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Generalized standardization lemma for the additives

(preliminary report)

Lorenzo Tortora de Falco [1]

Équipe de Logique Mathématique
Université Paris VII
Paris, France

E. N. T. C. S.

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Abstract

This is a preliminary report, in which we prove the so-called standardization lemma for the additives, in presence of a generalization (the natural one) of the commutative elementary reduction step. The technical tools developped to achieve this result suggest some remarks on the non confluence of full linear logic.

1 Introduction

All the connectives of linear logic (LL) come from a semantical decomposition of the usual intuitionistic and classical ones. But the additive fragment of LL is much less studied than its multiplicative brother, because of its unsatisfactory syntax (which was only recently improved in the work [Girard 95a]).

However, the additives are the natural candidates to type functions like the predecessor and the "if...then...else..." (see [Girard 95b] and [Girard 87]).

Another peculiarity of the additive connectives is that, while any fragment of LL without additives enjoys confluence property, this is no more true as soon as one adds the (full) additive fragment (even if the semantical soundness still holds: two different reducts of the same proof are semantically equal). Nevertheless, the system PN_2 of second order linear logic is strongly normalizing as J.Y. Girard proved in [Girard 87], making a strong use of the so called "standardization lemma".

[1] tortora@logique.jussieu.fr; supported by a fellowship of the Consiglio Nazionale delle Ricerche (CNR).

A proof of this lemma is sketched in [Girard 87]. A detailed proof for the multiplicatives, the exponentials and the quantifiers can be found in [Danos 90] and relies on the confluence of this fragment.

We propose a generalization of the commutative elementary reduction step (e.r.s.) for the additives, which can easily seen to be semantically sound with respect to usual linear logic coherent semantics and to preserve the proof-nets correcteness criterion: in our commutative step the proof-net swallowed by an additive box is not necessarily the empire of the occurrence of the cut-formula, but *any* subproof-net having this formula among its conclusions. Such a commutative e.r.s. is a very natural generalization of the usual one: the embedding in LL of the system for classical logic LK^{tq} introduced in [DJS 95] requires, to be "homoreductive" (that is to be a morphism for cut-elimination steps), a variant of [Girard 87] 's commutative e.r.s. for the additives which is a particular case of the one considered in this paper.

We give a (completely new) proof of the standardization lemma for the additives in PN_2 equipped with this generalized commutative e.r.s. (by the way, this gives a detailed proof of the standardization lemma for the additives with the commutative e.r.s. of [Girard 87]).

Provided the technical point evoked by remark 3.8 has been carefully checked, we obtain that this "revisited" PN_2 is strongly normalizing. An immediate consequence of this result is that the system $taLL$ of LL with "tamed" additives (used in [DJS 95] and [JST 95] to prove strong normalization for LK^{tq}) enjoys strong normalization (this was proven in [DJS 95] through an embedding of $taLL$ in its non additive subsystem).

2 Preliminaries and the independence lemma

The multiplicatives and the exponentials are treated as in [Danos 90] and the quantifiers as in [Girard 91a], while we treat the additives as in [Girard 87], except of course for the commutative reduction step.

If B is an additive box with front door $E\&F$ in a proof-net R, we will often speak of the first (resp. the second) component of B meaning the biggest subproof-net of B having E (resp. F) among its conclusions.

We take the notation $[ccad]$ instead of $(\& - CC)$ of [Girard 87] for the commutative additive e.r.s.: if $R \overset{[ccad]}{\rightarrow} R'$, we will say that the e.r.s. $[ccad]$ is B-attractive (or attractive for the box B), if B is the additive box of R that swallows (and duplicates) a subproof-net of R.

We shall say that a link, an edge or a subproof-net of a given proof-net R has exponential (resp. additive) depth n in R, if it is contained in n exponential (resp. additive) boxes of R.

2.1. Definition. Let R' be a one step reduct of the proof-net R.
Any logical or axiom link n of R' comes from a unique occurrence of (the same) logical or axiom link $\overleftarrow{n}$ in R (promotion and dereliction are considered as logical links). We call $\overleftarrow{n}$ the **lift of the link** n in R. Conversely, we define

the set of **residues of the** logical (resp. axiom) **link** l in R, as the set of all the occurrences l' of logical (resp. axiom) links in R' such that $\overleftarrow{l'}= l$. In the particular case where the logical link considered in R' is &, this link is actually an additive box B, which comes from *exactly* one occurrence of (the "same") additive box $\overleftarrow{B}$ in R (observe, however, that the two components of the additive box are not necessarily the same in R and in R'). We call $\overleftarrow{B}$ the **lift of the additive box** B in R. Conversely, we define the set of **residues of the additive box** B_1 of R as the set of all the occurrences B_1' of additive boxes in R' such that $\overleftarrow{B_1'}= B_1$.

Similarly, any occurrence of a cut-link c in R' comes from a unique occurrence $\overleftarrow{c}$ of a cut-link with the same active formulas (up to substitutions if the reduced cut is $\forall/\exists$) in R, except for those cut-links which are created by a logical e.r.s. (the e.r.s. dereliction/promotion is considered as a logical one). In this last case, we shall say that the created links have no lift in R and that the logical cut-link reduced has no residues in R'.

