# A proof of the Focusing Theorem via MALL proof nets*

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**Abstract.** We present a demonstration of Andreoli's *focusing theorem* for proofs of linear logic (MALL) that avoids directly reasoning on sequent calculus proofs. Following Andreoli-Maieli's strategy, exploited in the MLL case, we prove the focusing theorem as a particular sequentialization strategy for MALL proof nets that are in *canonical form*. Canonical proof nets satisfy the property that asynchronous links are always ready to sequentialization while synchronous focusing links represent clusters of links that are hereditarily ready to sequentialization.

**Keywords:** linear logic · sequent calculus · focusing proofs · proof nets.

## 1 Introduction

Focusing is an efficient proof-search procedure for Linear Logic [4], based on a proof normalization result (the "Focusing Theorem") that has been described by Andreoli in [1]. Focusing is described there in terms of the sequent system of (commutative) Linear Logic, which it refines in two steps: "Dyadic", resp. "Triadic" system. Basically, each refinement eliminates redundancies in proof-search due to irrelevant sequentializations of inference figures in the sequent-based representation of proofs. The expressive power of Focusing is captured in a crisp way in a fully representative fragment of Linear Logic, called "LinLog", introduced in [1] together with a normalization procedure from Linear Logic to LinLog. Usually the focusing theorem is proved in the linear sequent calculus and the proof is quite complex requiring an argument that makes use of a double induction. Andreoli and Maieli have shown in [2] that Focusing can also be interpreted in the proof net formalism, where it appears, at least in the multiplicative fragment of linear logic (MLL), to be a simple refinement of the "Splitting Lemma" for proof nets. The Splitting Lemma is at the core of the Sequentialization procedures for proof nets, and Focusing thus appears as a sequentialization strategy. This change of perspective allows the generalization of the Focusing result to (the multiplicative fragment of) any logic where the "Splitting Lemma" holds. Here we extend this idea of [2] to the case of MALL proof nets: we show how the focusing theorem for MALL can be interpreted as a refinement of a Focusing Lemma in which, in addition to the splitting case, it is also necessary to take into account clusters of tensor ($\otimes$) and plus ($\oplus_i$) links that are hereditarily ready to sequentialization (this is also known as the "critical synchronous section"). In order to show this result we first need to fix a syntax for MALL proof nets. Unlike what happens for MLL proof nets, the syntax of MALL proof nets

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is not so peaceful and univocal ([5, 6, 9]). There are essentially two syntaxes for MALL proof nets: that one by Girard [5] based on proof structures weighted with Boolean monomials, and a "more canonical" one by Hughes and Van Glabbeek [6]. Here, we choose the former syntax and we show how it is possible to transform monomial proof nets into *canonical forms*, in the same way as done by Hughes and Van Glabbeek. The canonical form of the (monomial) proof nets is now given by the adoption of an additive contraction link allowed only with atomic premises or with $\oplus$ premises coming from different instances of unary additive $\oplus$ links, i.e. $\oplus_1$ and $\oplus_2$. This new syntactic condition (on contraction links) allows to maximize the superposition of proof structures thus rendering them more canonical.

**Paper Contributions.** We characterize a (proper sub-)class of of (monomial) proof nets, that is in correspondence with the class of focusing MALL sequent proofs: this is called the class of proof nets in canonical form (CPN, Definition 6). The correspondence between CPNs and focusing proofs is established via the sequentialization Theorem 4 which relies on the Focusing Theorem 3, a refinement of the Splitting and Ready Lemmas 4 and 5. The canonical form of a given proof net $\pi$ ensures that the asynchronous conclusions (i.e., conclusions of type $\parr, \&$) of $\pi$ are always ready to sequentialization, while the Focusing Theorem 3 allows to identify those synchronous conclusions links (of type $\oplus, \otimes$ links) of $\pi$ that are hereditarily (i.e., recursively) ready to sequentialization.

