Cut Elimination for Proof Nets of the Purely **M**ultiplicative and **A**dditive Fragment of **L**inear **L**ogic

Roberto Maieli

Università degli Studi "Roma Tre"
maieli@uniroma3.it

joint work with Olivier Laurent
(CNRS, Paris)

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Linear Logic and Proof Theory

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- This task was traditionally carried out by means of sequent calculi with the consequence that the most part of these works were engrossed by tedious commutations of rules.

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  - additives: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\,\& \qquad \dfrac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \dfrac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2$

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This situation has changed with the new geometrical syntax for proofs:

proof nets

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- allowing super-positions (slices)

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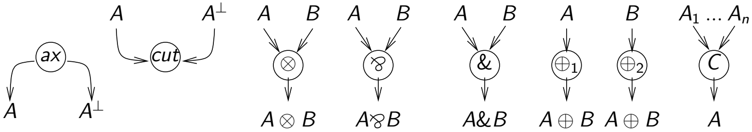

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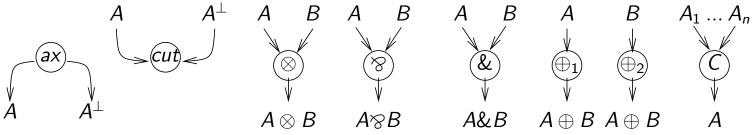

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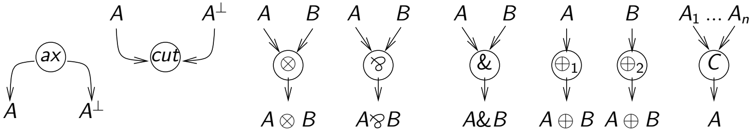

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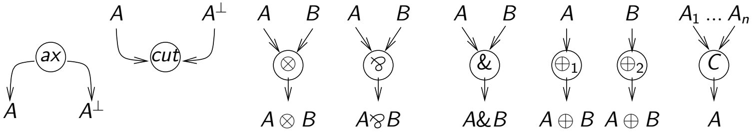

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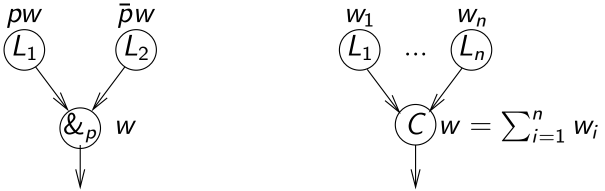
  3. all weights are *modulo* E;

MALL Proof Structures: weights assignment

3. two nodes have the same weight, if they have a common edge, except when the edge is the premise of a & or C node:



ϵ_p does not occur in w $\quad$ $\forall i \forall j, w_i w_j = 0 \; (1 \leq i, j \leq n)$

MALL Proof Structures: weights assignment

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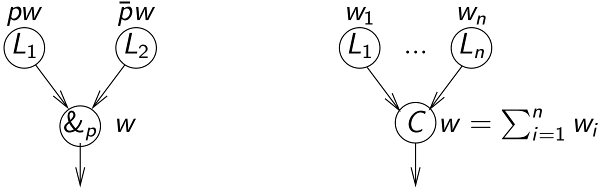

ϵ_p does not occur in w $\quad$ $\forall i \forall j, w_i w_j = 0 \; (1 \le i, j \le n)$

6. every conclusion node has weight 1;

MALL Proof Structures: technical condition

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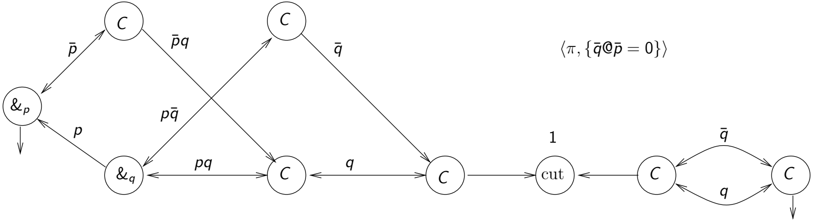
   - $(\sum_{i=1}^{n} w_i + \sum_{j=1}^{m} v_j)$ is a monomial weight $\mod E$;
   - all weights $w_1, ..., w_n, v_1, ...v_m$ are pairwise disjoint.