A subproof-net α of R', comes from *at most* a unique occurrence of (the same up to substitutions) subproof-net $\overleftarrow{\alpha}$ in R. If such a subproof-net exists in R, we shall say that $\overleftarrow{\alpha}$ is the **lift of the subproof-net** α. Conversely, we define the set of **residues of the subproof-net** β of R, as the set of all the occurrences β' of subproof-nets of R' such that $\overleftarrow{\beta'}= \beta$.

The notions of lift and of residue can be obviously generalized to a reduction sequence.

2.2. Lemma. ***(substitution lemma)*** Let T, T' be two proof-nets, s.t. $T \stackrel{\sigma}{\twoheadrightarrow} T'$. Let α be a subproof-net of T, and β any proof-net with the same conclusions of α.

If, for all $[x] \in \sigma$ s.t. $T \stackrel{\sigma_1}{\twoheadrightarrow} T_1 \stackrel{[x]}{\to} T_2 \stackrel{\sigma_2}{\twoheadrightarrow} T'$, and for all residue γ_1 of α in T_1, the following conditions are satisfied:

i) if $[x] = [ax]$ then x is not contained in γ_1 and the link ax erased by $[x]$ is not contained in γ_1

ii) if $[x] \neq [ax], [ccad]$ then both the formulas (better the occurrences of formula) active in x are not contained in γ_1 (in particular, the conclusions of γ_1 are not active in x)

iii) if $[x] = [ccad]$ then x is not contained in γ_1, $[x]$ is not attractive for a box of γ_1 and if $[x]$ duplicates a subproof-net of γ_1 then $[x]$ duplicates the whole γ_1

then, if $\alpha_1', \dots \alpha_n'$ are the residues of α in T', we have $T[\beta/\alpha] \stackrel{\sigma}{\twoheadrightarrow} T'[\beta/\alpha_1', \dots, \beta/\alpha_n']$ (up to substitutions). If $[\forall/\exists] \in \sigma$, then we have, more precisely, $T[\beta/\alpha] \stackrel{\sigma}{\twoheadrightarrow} T'[\beta_1/\alpha_1', \dots, \beta_n/\alpha_n']$, where $\beta_i = \beta$ in which we performed the substitution(s) required by the e.r.s. $[\forall/\exists]$.

proof: By induction on σ. If σ is empty, then $T = T'$ and the result is obvious.

Otherwise $T \stackrel{\sigma_1}{\twoheadrightarrow} T_1 \stackrel{[x]}{\to} T'$. By induction hypothesis, if $\alpha_1^1 \dots \alpha_1^m$ are the residues of α in T_1, we have $T[\beta/\alpha] \overset{\sigma_1}{\twoheadrightarrow} T_1[\beta/\alpha_1^1,\ldots,\beta/\alpha_1^m]$ (up to substitutions). To achieve the result, it is enough to check that for any residue α_1^i of α in T_1 and for all the residues α' of α_1^i in T', we have that $T_1[\beta/\alpha_1] \overset{[x]}{\rightarrow} T'[\beta/\alpha']$ (up to substitutions if $[x] = [\forall/\exists]$), where $[x]$ is any e.r.s. satisfying the conditions $i)$, $ii)$ and $iii)$. □

The following (obvious) remark is fundamental, and was actually the starting (technical) point of this work.

2.3. Remark. Let R be a proof-net and R' any reduct of R. If l (resp. α, B) is a logical axiom or cut link different from an exponential commutative cut (resp. a subproof-net different from an exponential box, an additive box) with exponential depth zero in R, then any residue of l (resp. α, B) in R' has exponential depth zero in R'.
This is because the only possibility for a link (different from a commutative exponential cut) to enter an exponential box is to be already in another exponential box that is active in an exponential commutative e.r.s..

2.4. Lemma. ***(Independence lemma).*** Let T, T' be two proof-nets, s.t. $T \overset{\sigma}{\twoheadrightarrow} T'$. Let $\mathbb{B}$ be an additive box with exponential depth 0 in T, and $\mathbb{B}'$ a residue of $\mathbb{B}$ in T'. If $\mathbb{B}'$ has additive depth p and $B_1,...,B_p$ are the additive boxes containing $\mathbb{B}'$ in T', then for all residue $\mathbb{B}''$ of $\mathbb{B}$ in T' (different from $\mathbb{B}'$), there exists $i \in \{1,...,p\}$ s.t. $\mathbb{B}''$ is a subproof-net of the component of B_i which does not contain $\mathbb{B}'$.
proof: Induction on σ. If $T' = T$, then the result is obvious.
Otherwise $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T'$. Let $\mathbb{B}_1$ be $\mathbb{B}'$'s lift in T_1.
By induction hypothesis $\mathbb{B}_1$ satisfies the lemma in T_1, and the only case that may change something to this situation, is the case $[x] = [ccad]$ in which $\mathbb{B}_1$ is a subproof-net of the one swallowed by an additive box. But in this case, what actually happens is simply that $\mathbb{B}_1$ has additive depth $p-1$, and whatever of the two $\mathbb{B}_1$'s residues in T' you choose for $\mathbb{B}'$, the lemma still holds. □

In the sequel we will speak of the main subproof-nets of T' (that will also be denoted by β_1,..., β_p) in the canonical representation of T' with respect to the residue $\mathbb{B}'$ of the additive box $\mathbb{B}$ of exponential depth zero in T, meaning the components of the additive boxes $B_1,...,B_p$ which do not contain $\mathbb{B}'$.