## 2 The MALL fragment of linear logic

In this paper, we consider only the pure (without units) multiplicative and additive fragment of Linear Logic (MALL). MALL formulas $A, B, ...$ are built from literals (propositional variables $P, Q, ...$ and their negations $P^\perp, Q^\perp, ...$) and the binary connectives $\otimes$ (*tensor*), $\parr$ (*par*), $\&$ (*with*) and $\oplus$ (*plus*). Negation $(.)^\perp$ extends to arbitrary formulas by the de Morgan laws: $(A \otimes B)^\perp = (A^\perp \parr B^\perp)$, $(A \parr B)^\perp = (A^\perp \otimes B^\perp)$, $(A \& B)^\perp = (A^\perp \oplus B^\perp)$, and $(A \oplus B)^\perp = (A^\perp \& B^\perp)$. A MALL sequent $\Gamma$ is a multiset of formulas $A_1, ..., A_n$. Sequents are one-sided, so we may omit turnstiles ($\vdash$). The rules of the proof system $\Sigma_1$ are depicted in the top part of Fig.1. In the MALL fragment we consider, the refined focused system described in [1] can be reduced to $\Sigma_2$ of Fig. 1: it is called the "Dyadic System $\Sigma_2$ for MALL". Connectives are split into two categories: *asynchronous* (or *negative*), $\parr$ and $\&$, corresponding to a kind of "don't care non-determinism" and *synchronous* (or *positive*), $\otimes$ and $\oplus$, corresponding to a kind of "true non-determinism" w.r.t. proof-search. Furthermore, we assume that the class of atomic formulas is split into two dual, disjoint sub-classes: the *positive* atoms $X, Y, Z, ...$ and their *negative* duals $X^\perp, Y^\perp, Z^\perp, ...$ with $X^{\perp\perp} = X$ (but this distinction is only conventional). Focusing sequents are of two types: "$\vdash \Gamma \Uparrow L$" and "$\vdash \Gamma \Downarrow F$", where $\Gamma$ is a multiset of non-asynchronous formulas, $L$ is a list of formulas and $F$ is a single formula called the "focus" of the sequent. The Focusing system is justified by the following theorem (stated and proved in [1]):

**Theorem 1 (Andreoli, 1992).** *Let $\Gamma$ be a multiset of non-asynchronous formulas and $L$ an ordered list of formulas: $\vdash_{\Sigma_1} \Gamma, L$ if and only if $\vdash_{\Sigma_2} \Gamma \Uparrow L$.*

The original proof consists in showing that any proof of $\Gamma, L$, in the standard sequent system $\Sigma_1$, can be mapped, by permutation of inferences and deletion of dummy subproofs, into a proof of $\Gamma \Uparrow L$ in the focusing system $\Sigma_2$ and vice-versa. In the following, The monadic sequent system $\Sigma_1$ for MALL:

$$\frac{}{A, A^{\perp}}\ \mathrm{ax} \qquad \frac{\Gamma, A \qquad \Delta, A^{\perp}}{\Gamma, \Delta}\ \mathrm{cut} \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\ \& \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \oplus_i A_2}\ \oplus_{i=1,2}$$

The dyadic (focused) sequent system $\Sigma_2$ for MALL:

- Logical rules: $\parr, \&, \otimes, \oplus_{i=1,2}$

$$\frac{\Gamma \Uparrow L, F, G}{\Gamma \Uparrow L, F \parr G}\ \parr \qquad \frac{\Gamma \Uparrow L, F \qquad \Gamma \Uparrow L, G}{\Gamma \Uparrow L, F \& G}\ \& \qquad \frac{\Gamma \Downarrow F \qquad \Delta \Downarrow G}{\Gamma, \Delta \Downarrow F \otimes G}\ \otimes \qquad \frac{\Gamma \Downarrow F_i}{\Gamma \Downarrow F_1 \oplus F_2}\ \oplus_{i=1,2}$$

