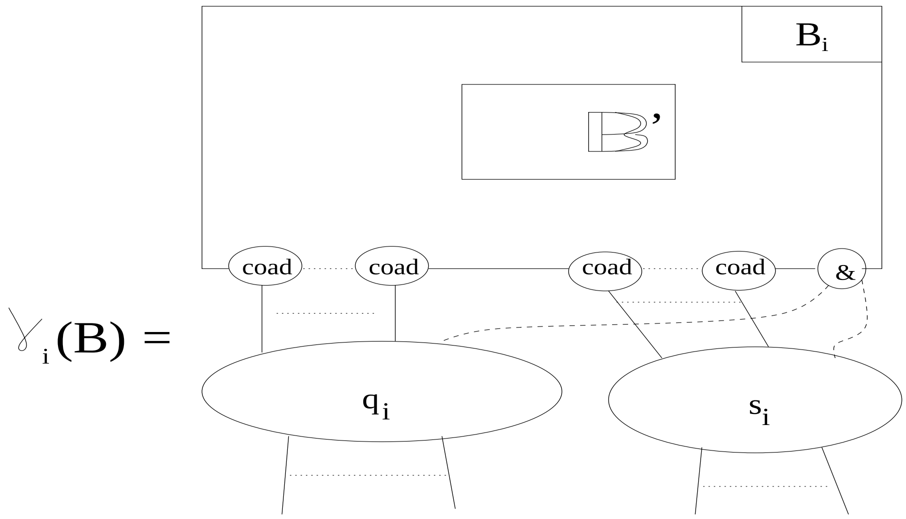

Figure 10.1

step 2.3: We now define $C_2^{\mathbb{B}'}(B)$.

case 2.3.1: If B is an S-box, we define $C_2^{\mathbb{B}'}(B) = C_2^{\mathbb{B}'}(\gamma_i(B))$, which satisfies ($\bullet$).

case 2.3.2: If B is a Q-box, the component $\gamma_j(B)$ of B (which by definition does not contain $\mathbb{B}'$) is a main subproof-net of T'. This is the case in which we have to define also $C_2^{\mathbb{B}'}(\gamma_j(B))$. We already noticed (at the beginning of case 2), that there exists a special cut c_j in $\gamma_j(B)$. By lemma 2.4, either c_j has additive depth 0 in $\gamma_j(B)$ and it is the unique special cut of $\gamma_j(B)$ (**case 2.3.2.1**), or there exists an additive box B_j of $\gamma_j(B)$ containing all the special cuts of T' that are links of $\gamma_j(B)$ (**case 2.3.2.2**).

case 2.3.2.1: In this case, $\gamma_j(B)$ is the proof-net in figure 9.3, where B_j does not contain any S-link (by lemma 3.18), and q_j (resp. s_j) is a subgraph of T' containing only Q-links (resp. S-links) that are not S-links (resp. Q-links). We can apply definition 6.7 to the box B_j. For the usual reasons, the graph R_j obtained from $\gamma_j(B)$ by erasing s_j is a proof-net, and we can define $C_2^{\mathbb{B}'}(\gamma_j(B)) = R_j[C_2(B_j)/B_j]$. The conclusions of $C_2^{\mathbb{B}'}(\gamma_j(B))$ thus defined, are the conclusions of $\gamma_j(B)$ that are Q-edges and an edge labeled by C.