2.5. Remark. (i) The independence lemma says (in particular) that, in any reduct of a proof-net T, two different residues of an additive box of depth zero in T *cannot interract* (they will never belong to two different proof-nets connected by a $\otimes$-link, for example). Observe that this is not at all the case for the residues of an exponential box. Intuitively, this means that the duplication involved in an additive commutative e.r.s. is of a completely different nature from the one involved in a contraction e.r.s..

(ii) From the independence lemma, it follows that a residue $\mathbb{B}'$ of an additive box $\mathbb{B}$ of exponential depth 0 in T cannot be a subproof-net of a residue $\mathbb{B}''$ of $\mathbb{B}$ different from $\mathbb{B}'$.

2.6. Lemma. *Let T, T' be two proof-nets such that $T \overset{\sigma}{\twoheadrightarrow} T'$, let $\mathbb{B}$ be an additive box with exponential depth zero in T and $\mathbb{B}'$ a residue of $\mathbb{B}$ in T'. Let's consider any proof-net appearing in σ in its canonical representation with respect to $\mathbb{B}'$'s lift in this proof-net.*
Let T_1 and T_2 be two proof-nets in the following decomposition of σ: $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{\sigma_2}{\twoheadrightarrow} T_2 \overset{\sigma_3}{\twoheadrightarrow} T'$. If β is a main subproof-net of T_1, then all its residues in T_2 are subproof-nets of main subproof-nets of T_2.
proof: By induction on σ_2.
If $T_1 = T_2$, then the lemma holds.
Otherwise, $T_1 \overset{\sigma_2'}{\twoheadrightarrow} T_2' \overset{[x]}{\rightarrow} T_2$. Let β be a main subproof-net of T_1, β'' any residue of β in T_2 and β' its lift in T_2'. (Obviously, if β has no residues in T_2 then the lemma holds).
By induction hypothesis, β' is a subproof-net of a main subproof-net γ of T_2'.
If γ has no residue in T_2, then $\gamma \overset{[x]}{\rightarrow} \delta$ (γ cannot desappear in a $[\&/\oplus]$ e.r.s., because in this case β' would desappear too) where δ is a main subproof-net of T_2, and β'' is a subproof-net of δ.
Otherwise, β'' is a subproof-net of a residue of γ in T_2. Then, it is enough to check that for all possible $[x]$, any residue of γ in T_2 is a subproof-net of a main subproof-net of T_2. □

3 Proof of the standardization lemma

The following formulation of the standardization lemma for the additives is not exactly the one of [Girard 87], but it is more in the spirit of the one given in [Danos 90] for the multiplicative and exponential fragment. Anyway, both the formulations are clearly enough to prove the strong normalization property.

Observe that, of course, for the following theorem there is also the symmetrical result (just take $\oplus_1$ instead of $\oplus_0$).

3.1. Theorem. *Let $\mathbb{B}$ be an additive box with front door $D\&C$ and auxiliary doors $A_1, ..., A_n$, let Q_i ($i \in \{1, ..., n\}$) be a proof-net with $A_i^\perp$ among its conclusions, and let S be a proof-net with the link $\oplus_0$ (with premisse $D^\perp$ and conclusion $D^\perp \oplus C^\perp$) among its last links.*
Let 's call T the proof-net obtained by cutting the box $\mathbb{B}$ with all the Q_i out of the auxiliary doors and with S out of the front door, and let 's call $C_1(T)$ (resp. $C_2(T)$) the proof-net obtained from T by eliminating the logical cut between $\mathbb{B}$ and S (resp. by connecting directly $\mathbb{B}$'s second component with the Q_i).
If $C_1(T)$ and $C_2(T)$ are strongly normalizable, then so is T.

From now on, T, $C_1(T)$, $C_2(T)$ and $\mathbb{B}$ will denote proof-nets of the previous form. If $T \stackrel{\sigma}{\twoheadrightarrow} T'$, we shall call special cut (resp. special link $\oplus$) any residue of the logical cut between $\mathbb{B}$ and S (resp. any residue of the link $\oplus_0$ of S) in T'. Observe that $\mathbb{B}$, $\oplus_0$ and the logical cut have exponential depth zero in T, so that remark 2.3, the independence lemma and lemma 2.6 apply.
To prove the theorem, we define a size on any reduct of T, and show that this size shrinks when performing any e.r.s..

In the following lemma, the notion of path is the (obvious) generalization to the case of full LL of the notion of *straight* path of [DanReg 95].