- Identity $[id]$ : if $F$ is a positive atom
- Reaction $[R \Uparrow]$ : if $F$ is not asynchronous
- Reaction $[R \Downarrow]$ : if $F$ is neither synchronous nor a positive atom
- Decision $[D]$ : if $F$ is synchronous or a positive atom

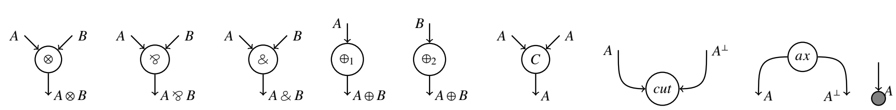
$$\frac{}{F^{\perp} \Downarrow F}\ id \qquad \frac{\Gamma, F \Uparrow L}{\Gamma \Uparrow L, F}\ R \Uparrow \qquad \frac{\Gamma \Uparrow F}{\Gamma \Downarrow F}\ R \Downarrow \qquad \frac{\Gamma \Downarrow F}{\Gamma, F \Uparrow}\ D$$

**Fig. 1.** the monadic $\Sigma_1$ (top) and dyadic $\Sigma_2$ (bottom) systems for MALL

we show a different proof of this result as a refinement of the sequentialization of MALL proof nets. Focusing basically appears as a strategy in the choice of the sequentializable formulas in the Sequentialization procedure. For doing that, we need first to choose a syntax for MALL proof structures which, unlike the MLL case, is neither standard nor univocal ([5, 6, 9]). For several reasons[1] we prefer the syntax of [8] (refinement of [5]).

## 3 Proof structures

**Definition 1 (pre-proof structure).** *A MALL* pre-proof structure *(PPS) $\pi$ is a directed graph such that each edge is labelled by a MALL formula, each node has a type in $\{ax, cut, \otimes, \parr, \&, \oplus_1, \oplus_2, C, \bullet\}$ and built according to the following typing constraints:*




1. *the entering (resp., outgoing or exiting) edges of a node L are the* premises *(resp., the* conclusions*) of L;*
2. *each edge must be conclusion of exactly one node and premise of at most one node;*

[1] Compared with the syntax of [6], monomial proof structures [8] are technically simpler; they allow us to easily extend to the MALL case arguments originally used for the MLL case such as Laurent's Splitting Lemma [7] and Andreoli-Maieli's Focusing Theorem [2]. Monomial proof structures have a natural presentation in terms of Coherent Spaces [4] and their correctness criterion can be also formulated in terms of "graph retraction steps" *à la Danos* [9].

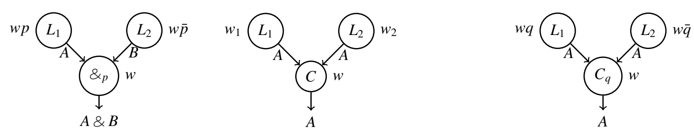
3. *"•" denotes a dummy node whose unique premise is also called* conclusion of $\pi$*;*
4. *a node whose conclusion (resp., conclusions) is conclusion (resp. are conclusions) of* $\pi$ *are called* terminal *or* conclusion nodes *of* $\pi$.

*We call* link *the pseudo-graph made by a node together with its premise(s) and its conclusion(s) (if any); e.g. the previous figure displayed the so called* MALL links.

*Proof structures* are PPSs equipped with boolean weights. Assume a set $\mathcal{B}$ of Boolean variables denoted by $p, q, r, ...$, then a *monomial weight* (simply, a weight) $w, v, ...$, over $\mathcal{B}$ is a product (conjunction) of variables or negation of variables of $\mathcal{B}$. We replace $p.p$ by p. Often, in a product of weights, $v$ and $w$, we omit "." and we write "$vw$" instead of "$v.w$". As usual in a Boolean algebra, we define the standard order relation "$\leq$" between two weights $v$ and $w$ as follows: $v \leq w$ if there exists a weight $v'$ s.t. $v = v'.w$. We also assume the following notation: 1 for the empty product, 0 for a product where both $p$ and its negation $\bar{p}$ appear and $\epsilon_p$ for a variable $p$ or $\bar{p}$. We say that a weight $w$ *depends on a variable* $p$ when $\epsilon_p$ appears in $w$; two weights, $v$ and $w$, are *disjoint* when $v.w = 0$.