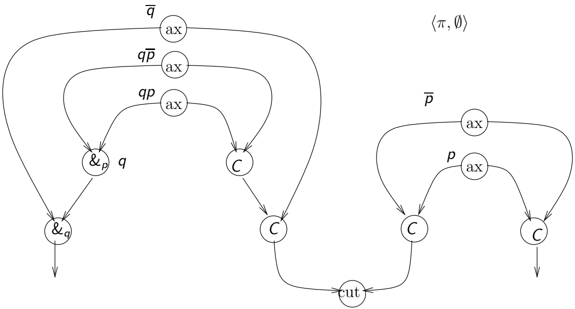
MALL Proof Structures: example 1

The following pair $\langle \pi, \{(\bar{q}@\bar{p} = 0)\}\rangle$ is a proof structure:



MALL Proof Structures: example 2

The following pair $\langle \pi, \emptyset \rangle$ is not a proof structure



it violates the *technical condition* of PS definition: there exists a (axiom) node whose weight is $\bar{p}$ but $\bar{p} \not\leq q$, where q is the weight of the unique $\&_p$ node.

The Request of a Correctness Criterion

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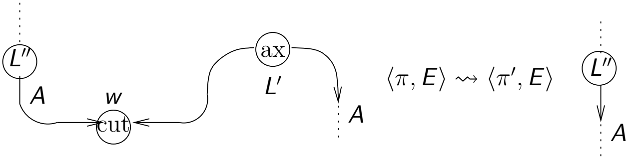
- we are interested on those proof structures that correspond to proofs of the sequent calculus;
- those proof structures will be called **proof nets**
- there exists a **Correction Crietrion** that detect these PNs
- cut elimination can be defined directly on PSs
- then we have to show that the Correction Crietrion is preserved by the cut elimination

Cut Elimination

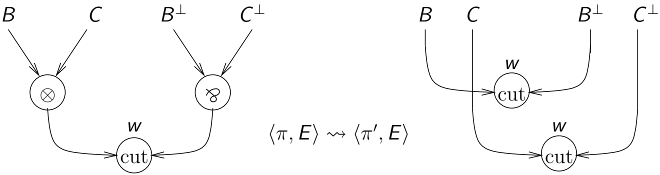
Cut Elimination: *ax*-step

If L' (resp., L'') is an axiom node of π, then $\langle \pi, E \rangle \rightsquigarrow \langle \pi', E \rangle$, where π' is obtained by removing in π both formulas A and $A^{\perp}$ (as well as L) and giving a new conclusion to L'' (resp., L'), the other conclusion of L' (resp., L'')



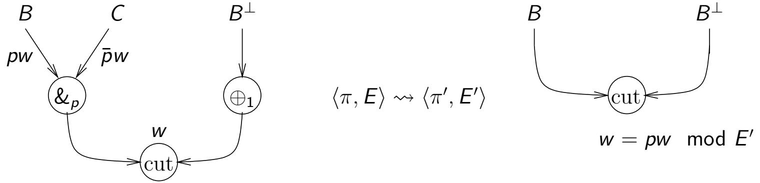
Cut Elimination: ($\otimes/\parr$)-step

If L' is a $\otimes$ node with premises B and C and L'' is a $\parr$ node with premises $B^\perp$ and $C^\perp$, then $\langle \pi, E \rangle \leadsto \langle \pi', E \rangle$, where π' is obtained by removing in π the formulas A and $A^\perp$ as well as the cut node L with L' and L'' and adding two new cut nodes with premises, respectively, $B, B^\perp$ and $C, C^\perp$



Cut Elimination: $(\oplus_i/\&)$-step

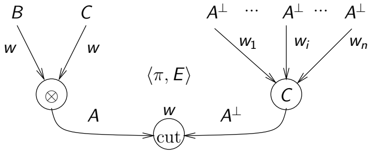
If L' is a $\&_p$ node with weight w and B and C as premises whose weights are, respectively, pw and $\bar{p}w$, and L'' is a $\oplus_1$ node with premise $B^\perp$ in π, then $\langle \pi, E\rangle \rightsquigarrow \langle \pi', E'\rangle$ as below



- $E' = E \cup \{\bar{p}@w = 0\}$;
- π' is what remains still nonzero, $\mod E'$, w.r.t. π.