3.2. Lemma. *Let* T' be such that $T \stackrel{\sigma}{\twoheadrightarrow} T'$. Then

1) Let & (resp. $\oplus_0$) be the front door of a $\mathbb{B}$'s residue (resp. a special $\oplus_0$ link) in T' and Φ a path with starting link & (resp. $\oplus_0$) and starting edge the conclusion of the front door of the residue of $\mathbb{B}$ (resp. the conclusion of the special $\oplus_0$ link).
If n is the first link crossed by Φ such that $n \neq \mathit{coad}$, then n is a special cut.
2) Let c be a special cut in T' and Φ a path having c as starting link.
If n is the first link crossed by Φ such that $n \neq \mathit{coad}$, then n is either a special $\oplus_0$ link either the front door of a residue of $\mathbb{B}$.

proof: The main point here is that all the residues of $\mathbb{B}$, all the special $\oplus_0$ links and all the special cuts, have exponential depth 0 in T'. □

3.3. Definition. Let T' be such that $T \stackrel{\sigma}{\twoheadrightarrow} T'$.
For any residue $\mathbb{B}_{T'}$ of $\mathbb{B}$ in T' with first component $\gamma_1(\mathbb{B}_{T'})$, and for any subproof-net γ of T' with a special link $\oplus_0$ among its last links, we define

$$C_1(T') := T'\left[\gamma_1(\mathbb{B}_{T'})\Big/_{\mathbb{B}_{T'}}, \gamma\backslash\{\oplus_0\}\Big/_{\gamma}\right]$$

where $\gamma\backslash\{\oplus_0\}$ is obtained from γ by eliminating the special link $\oplus_0$ (then, up to substitutions, $\gamma\backslash\{\oplus_0\}$ will have $D^\perp$ instead of $D^\perp \oplus C^\perp$ among its conclusions).

3.4. Remark. (i) $C_1(T')$ is well-defined (i.e. it is a proof-net) thanks to the independence lemma (see ii) of remark 2.5) and to lemma 3.2: special cuts are replaced by cuts with active formula D (up to substitutions).

(ii) For $T' = T$, $C_1(T)$ is the proof-net previously defined.

(iii) If in T' there is no residue of $\mathbb{B}$, then from lemma 3.2 $C_1(T') = T'$

(iv) Let x be a cut-link in the reduct T' of T. If x is not a *logical* special cut, and if x does not appear in the second component of any residue of $\mathbb{B}$ in T', then we have "the same" cut-link x in $C_1(T')$.

3.5. Proposition. *Let* T_1 *and* T_2 *be two proof-nets such that* $T \stackrel{\sigma_1}{\twoheadrightarrow} T_1 \stackrel{[x]}{\rightarrow} T_2$.

situation (a): $[x] = [ccad]$ attractive for a residue of $\mathbb{B}$ in T_1, or x is a logical special cut, or x is a cut-link in the second component of any residue of $\mathbb{B}$ in T_1. In this case, $C_1(T_1) = C_1(T_2)$.

situation (b): otherwise. In this case, $C_1(T_1) \stackrel{[x]}{\to} C_1(T_2)$.

In particular, if T' is a proof-net such that $T \twoheadrightarrow T'$ and if $C_1(T)$ is strongly normalizable, then $C_1(T')$ is also strongly normalizable.

proof: If there is no residue of $\mathbb{B}$ in T_1, then there is no residue of $\mathbb{B}$ in T_2; we have that $C_1(T_1) = T_1$ and $C_1(T_2) = T_2$, and from lemma 3.2 there is no special cut in T_1: we are in situation (b) and we have that $C_1(T_1) \stackrel{[x]}{\to} C_1(T_2)$.
If x is a logical special cut, then from the definition of C_1, one clearly deduce $C_1(T_1) = C_1(T_2)$.
If x is a cut-link in the first (resp. the second) component of a residue of $\mathbb{B}$ in T_1, then we are in situation (b) (resp. in situation (a)) and $C_1(T_1) \stackrel{[x]}{\to} C_1(T_2)$ (resp. $C_1(T_1) = C_1(T_2)$).
If $[x] = [ccad]$ attractive for a residue of $\mathbb{B}$ in T_1, then the only effect of $[x]$ is to increase the number of links contained in the residue (without duplicating any residue of $\mathbb{B}$, because of the independence lemma), but the definition of C_1 doesn't care: we are in situation a) and $C_1(T_1) = C_1(T_2)$.
Otherwise, on the one hand from remark 3.4 x appears in $C_1(T_1)$, and on the other hand (because x is neither logical special neither contained in a residue of $\mathbb{B}$ in T_1 and, if $x = ccad$, then $[x]$ is not attractive for any residue of $\mathbb{B}$ in T_1, and it is impossible for $[x]$ to duplicate a subproof-net of a residue $\mathbb{B}_1$ of $\mathbb{B}$ in T_1 without duplicating the whole $\mathbb{B}_1$) the hypothesis of the substitution lemma (lemma 2.2) are satisfied. Then, if $\mathbb{B}_1^1, \ldots, \mathbb{B}_1^n$ are all the residues of $\mathbb{B}$ in T_1 and $\alpha_1, \ldots, \alpha_n$ are any proof-nets s.t. $\mathbb{B}_1^i$ and α_i have the same conclusions, we have, up to substitutions if $[x] = [\forall/\exists]$ (and again thanks to the independence lemma), that $T_1[\alpha_1/\mathbb{B}_1^1, \ldots, \alpha_n/\mathbb{B}_1^n] \xrightarrow{[x]} T_2[\alpha_1/\mathbb{B}_1^{1'}, \ldots, \alpha_n/\mathbb{B}_1^{n'}]$ where $\mathbb{B}_1^{i'}$ is any residue of $\mathbb{B}_1^i$ in T_2 (there are at most two for each i). In particular, thanks to lemma 3.2, we can substitute to any residue of $\mathbb{B}$ in T_1 its first component, provided we substitute any special cut by a cut with active formula D (up to substitutions): we are in situation b) and $C_1(T_1) \stackrel{[x]}{\to} C_1(T_2)$. $\square$