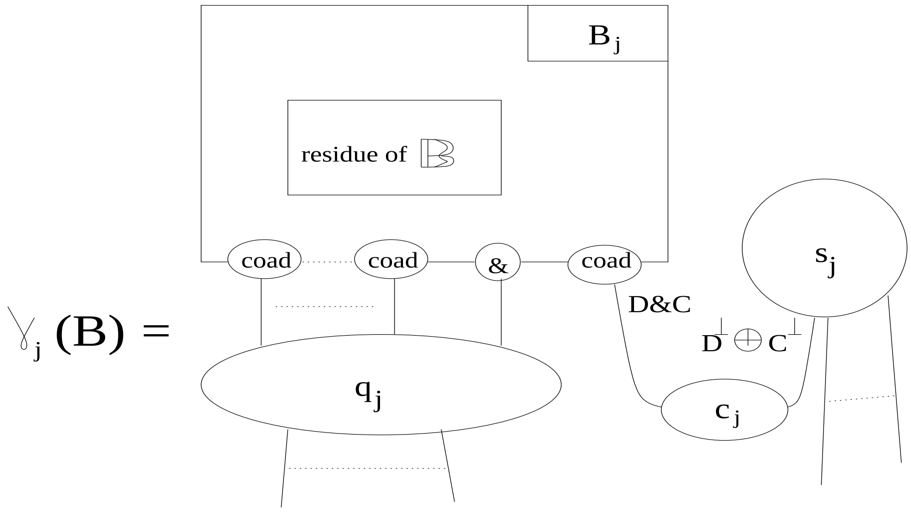

Figure 9.3

case 2.3.2.2: In this case, $\gamma_j(B)$ is the proof-net in figure 10.2, where B_j is the biggest additive box of $\gamma_j(B)$ containing all the special cuts of T' that are links of $\gamma_j(B)$. The conclusion of the front door of B_j is an edge of q_j (resp. of s_j) when B_j is a Q-box (resp. an S-box), and q_j (resp. s_j) is a subgraph of T' containing only Q-links (resp. S-links) that are not S-links (resp. Q-links). By lemma 6.3 applied to the subproof-net $\alpha_j = B_j$ of T', there exists a proof-net R and a decomposition of σ s.t. $T \overset{\sigma'}{\twoheadrightarrow} R \overset{\sigma''}{\twoheadrightarrow} T'$, where σ'' is not empty, and the lift $\alpha_R = B_R$ in R of the subproof-net $\alpha_j = B_j$ of T' is an additive box containing the lift $\mathbb{B}_R$ of $\mathbb{B}'$ in R. By induction hypothesis on σ (this is the only place where we use this hypothesis, but it is crucial), $C_2^{\mathbb{B}_R}(B_R)$ is a proof-net satisfying ($\bullet$). Following definition 3.19 of residue of a subproof-net given in [Tortora 00a], B_j and B_R have the same conclusions (up to substitutions), and the Q-edges conclusions of B_R are "the same" as the Q-edges conclusions of B_j (remember that the residue of a subproof-net is untouched by normalization). As usual, let R_j be the subproof-net of T' obtained from $\gamma_j(B)$ by erasing s_j. We define $C_2^{\mathbb{B}'}(\gamma_j(B)) = R_j[C_2^{\mathbb{B}_R}(B_R)'/B_j]$, where the

" ' " indicates that we have performed in $C_2^{\mathbb{B}_R}(B_R)$ the substitutions required by σ'' (due to the possible [$\forall/\exists$] e.r.s. of σ''). The conclusions of $C_2^{\mathbb{B}'}(\gamma_j(B))$ thus defined, are the conclusions of $\gamma_j(B)$ that are Q-edges and an edge labeled by C.

In both cases (2.3.2.1 and 2.3.2.2), let n be the front door of B and suppose that the premise a_i (resp. a_j) of n which is conclusion of $\gamma_i(B)$ (resp. of $\gamma_j(B)$) is labeled by A_i (resp. A_j) and that the other conclusions of B that are Q-edges are labeled by $E_1, \ldots, E_k$. $C_2^{\mathbb{B}'}(\gamma_i(B))$ (resp. $C_2^{\mathbb{B}'}(\gamma_j(B))$) is a proof-net whose conclusions are labeled by $A_i, E_1, \ldots, E_k, C$ (resp. $A_j, E_1, \ldots, E_k, C$). We can then define $C_2^{\mathbb{B}'}(B)$ as the additive box having as components $C_2^{\mathbb{B}'}(\gamma_i(B))$ and $C_2^{\mathbb{B}'}(\gamma_j(B))$: this box is a proof-net satisfying ($\bullet$).

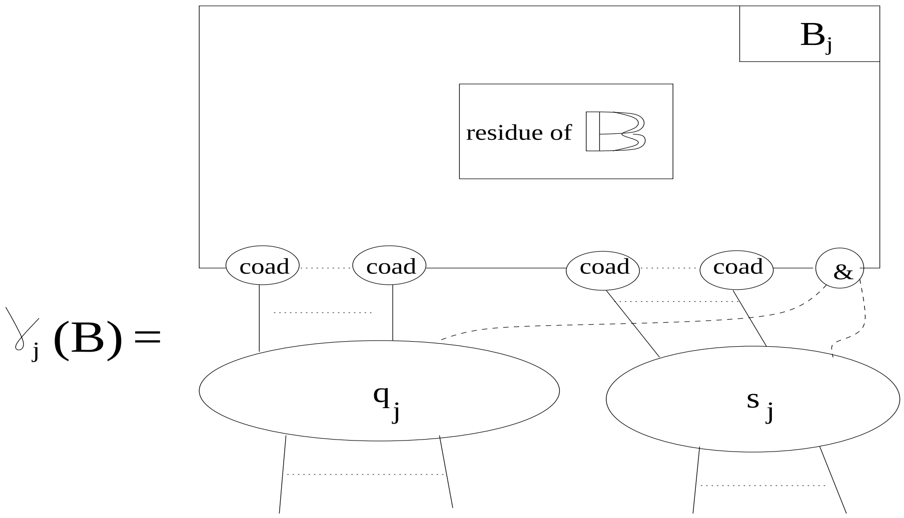

Figure 10.2

Finally, let B be the biggest additive box of T' containing $\mathbb{B}'$. We are supposing that B contains some S-links (remember that we are in case 2). Then T' is the proof-net in figure 10.3, where q (resp. s) is a subgraph of T' containing only Q-links (resp. S-links) that are not S-links (resp. Q-links). As usual, let R be the subproof-net of T' obtained from T' by erasing s. We define $C_2^{\mathbb{B}'}(T') = R[C_2^{\mathbb{B}'}(B)/B]$. The conclusions of $C_2^{\mathbb{B}'}(T')$ are the Q-edges conclusions of T' and an edge labeled by C.