Cut Elimination: $(\otimes/C)$-step

If L' is a C node and L'' is a $\otimes$ node



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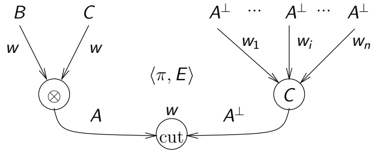

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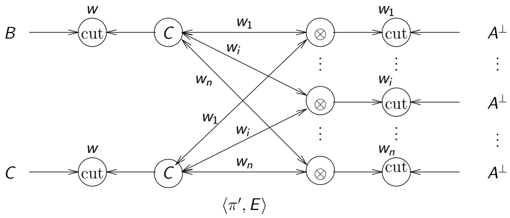

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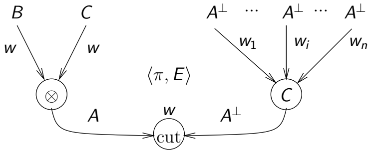

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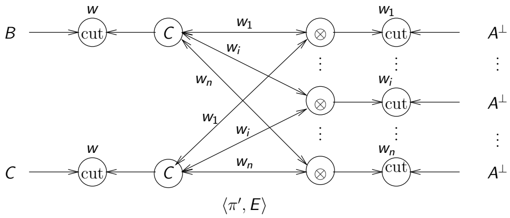

Case $(⅋/C)$-step is analogous (replace $\otimes$s with ⅋s).

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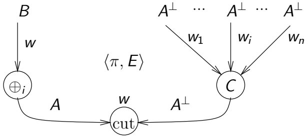

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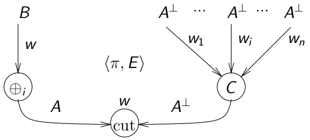

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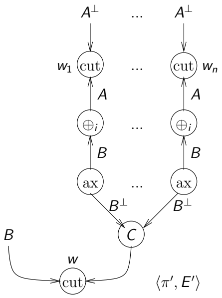

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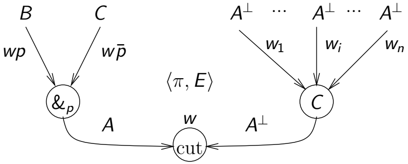

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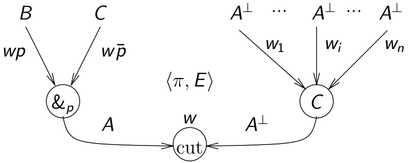

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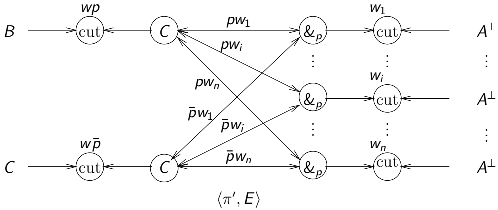

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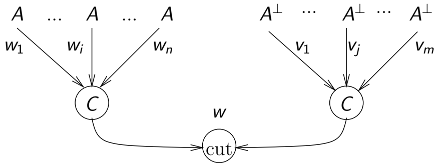

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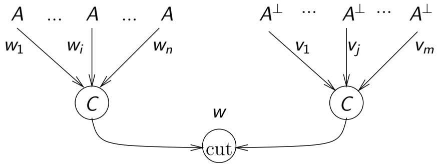

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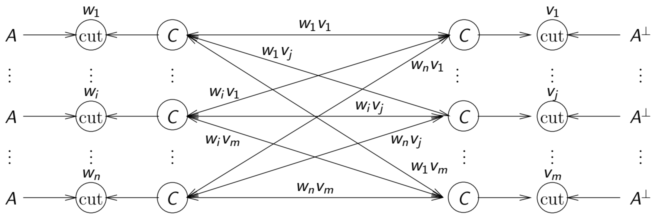

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If a PS $\langle \pi, E \rangle$ reduces in one step to $\langle \pi', E' \rangle$, then $\langle \pi', E' \rangle$ is a PS too.

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Theorem (Stability of PS)

If a PS $\langle \pi, E \rangle$ *reduces in one step to* $\langle \pi', E' \rangle$*, then* $\langle \pi', E' \rangle$ *is a PS too.*

Theorem (Stability of Correctness Criterion)

If a PS $\langle \pi, E \rangle$ *is correct and it reduces in one step to* $\langle \pi', E' \rangle$*, then* $\langle \pi', E' \rangle$ *is still a correct PS.*

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  - $|w_j|$, for $j = 1, 2$, is the length of w_j.

□

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then there exists a proof net $\langle \pi^*, E^* \rangle$ which $\langle \pi_i, E_i \rangle$, for $1 \leq i \leq 2$, reduces to in at most one step.

conclusions