Discussion: The first idea would now be to define in a similar way $C_2(T')$ (taking the second component of $\mathbb{B}$'s residues and erasing the first) for any reduct T' of T, and then to prove the analogue of the previous proposition for C_2. But here is where the dissymmetry between the role played by the first and the second component of $\mathbb{B}$ appears (in theorem 3.1 we decided to choose a special link $\oplus_0$ and not $\oplus_1$): when we reduce a logical special cut, it is always the *second* component of the residue of $\mathbb{B}$ which desappears.
Actually, consider for example the following reduction $T \stackrel{[ccad]}{\to} T_1 \stackrel{[\&/\oplus_0]}{\to} T_2$, where:

- T_1 is obtained from T applying an e.r.s. $[ccad]$ attractive for a box B contained in one of the Q_i, in such a way that the subproof-net swallowed by the box B contains $\mathbb{B}$ and the logical special cut,

- T_2 is obtained from T_1 by reducing precisely one of the two logical special cuts (that are residues in T_1 of T 's logical cut between $\mathbb{B}$ and S).

In T_1, we have two residues of $\mathbb{B}$ as subproof-nets of, respectively, the first and the second component of the residue B_1 of B in T_1; but in T_2, we have only one residue of $\mathbb{B}$, that is a subproof-net of (only) one of the two components of the residue B_2 of B in T_2.

In T_1, when we take the second component of the two residues of $\mathbb{B}$, we obtain two proof-nets with C among their conclusions, and we can connect the two edges by a link *coad* (which will be added to the box B_1) to define $C_2(T_1)$. But in T_2, we have only one residue of $\mathbb{B}$, so that when we take the second component of this residue of $\mathbb{B}$, we obtain a proof-net with C among its conclusions, and there is no way to obtain a proof-net with C among its conclusions from the component of B_2 which does not contain any $\mathbb{B}$'s residue. This is because (in the canonical representation of T_1 with respect to the residue of $\mathbb{B}$ that has a residue in T_2) we have reduced a cut in a *main subproof-net* of T_1.

The reader is invited to draw a small picture which should clarify the whole thing.

Then, if $T \overset{\sigma}{\twoheadrightarrow} T'$, we will select a residue $\mathbb{B}_{T'}$ of $\mathbb{B}$ in T', and will distinguish the case in which (taking for any proof-net appearing in σ its canonical representation with respect to the lift of $\mathbb{B}_{T'}$ in this proof-net) σ doesn't contain any reduction of a main subproof-net so that it will be possible to define $C_2^{\mathbb{B}_{T'}}(T')$, from the one in which σ contains a reduction of a main subproof-net, that will be handled in a different way (see proposition 3.7).

3.6. Proposition. *Let T' be such that $T \overset{\sigma}{\twoheadrightarrow} T'$. Let $\mathbb{B}_{T'}$ be a residue of $\mathbb{B}$ in T', and let's consider for any proof-net appearing in the reduction sequence σ, its canonical representation with respect to the (unique) lift of $\mathbb{B}_{T'}$ in this proof-net.*

If σ doesn't contain any e.r.s. reducing a main subproof-net, then it is possible to define $C_2^{\mathbb{B}_{T'}}(T')$ in such a way that:

- *the second component of $\mathbb{B}_{T'}$ is a subproof-net of $C_2^{\mathbb{B}_{T'}}(T')$*
- *$C_2(T) \twoheadrightarrow C_2^{\mathbb{B}_{T'}}(T')$.*

In particular, if $C_2(T)$ is strongly normalizable, then so is $C_2^{\mathbb{B}_{T'}}(T')$.

idea of proof: The idea is to specify the independence lemma (lemma 2.4) to the particular case of T and T', and then to define $C_2^{\mathbb{B}_{T'}}$ of all the (additive) boxes of T' containing $\mathbb{B}_{T'}$, "keeping all what comes from any of the Q_i and erasing all what comes from S". In order to do this, we first define (by induction on σ) $C_2^{\mathbb{B}_{T'}}$ of all the main subproof-nets of T' contained in the boxes that are residues in T' of additive boxes contained in any of the subproof-nets Q_i of T. (Thanks to the hypothesis "σ doesn't contain any e.r.s. reducing a main subproof-net", we don't stumble on the problem mentioned in the previous discussion).