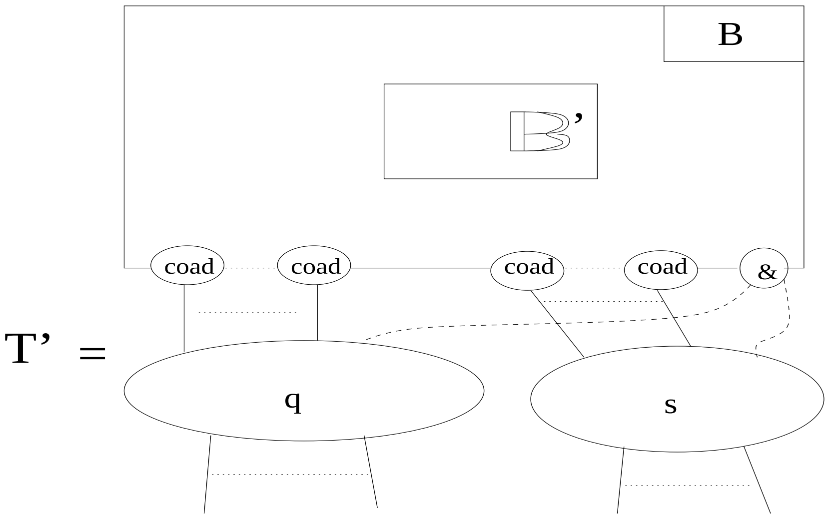

Figure 10.3

6.8. REMARK. Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let's consider for every proof-net appearing in the reduction sequence σ, its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net. Suppose that σ does not contain any e.r.s. reducing a cut inside a main subproof-net.
(i) Let $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$ be a decomposition of σ, and let $\mathbb{B}_1$ be the lift of of $\mathbb{B}'$ in T_1. If x is a Q-link which is not a link of $\gamma_1(\mathbb{B}_1)$, and if x is not a special cut, then we have "the same" cut link x in $C_2^{\mathbb{B}_1}(T_1)$.
(ii) None of the S-links of T_1 is a link of $C_2^{\mathbb{B}_1}(T_1)$.

(iii) There may be some Q-links of T_1 which *are not* links of $C_2^{\mathbb{B}_1}(T_1)$.

proof: (i) If x is a link of $\mathbb{B}_1$, then the result is obvious.
Otherwise let B_1 be the biggest additive box of T_1 containing $\mathbb{B}_1$. If x is not a link of B_1, the result is a consequence of definition 6.7. If x is a link of B_1, and if B_1' is the smallest additive box containing x and $\mathbb{B}_1$, then x is contained in the component of B_1' which contains $\mathbb{B}_1$ (otherwise x would be a cut link contained in a main subproof-net of T_1) and is then (still by definition 6.7) a link of $C_2^{\mathbb{B}_1}(B_1')$ and then a link of $C_2^{\mathbb{B}_1}(T_1)$.
(ii) is an immediate consequence of the definition of $C_2^{\mathbb{B}_1}(T_1)$.
(iii) is also a consequence of the definition of $C_2^{\mathbb{B}_1}(T_1)$: in case 2.3.1 of 6.7 we erase an entire component of an S-box, which might contain some Q-links.
□

The following proposition is the analogue of proposition 5.3: $C_2^{\mathbb{B}'}$ "projects" a sequence of reductions onto a sequence of reductions/equalities.

6.9. Proposition. Let $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T', and let's consider for every proof-net appearing in the reduction sequence σ, its canonical representation with respect to the lift of $\mathbb{B}'$ in this proof-net. Suppose that σ does not contain any e.r.s. reducing a cut inside a main subproof-net.
Let $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$ be a decomposition of σ and let $\mathbb{B}_1$ (resp. $\mathbb{B}_2$) be the lift of $\mathbb{B}'$ in T_1 (resp. T_2).
One has $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$ when:

- $[x]$ is a *[ccad]* e.r.s. $\mathbb{B}_1$-attractive
- $[x]$ is a *[ccad]* e.r.s. B-attractive, B is a Q-box of T_1, and the subproof-net of T_1 duplicated by $[x]$ contains only S-links
- $[x]$ is a *[ccad]* e.r.s. B-attractive and B is an S-box of T_1
- x is an S-link
- x is a cut link of $\gamma_1(\mathbb{B}_1)$.