Then we define $C_2^{\mathbb{B}_{T'}}(T')$ as $C_2^{\mathbb{B}_{T'}}(B_1)$ (if B_1 is the additive box of (additive) depth zero containing $\mathbb{B}_{T'}$) connected with "what remains of the Q_i at depth

zero in T'".
Finally, we check that $C_2(T) \twoheadrightarrow C_2^{\mathbb{B}_{T'}}(T')$ as usual by induction on σ.
Observe that $C_2^{\mathbb{B}_{T'}}(T')$ has always C (up to substitutions) among its conclusions. □

3.7. Proposition. *Let T' be such that $T \overset{\sigma}{\twoheadrightarrow} T'$.*
If $C_2(T)$ is strongly normalizable, then for any residue $\mathbb{B}_{T'}$ of $\mathbb{B}$ in T', $\gamma_2(\mathbb{B}_{T'})$ is strongly normalizable, where $\gamma_2(\mathbb{B}_{T'})$:= second component of $\mathbb{B}_{T'}$.
proof: Induction on $l(\sigma)$ =number of e.r.s. of σ.
If $l(\sigma) = 0$, then $T' = T$ and $\gamma_2(\mathbb{B})$ is a subproof-net of $C_2(T)$ which is strongly normalizable by hypothesis.
Otherwise, $l(\sigma) = n + 1$. Let $\mathbb{B}_{T'}$ be a residue of $\mathbb{B}$ in T', we are going to prove that $\gamma_2(\mathbb{B}_{T'})$ is strongly normalizable. In the sequel of the proof, all the proof-nets will be considered in their canonical form with respect to the lift of $\mathbb{B}_{T'}$ they contain.
If σ doesn't contain any e.r.s. reducing a main subproof-net, then from the previous proposition, we deduce that $\gamma_2(\mathbb{B}_{T'})$ (as a subproof-net of $C_2^{\mathbb{B}_{T'}}(T')$) is strongly normalizable.
Otherwise let $[x]$ be the last e.r.s. of σ reducing a main subproof-net, more precisely let's consider the following decomposition of σ: $T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\to} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'$ where σ_2 doesn't contain any e.r.s. reducing a main subproof-net, and suppose that $\beta \overset{[x]}{\to} \gamma$ where β (resp. γ) is a main subproof-net of T_1 (resp. of T_2).
From lemma 2.6, any residue of γ in any of the proof-nets appearing in σ_2 (including T') is a subproof-net of a main subproof-net of this proof-net. But $[x]$ is the last e.r.s. of σ reducing a main subproof-net, so that if $[ccad] \in \sigma_2$ this e.r.s. cannot duplicate any subproof-net of a residue of γ without duplicating the whole residue of γ. Then the substitution lemma (lemma 2.2) applies: if δ is any proof-net with the same conclusions of γ and if $\gamma_1', \ldots, \gamma_n'$ are the residues of γ in T', we have that $T_2\Big[\delta/\gamma\Big] \overset{\sigma_2}{\twoheadrightarrow} T'\Big[\delta/\gamma_1', \ldots, \delta/\gamma_n'\Big]$. In particular, if we take $\delta = \beta$, we have that $T_2\Big[\beta/\gamma\Big] \overset{\sigma_2}{\twoheadrightarrow} T'\Big[\beta/\gamma_1', ..., \beta/\gamma_n'\Big]$ i.e. $T_1 \overset{\sigma_2}{\twoheadrightarrow} T'\Big[\beta/\gamma_1', ..., \beta/\gamma_n'\Big]$.
If we define $\sigma' := \sigma \backslash \{[x]\}$, we have that $T \overset{\sigma'}{\twoheadrightarrow} T'\Big[\beta/\gamma_1', ..., \beta/\gamma_n'\Big] \overset{[x]^n}{\twoheadrightarrow} T'$, with $l(\sigma') < l(\sigma)$.
Then, there are two possibilities:

1) there are no residues of γ in T':

in this case we can eliminate the e.r.s. $[x]$: we have that $T \overset{\sigma'}{\twoheadrightarrow} T'$, with $l(\sigma') < l(\sigma)$, and then, by induction hypothesis, $\gamma_2(\mathbb{B}_{T'})$ is strongly normalizable.