Otherwise, one has $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\rightarrow} C_2^{\mathbb{B}_2}(T_2)$.
In particular, if $C_2(T)$ is strongly normalizable, then so is $C_2^{\mathbb{B}'}(T')$.

proof: If $[x]$ is a *[ccad]* e.r.s. $\mathbb{B}_1$-attractive, or if $[x]$ is a *[ccad]* e.r.s. B-attractive (where B is a Q-box of T_1, and the subproof-net of T_1 duplicated by $[x]$ contains only S-links), or if $[x]$ is a $[ccad]$ e.r.s. B-attractive and B is an S-box, or if x is an S-link, or if x is a cut link of $\gamma_1(\mathbb{B}_1)$, then we say that we are in situation (a). Otherwise we say that we are in situation (b).

If x is a cut link of $\gamma_1(\mathbb{B}_1)$ (resp. of $\gamma_2(\mathbb{B}_1)$), then we are in situation (a) (resp. in situation (b)) and clearly $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$ (resp. $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(T_2)$). Otherwise, there exists an additive box of T_1 containing $\mathbb{B}_1$ which does not contain x (this box might be $\mathbb{B}_1$ itself). Let B_x be the biggest among these boxes of T_1. If B_x has additive depth greater than zero in T_1, let's call B_x^1 the smallest additive box of T_1 containing B_x and different from B_x. Let $R_1 = T_1$ if B_x has additive depth zero, and let R_1 be the component of B_x^1 which contains B_x otherwise. x is a cut link of R_1 with additive depth zero in R_1. For the reasons evoked several times in 6.7 (and mainly thanks to proposition 3.9), R_1 is the proof-net of figure 9.4 or 10.4, where q (resp. s) contains only Q-links (resp. S-links) that are not S-links (resp. Q-links).
Observe that R_1 might contain only Q-links that are not S-links: in this case R_1 is the proof-net of figure 10.4, where s is empty (and B_x might be the box $\mathbb{B}_1$). Observe also that when $B_x = \mathbb{B}_1$ and R_1 is the proof-net in figure 9.4, then we have to replace the box B_x by the box $\mathbb{B}_1$, and the coad link with conclusion $D\&C$ by a with link with the same conclusion.
Because x is a cut link with additive depth 0 in R_1, it is a link of q or a link of s, or (only in case of figure 9.4) the special cut c_1. We have that $R_1 \overset{[x]}{\to} R_2$, $T_2 = T_1[R_2/R_1]$, and $C_2^{\mathbb{B}_2}(T_2) = C_2^{\mathbb{B}_1}(T_1)[C_2^{\mathbb{B}_2}(R_2)/C_2^{\mathbb{B}_1}(R_1)]$.
Part I: Suppose first that $[x]$ is not a $[ccad]$ e.r.s.. In this case, by lemma 6.2, x cannot be (in case of figure 9.4) the special cut c_1 of T_1.
If x is a link of s, then by definition 6.7 $C_2^{\mathbb{B}_1}(R_1) = C_2^{\mathbb{B}_2}(R_2)$: we are in situation (a) and $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$. Observe that in this case, if $[x]$ opens the box B_x (we are necessarily in case of figure 10.4), then B_x is an S-box of T_1; and the previous equalities hold because we know that the component of B_x which is erased by $[x]$ is the one which does not contain $\mathbb{B}_1$ (remember that $\mathbb{B}_1$ has a residue $\mathbb{B}_2$ in T_2): this is precisely the component that we erase when we define $C_2^{\mathbb{B}_1}(B_x)$. To be more precise, suppose that $\mathbb{B}_1$ is contained in the first component of B_x and that α is the residue in T_2 of $\gamma_1(B_x)$. Then we have $C_2^{\mathbb{B}_1}(B_x) = C_2^{\mathbb{B}_1}(\gamma_1(B_x)) = C_2^{\mathbb{B}_2}(\alpha)$.
If x is a link of q, then we are in situation (b). By remark 6.8 x is also a link of $C_2^{\mathbb{B}_1}(R_1)$. If $[x]$ does not open B_x, then (as a subprof-net) B_x has a unique residue B_x^2 in T_2 and clearly $C_2^{\mathbb{B}_1}(B_x) = C_2^{\mathbb{B}_2}(B_x^2)$. Then $C_2^{\mathbb{B}_1}(R_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(R_2)$, and $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(T_2)$. If $[x]$ opens the box B_x, then B_x is

a Q-box and $[x]$ opens also the box $C_2^{\mathbb{B}_1}(B_x)$. Definition 6.7 implies then $C_2^{\mathbb{B}_1}(R_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(R_2)$, and $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(T_2)$.

Part II: Suppose now that $[x]$ is a $[ccad]$ e.r.s..

If $[x]$ is B_x-attractive, let's call B_x^2 the unique residue of B_x in T_2. The subproof-net of T_1 duplicated by $[x]$ contains only either Q-links that are not S-links, or S-links that are not Q-links (except the special cut c_1 when R_1 is the proof-net in figure 9.4). We then have exactly 3 possibilities:

(1) B_x is an S-box or $B_x = \mathbb{B}_1$: then the only effect of $[x]$ is to increase the number of links of B_x, and because B_x does not exist in $C_2^{\mathbb{B}_1}(T_1)$ this is "invisible": we are in situation (a) and $C_2^{\mathbb{B}_1}(R_1) = C_2^{\mathbb{B}_2}(R_2)$, so that $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$.

(2) B_x is a Q-box and the subproof-net of T_1 duplicated by $[x]$ contains only S-links: then the only effect of $[x]$ is to increase the number of S-links of B_x, and all these links do not exist in $C_2^{\mathbb{B}_1}(T_1)$: we are in situation (a) and $C_2^{\mathbb{B}_1}(B_x) = C_2^{\mathbb{B}_2}(B_x^2)$, $C_2^{\mathbb{B}_1}(R_1) = C_2^{\mathbb{B}_2}(R_2)$, and $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$.

(3) B_x is a Q-box and the subproof-net of T_1 duplicated by $[x]$ contains only Q-links that are not S-links: then the only effect of $[x]$ is to increase the number of links of B_x that are Q-links and not S-links, and all these links do exist in $C_2^{\mathbb{B}_1}(T_1)$: we are in situation (b), $C_2^{\mathbb{B}_1}(R_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(R_2)$, and $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(T_2)$.

The last case to analyze is the one in which $[x]$ is not B_x-attractive.

If $[x]$ does not duplicate B_x, then x is not a special cut link. In this case, it is immediate from definition 6.7 that when x is a link of q (resp. of s), then we are in situation (b) (resp. in situation (a)) and $C_2^{\mathbb{B}_1}(R_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(R_2)$ (resp. $C_2^{\mathbb{B}_1}(R_1) = C_2^{\mathbb{B}_2}(R_2)$), which implies that $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(T_2)$ (resp. $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$).

If $[x]$ is B_1-attractive (for some additive box B_1 of R_1) and $[x]$ duplicates the box B_x, let B_2 be the unique residue of B_1 in T_2. Let α_1 be the subproof-net (containing B_x) of T_1 duplicated by $[x]$, and let β_1 be the subproof-net of T_1 consisting in the box B_1, the cut link x and α_1. α_1 has two residues α_2^1 and α_2^2 in T_2 which are subproof-nets of, respectively, the first and the second component of B_2. More precisely, for $i \in \{1, 2\}$, $\gamma_i(B_2)$ is obtained by cuting α_2^i and the residue in T_2 of $\gamma_i(B_1)$, by means of the cut link x_2^i (itself a residue of x in T_2). We have to distinguish the case in which B_1 is an S-box (case 1) from the one in which B_1 is a Q-box (case 2).

In case 1 (in this case x might be the special cut c_1 of figure 9.4), B_1 is an S-box contained in s which contains no Q-link and x is an S-link. All the Q-links of $\gamma_i(B_2)$ that are not S-links, are then links of α_2^i. If $\mathbb{B}_2$ is a box

of (for example) $\gamma_1(B_2)$, then we have $C_2^{\mathbb{B}_2}(B_2) = C_2^{\mathbb{B}_2}(\gamma_1(B_2)) = C_2^{\mathbb{B}_2}(\alpha_2^1) = C_2^{\mathbb{B}_1}(\alpha_1) = C_2^{\mathbb{B}_1}(\beta_1)$, from which one deduces $C_2^{\mathbb{B}_1}(R_1) = C_2^{\mathbb{B}_2}(R_2)$: we are in situation (a) and $C_2^{\mathbb{B}_1}(T_1) = C_2^{\mathbb{B}_2}(T_2)$.
In case 2, x is a link of q and B_1 is a Q-box contained in q: $C_2^{\mathbb{B}_1}(\gamma_i(B_1)) = \gamma_i(B_1)$. Then $C_2^{\mathbb{B}_2}(\gamma_i(B_2))$ is obtained by cuting $C_2^{\mathbb{B}_1}(\gamma_i(B_1)) = \gamma_i(B_1)$ and $C_2^{\mathbb{B}_1}(\alpha_1)$. This means that $C_2^{\mathbb{B}_2}(B_2)$ can be obtained by applying the "same" e.r.s. $[x]$ to $C_2^{\mathbb{B}_1}(\beta_1)$: $C_2^{\mathbb{B}_1}(\beta_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(B_2)$. Then $C_2^{\mathbb{B}_1}(R_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(R_2)$: we are in situation (b) and $C_2^{\mathbb{B}_1}(T_1) \overset{[x]}{\to} C_2^{\mathbb{B}_2}(T_2)$. □