2) there is a residue of γ in T':

by induction hypothesis, if $\mathbb{B}_{T'[\beta/\gamma_1',...,\beta/\gamma_n']}$ is the lift of $\mathbb{B}_{T'}$ in $T'\Big[\beta/\gamma_1', ..., \beta/\gamma_n'\Big]$ (w.r.t. the reduction sequence $[x]^n$), then $\gamma_2(\mathbb{B}_{T'[\beta/\gamma_1',...,\beta/\gamma_n']})$ is strongly normalizable; and because the n e.r.s. $[x]$ reduce main subproof-nets, we have that $\mathbb{B}_{T'[\beta/\gamma_1',...,\beta/\gamma_n']} = \mathbb{B}_{T'}$ and $\gamma_2(\mathbb{B}_{T'[\beta/\gamma_1',...,\beta/\gamma_n']}) = \gamma_2(\mathbb{B}_{T'})$. □

Let's prove now the (generalized) standardization lemma (theorem 3.1): Consider the following reduction sequence starting from T:

$$T \overset{\sigma_1}{\twoheadrightarrow} T_1 \overset{[x]}{\rightarrow} T_2 \overset{\sigma_2}{\twoheadrightarrow} T'.$$

For any proof-net R appearing in this reduction sequence and for any residue $\mathbb{B}'$ of $\mathbb{B}$ in R with conclusions $D\&C$ (up to substitutions) and $A_1,\dots,A_{n'}$, we define:

-$\Re(R)$:={ $\mathbb{B}_R$: $\mathbb{B}_R$ is a residue of $\mathbb{B}$ in $R\}$ and $\|\Re(R)\| := card(\Re(R))$

-$\Lambda_{\mathbb{B}'}(R)$:={ $\gamma_1,\dots,\gamma_{n'}$: γ_i is the biggest subproof-net of R connected to $\mathbb{B}'$ through a commutative cut $ccad$ with active formula $A_i\}$. Observe that for some i γ_i may be the empty proof-net.

-$\|\Lambda_{\mathbb{B}'}(R)\|$:=$n(\gamma_1) + \dots + n(\gamma_{n'})$ where $n(\gamma_i)$=number of links of γ_i.

-$\|\Lambda(R)\| := \Sigma_{B\in\Re(R)}\|\Lambda_B(R)\|$

We prove that

$$(\|C_1(T_2)\|, \|\Re(T_2)\|, \|\Lambda(T_2)\|, \Sigma_{B\in\Re(T_2)}\|\gamma_2(B)\|)$$
$$<_{LEX}$$
$$(\|C_1(T_1)\|, \|\Re(T_1)\|, \|\Lambda(T_1)\|, \Sigma_{B\in\Re(T_1)}\|\gamma_2(B)\|)$$

where $\gamma_2(B)$ is the second component of B, and $\|C_1(T_i)\|$ (resp. $\|\gamma_2(B)\|$) is the maximal length of the reduction sequences starting from $C_1(T_i)$ (resp. $\gamma_2(B)$).

From proposition 3.5, if $[x] \neq [ccad]$ attractive for a residue of $\mathbb{B}$ in T_1, if x is not a logical special cut and if x is not a cut in the second component of a residue of $\mathbb{B}$ in T_1, then $\|C_1(T_2)\| < \|C_1(T_1)\|$.

If x is a logical special cut, then by proposition 3.5, $\|C_1(T_2)\| = \|C_1(T_1)\|$, and $\|\Re(T_2)\| < \|\Re(T_1)\|$.

If $[x] = [ccad]$ $\mathbb{B}_1$-attractive for a residue $\mathbb{B}_1$ of $\mathbb{B}$ in T_1, then from proposition 3.5 $\|C_1(T_2)\| = \|C_1(T_1)\|$. Observe that, because of the independence lemma, the number of residues of $\mathbb{B}$ cannot increase in T_2 with respect to T_1, and we have that $\|\Lambda(T_2)\| < \|\Lambda(T_1)\|$.

If x is a cut in $\gamma_2(\mathbb{B}_1)$ for a residue $\mathbb{B}_1$ of $\mathbb{B}$, then from proposition 3.5 $\|C_1(T_2)\| = \|C_1(T_1)\|$. We also have $\|\Re(T_2)\| = \|\Re(T_1)\|$, while the independence lemma gives $\|\Lambda(T_2)\| = \|\Lambda(T_1)\|$.

If $\mathbb{B}_1'$ is the (unique) residue of $\mathbb{B}_1$ in T_2, then $\|\gamma_2(\mathbb{B}_1')\| < \|\gamma_2(\mathbb{B}_1)\|$, so that $\Sigma_{B\in\Re(T_2)}\|\gamma_2(B)\| < \Sigma_{B\in\Re(T_1)}\|\gamma_2(B)\|$. □

3.8. Remark. Observe that to have a complete proof of strong normalization for this revisited PN_2, one still has to prove that the standardization lemma for the multiplicative and exponential fragment proven in [Danos 90] still holds in presence of the additives. We plan to be more precise concerning this point in future work.

4 LL and the Church-Rosser property

As stated in the introduction, full linear logic does not enjoy the confluence property. This can be easily seen for example by eliminating the cut between the two canonical proofs of $\vdash A^{\perp}\&A^{\perp}, A$ and $\vdash A\&A, A^{\perp}$. The point is that both the active formulas in the *cut*-link are conclusions of an auxiliary door of an additive box: one has to choose which box swallows and duplicates the other one (observe that this conflict occurs also with the original definition of $[ccad]$ given in [Girard 87]).

Nevertheless, the results of the previous sections (and specially the independence lemma) seem to indicate that this lack of confluence is not, so to speak, very damaging. The aim of this section is precisely to discuss a mathematical version of this informal claim.