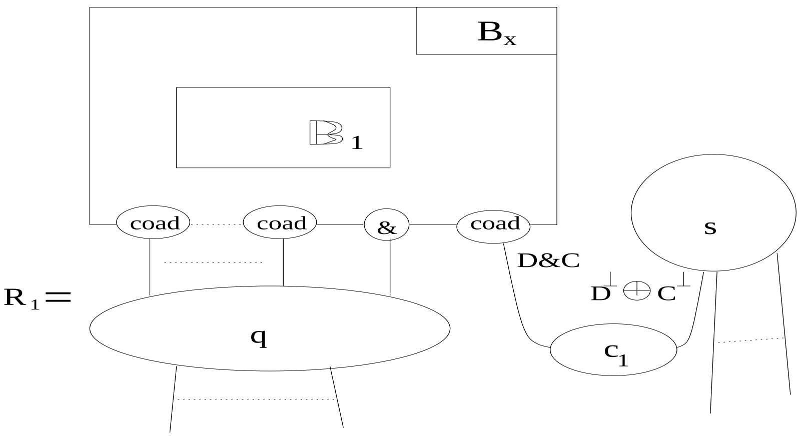

Figure 9.4

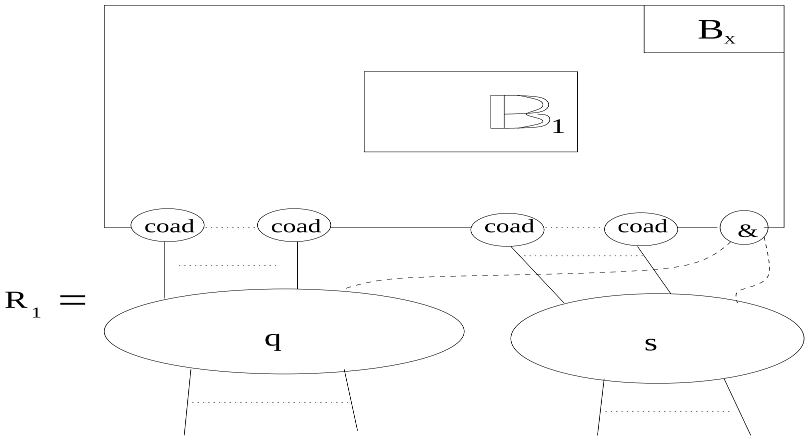

Figure 10.4

7 Proof of the standardization theorem

In section 4, we wrote that to prove the additive standardization theorem (theorem 4.2), we intended to define a size for every reduct of T, and then show that this size shrinks when performing every e.r.s.. This is precisely what we are going to do in this final section.

The last ingredient for our definition of size is the maximal length of the reduction sequences starting from any of the second component of a residue of $\mathbb{B}$. The following proposition states that this length is always finite.

7.1. PROPOSITION. *Let T' be such that $T \overset{\sigma}{\twoheadrightarrow} T'$.*
If $C_2(T)$ is strongly normalizable, then for every residue $\mathbb{B}'$ of $\mathbb{B}$ in T', $\gamma_2(\mathbb{B}')$ is strongly normalizable.

proof: By induction on σ.
If $T' = T$, then $\gamma_2(\mathbb{B})$ is a subproof-net of $C_2(T)$ which is strongly normalizable by hypothesis.
Otherwise, let $\mathbb{B}'$ be a residue of $\mathbb{B}$ in T'; we are going to prove that $\gamma_2(\mathbb{B}')$ is strongly normalizable. In the sequel of the proof, all the proof-nets will be considered in their canonical form with respect to the lift of $\mathbb{B}'$ they contain.
If σ doesn't contain any e.r.s. reducing a cut inside a main subproof-net, then from proposition 6.9, we deduce that $\gamma_2(\mathbb{B}')$ (as a subproof-net of $C_2^{\mathbb{B}'}(T')$) is strongly normalizable.
Otherwise let $[x]$ be the last e.r.s. of σ reducing a cut inside a main subproof-net, more precisely let's consider the following decomposition of σ: $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$ where σ_2 doesn't contain any e.r.s. reducing a cut inside a main subproof-net, and suppose that $\beta \overset{[x]}{\rightarrow} \gamma$ where β (resp. γ) is a main subproof-net of T_1 (resp. of T_2).
From lemma 2.8, any residue of γ in any of the proof-nets appearing in σ_2 (including T') is a subproof-net of a main subproof-net of this proof-net. But $[x]$ is the last e.r.s. of σ reducing a cut inside a main subproof-net, so that if $[ccad] \in \sigma_2$ this e.r.s. cannot duplicate any subproof-net of a residue of γ without duplicating the residue of γ itself. Then the substitution lemma (lemma 2.2) applies: if δ is any proof-net with the same conclusions of γ and if $\gamma'_1, \ldots, \gamma'_n$ are the residues of γ in T', we have that $T_2\left[\delta/\gamma\right] \overset{\sigma_2}{\twoheadrightarrow} T'\left[\delta/\gamma'_1, \ldots, \delta/\gamma'_n\right]$. In particular, if we take $\delta = \beta$, we have that $T_2\left[\beta/\gamma\right] \overset{\sigma_2}{\twoheadrightarrow} T'\left[\beta/\gamma'_1, ..., \beta/\gamma'_n\right]$ i.e. $T_1 \overset{\sigma_2}{\twoheadrightarrow} T'\left[\beta/\gamma'_1, ..., \beta/\gamma'_n\right]$. If we define $\sigma' := \sigma \backslash \{[x]\}$, we have that $T \overset{\sigma'}{\twoheadrightarrow} T'\left[\beta/\gamma'_1, ..., \beta/\gamma'_n\right] \overset{[x]^n}{\twoheadrightarrow} T'$.
Then, there are two possibilities:

1) there is no residue of γ in T':

this means that they have all been erased during the normalization process. I In this case we can eliminate the e.r.s. $[x]$: we have that $T \overset{\sigma'}{\twoheadrightarrow} T'$, and then, by induction hypothesis, $\gamma_2(\mathbb{B}')$ is strongly normalizable.

2) there is a residue of γ in T':

by induction hypothesis, if $\mathbb{B}_{T'[\beta/\gamma'_1,...,\beta/\gamma'_n]}$ is the lift of $\mathbb{B}'$ in $T'\left[\beta/\gamma'_1, ..., \beta/\gamma'_n\right]$ (w.r.t. the reduction sequence $[x]^n$), then $\gamma_2(\mathbb{B}_{T'[\beta/\gamma'_1,...,\beta/\gamma'_n]})$ is strongly normalizable; and because the n e.r.s. $[x]$ reduce cuts inside main subproof-nets, we have that $\mathbb{B}_{T'[\beta/\gamma'_1,...,\beta/\gamma'_n]} = \mathbb{B}'$ and $\gamma_2(\mathbb{B}_{T'[\beta/\gamma'_1,...,\beta/\gamma'_n]}) = \gamma_2(\mathbb{B}')$. $\square$

Let's prove now the (generalized) standardization theorem (theorem 4.2): we have to show that if $C_1(T)$ and $C_2(T)$ are strongly normalizable, then so is T.
Consider the following reduction sequence starting from T:

$$T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'.$$

For every proof-net R appearing in this reduction sequence and for every residue $\mathbb{B}'$ of $\mathbb{B}$ in R with conclusions labeled by $D\&C$ and A_1,...,$A_{n'}$, we define:

-$\Re(R)$:={ $\mathbb{B}_R$: $\mathbb{B}_R$ is a residue of $\mathbb{B}$ in R} and $\|\Re(R)\| := card(\Re(R))$

-$\Lambda_{\mathbb{B}'}(R)$:={ γ_1,...,$\gamma_{n'}$: γ_i is the biggest subproof-net of R connected to $\mathbb{B}'$ through a commutative cut *ccad* having among its premises the conclusion of $\mathbb{B}'$ labeled by A_i}. Observe that for some i γ_i may be the empty proof-net.

-$\|\Lambda_{\mathbb{B}'}(R)\|$:=$n(\gamma_1) + ... + n(\gamma_{n'})$ where $n(\gamma_i)$=number of links of γ_i.

-$\|\Lambda(R)\| := \Sigma_{B\in\Re(R)}\|\Lambda_B(R)\|$

We prove that

$$\begin{gathered}(\|C_1(T_2)\|, \|\Re(T_2)\|, \|\Lambda(T_2)\|, \Sigma_{B\in\Re(T_2)}\|\gamma_2(B)\|)\\ <_{LEX}\\ (\|C_1(T_1)\|, \|\Re(T_1)\|, \|\Lambda(T_1)\|, \Sigma_{B\in\Re(T_1)}\|\gamma_2(B)\|)\end{gathered}$$

where $\|C_1(T_i)\|$ (resp. $\|\gamma_2(B)\|$) is the maximal length of the reduction sequences starting from $C_1(T_i)$ (resp. $\gamma_2(B)$). Notice that $\|\gamma_2(B)\|$ is well definded thanks to proposition 7.1.
From proposition 5.3, if $[x]$ is not a $[ccad]$ e.r.s. which is attractive for a residue of $\mathbb{B}$ in T_1, if x is not a logical special cut and if x is not a cut in the second component of a residue of $\mathbb{B}$ in T_1, then $\|C_1(T_2)\| < \|C_1(T_1)\|$.
If x is a logical special cut, then by proposition 5.3, $\|C_1(T_2)\| = \|C_1(T_1)\|$, and $\|\Re(T_2)\| < \|\Re(T_1)\|$.
If $[x]$ is a $[ccad]$ e.r.s. $\mathbb{B}_1$-attractive for a residue $\mathbb{B}_1$ of $\mathbb{B}$ in T_1, then from proposition 5.3 $\|C_1(T_2)\| = \|C_1(T_1)\|$. Observe that, because of the separation property, the number of residues of $\mathbb{B}$ cannot increase in T_2 with respect to T_1, and we have that $\|\Lambda(T_2)\| < \|\Lambda(T_1)\|$.

If x is a cut in $\gamma_2(\mathbb{B}_1)$ for a residue $\mathbb{B}_1$ of $\mathbb{B}$, then from proposition 5.3 $\|C_1(T_2)\| = \|C_1(T_1)\|$. We also have $\|\Re(T_2)\| = \|\Re(T_1)\|$, while the separation property gives $\|\Lambda(T_2)\| = \|\Lambda(T_1)\|$. If $\mathbb{B}'_1$ is the (unique) residue of $\mathbb{B}_1$ in T_2, then $\|\gamma_2(\mathbb{B}'_1)\| < \|\gamma_2(\mathbb{B}_1)\|$, so that $\Sigma_{B\in\Re(T_2)}\|\gamma_2(B)\| < \Sigma_{B\in\Re(T_1)}\|\gamma_2(B)\|$.
$\square$

7.2. Remark. Observe that to have a complete proof of strong normalization for this revisited PN_2, one still has to prove that the standardization lemma for the multiplicative and exponential fragment proven in [Danos 90] still holds in our framework.

Aknowledgements

I wish to thank Vincent Danos for all the discussions we had on the subject, I am grateful to the anonymous referee for his (very!) careful reading and for his stimulating and penetrating remarks.
This version also beneficiated of the listening and the comments of J.-B. Joinet and H. Schellinx.

References

[Danos 90] Danos V., La logique linéaire appliquée à divers processus de normalisation (principalement du lambda-calcul). Thèse de doctorat, Université de Paris 7, 1990

[DanReg 89] Danos V., Regnier L., The structure of multiplicatives. Archives for Mathematical Logic, 28, pp. 181-203, 1989

[DanReg 95] Danos V., Regnier L., Proof-nets and the Hilbert space. In: J.Y. Girard, Y. Lafont and L. Regnier editors, Advances in Linear Logic, pp. 307-328. Cambridge Univerity Press, 1995. London Mathematical Society Lecture Note serie 222, Proceedings of the 1993 workshop on Linear Logic, Cornell University, Ithaca.

[DJS 97] Danos V., Joinet J.-B., Schellinx H., A new deconstructive logic: linear logic, the Journal of Symbolic Logic 62(3), pp. 755-807, september 1997

[Girard 87] Girard J.Y., Linear logic. Theoretical Computer Science, 50:1-101, 1987.

[Girard 91] Girard J.Y., Quantifiers in linear logic II. In: Corsi, G. and Sambin, G. editors, Nuovi problemi della logica e della filosofia della scienza, Volume II. Proceedings of the conference with the same name, Viareggio, 8-13 gennaio 1990. CLUEB, Bologna (Italy).

[Girard 95a] Girard J.Y., Proof-nets: the parallel syntax for proof-theory, In Ursini and Agliano editors, Logic and Algebra, New-York, 1995. Marcel Dekker.

[Girard 95b] Girard J.Y., Light Linear Logic, Information and Computation, vol. 143, n. 2, p.175-204, 1998

[JST 00] Joinet J.-B., Schellinx H., Tortora de Falco L., SN and CR for free-style LK^{tq}: linear decorations and simulation of normalization, to appear in the Journal of Symbolic Logic, 2000

[Laurent 99] Laurent O., Polarized proof-nets: proof-nets for LC, extended abstract, Proceedings of the conference "Typed Lambda-Calculus and Applications '99", L'Aquila Italy, april 1999, LNCS 1581 pp. 213-227, Springer-Verlag 1999

[LQT 00] Laurent O., Quatrini M., Tortora de Falco L., Polarized and focalized classical and linear proofs, preprint 24, Institut de Mathématiques de Luminy, 2000 and submitted to the Annals of Pure and Applied Logic

[Tortora 00a] Tortora de Falco L., "Additives of Linear Logic and Normalization -Part I: a (restricted) Church-Rosser property, to appear in Theoretical Computer Science, 2000, also available at the address http://www.logique.jussieu.fr/www.tortora/index.html

[Tortora 00b] Tortora de Falco L., "Réseaux, cohérence et expériences obsessionnelles", Thèse de doctorat, University Paris 7, January 2000, available at the address http://www.logique.jussieu.fr/www.tortora/index.html