In the sequel, we 'll denote by $\xrightarrow{\neg[ccad]}$ the rewriting rule obtained from the usual one by forbidding $[ccad]$ e.r.s., and by $\overset{\neg[ccad]}{\twoheadrightarrow}$ its reflexive and transitive closure. $\overset{lazy}{\twoheadrightarrow}$ will denote the reflexive and transitive closure of the rewriting rule $\xrightarrow{lazy}$ obtained from $\xrightarrow{\neg[ccad]}$ by restricting to the elimination of the *cut*-links with additive depth 0.

Conjecture: If R is a proof-net with no occurrences of the connective & nor of the quantifier $\exists^2$ (i.e. existential quantifier of the second order) in its conclusions, then R has a unique normal form.

4.1. Remark. One has to forbid to the second order existential quantifier to occur in the conclusions of the proof-net, simply because it can "hide" occurrences of the connective &.

4.2. Proposition. *In PN_2 the rewriting rule $\overset{\neg[ccad]}{\twoheadrightarrow}$ enjoys the Church-Rosser property.*
proof: This is a consequence of strong normalization and local confluence of $\xrightarrow{\neg[ccad]}$. Local confluence is proven by a simple inspection of cases, SN for $\xrightarrow{\neg[ccad]}$ in $MALL_q$ (multiplicative and additive linear logic with first and second order quantifiers) follows from an easy variant of theorem 3.1 (and of course from Girard 's SN theorem of [Girard 87]), but the present paper does not justify completely SN for $\xrightarrow{\neg[ccad]}$ in our revisited PN_2 (the analogue of Remark 3.8 applies, even if the way out is easier in this case). $\square$

4.3. Lemma. *Let R be a proof-net of $MALL_q$ with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions.*
If $R \xrightarrow{[x]} R_1$ where $[x]$ is a non lazy e.r.s. (of course the most important case is $[x] = [ccad]$), then there exist two lazy reduction sequences σ_1 and σ_2 and a proof-net R', such that $R \overset{\sigma_1}{\twoheadrightarrow} R'$ and $R_1 \overset{\sigma_2}{\twoheadrightarrow} R'$. In other words, it is possible to "avoid" non lazy e.r.s. in a lazy way.

sketch of proof: The main point is to prove that for any subproof-net α of R consisting in an additive box B connected by a *cut*-link out of an auxiliary door to a proof-net γ, there exist a lazy reduction sequence σ_α and a proof-net R_{σ_α} such that:

- σ_α satisfies the hypothesis of the substitution lemma (lemma 2.2) w.r.t. the subproof-net α of R
- $R \overset{\sigma_\alpha}{\twoheadrightarrow} R_{\sigma_\alpha}$
- all the residues of α in R_{σ_α} have additive depth 0 and all the main doors of B 's residues are active in a logical *cut*-link.

This can be proven by induction on $\|R\|_{\neg[ccad]}$, where for any proof-net T, $\|T\|_{\neg[ccad]}$ is the maximal length of the reduction sequences $\neg[ccad]$ starting from T. □

4.4. Theorem. *If R is a proof-net of $MALL_q$ with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions, then R has a unique normal form that can be reached in a lazy way.*

sketch of proof: One first proves that $\overset{lazy}{\twoheadrightarrow}$ is confluent, by checking local confluence of $\overset{lazy}{\longrightarrow}$. Then, if R' is a normal form of R and $R \overset{\sigma}{\twoheadrightarrow} R'$, one proves by induction on $l(\sigma)$ = number of e.r.s. of σ (using lemma 4.3 and confluence of $\overset{lazy}{\twoheadrightarrow}$) that R' can be reached from R in a lazy way. Finally, one proves (using again lemma 4.3 and confluence of $\overset{lazy}{\twoheadrightarrow}$) by induction on $l(\theta)$ that if $R \overset{\theta}{\twoheadrightarrow} R_\theta$ then $R_\theta \overset{lazy}{\twoheadrightarrow} R'$. □

4.5. Remark. (i) We feel that the same proof holds in the general case of the conjecture (i.e. for full linear logic), by substituting "$\neg[ccad]$" to "lazy"in lemma 4.3 and in the previous theorem. However, there is a technical difficulty that we did not solve yet.

(ii) There is also a semantical approach to the proof of the conjecture: it would be enough to prove that two different normal proof-nets of PN_2 without occurrences of & and of $\exists^2$ are interpreted by two different cliques (in the usual or even in the multiset semantics). A first attempt would be to prove that this holds for two different normal proof-nets in the multiplicative and exponential fragment, but as far as we know even this is not yet proven.

4.6. Theorem. *(Girard 95) If R is a proof-net of $MALL_q$, then R has a unique lazy normal form R'.*
If the conclusions of R don't contain any occurrence of the connective & nor of the quantifier $\exists^2$, then the unique lazy normal form R' is also a normal form and it can be reached from R in a time which is linear in the size (number of links) of R.

Both theorems combine in the following

4.7. Theorem. *Any proof-net in $MALL_q$ with no occurrences of the connective & nor of the quantifier $\exists^2$ in its conclusions has a unique normal form which can be reached in linear time.*

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