**Definition 2 (proof structure).** *A* MALL proof structure *(shortly, PS) is any PPS* $\pi$ *whose nodes are equipped with monomial weights as follows:*

1. *we associate a* Boolean variable *($p, q, ...$), called* eigen weight*, to each* &*-node of* $\pi$ *(eigen weights are supposed to be different);*
2. *we associate a* weight $w$*, i.e., a product (conjunction) of eigen weights or negations of eigen weights of* $\pi$ *($p, \overline{p}, q, \overline{q}...$), to each node with the constraint that two nodes have the same weight if they have a common edge, except when the edge is the premise of a* & *or C-node: in these cases we proceed as follows:*
   (a) *if $w$ is the weight of a* &*-link and $p$ is its eigen weight then $w$ does not depend on $p$ and its premise links, $L_1$ and $L_2$, must have weights resp., $w.p$ and $w.\overline{p}$;*
   (b) *if $w$ is the weight of a C-link and $w_1, w_2$ are the weights of its premise links, $L_1$ and $L_2$, then $w = w_1 + w_2$ and $w_1 w_2 = 0$ (see the two l.h.s. pictures below);*




*$p$ does not occur in $w = wp + w\bar{p}$* $\quad$ $w = w_1 + w_2$ *and* $w_1 w_2 = 0$ $\quad$ *$\exists q$ s.t. $w_1 = wq$ and $w_2 = w\bar{q}$*

3. *every node that is conclusion of* $\pi$ *has weight* 1 *(dummy nodes • have weight* 1*);*
4. *(*dependence condition[2]*) if $w$ is the weight of a* &*-link with eigen weight $p$ and $w'$ is a weight depending on $p$ and appearing in the proof-structure then $w' \leq w$.*

**Fact 1** *Since the weights associated to a PS are products (*monomials*) of the Boolean algebra generated by the eigen weights associated to a proof structure then, for each weight $w$ associated to a contraction node, there exists a unique eigen weight $q$ that* splits $w$ into $w_1 = wq$ and $w_2 = w\bar{q}$. *We sometimes index a contraction node C with its* splitting variable $q$*, that is $C_q$ as in the rightmost hand side picture above.*

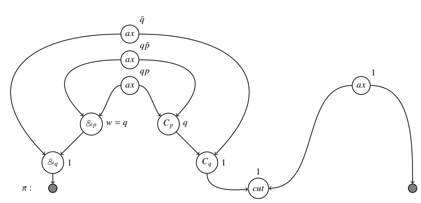
[2] The dependence condition corresponds to the *resolution condition* of [6].

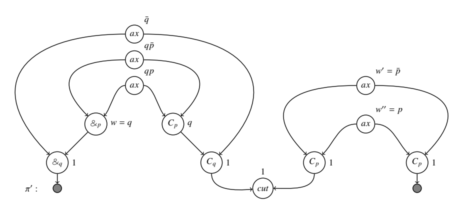
**Definition 3 (valuation, slice, switching).** *A* valuation $\varphi$ *for a PS* $\pi$ *is a function from the set of all weights of* $\pi$ *into* $\{0, 1\}$. *Fixed a valuation* $\varphi$ *for* $\pi$ *then:*

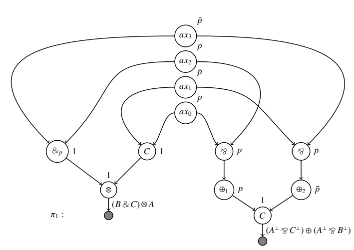
- *a* slice $\varphi(\pi)$ *is the graph obtained from* $\pi$ *by keeping only those nodes with weight* 1 *together with its outgoing edges (conclusion(s));*
- *a* multiplicative switching (induced by $\varphi$), $S_m(\pi)$ *of* $\pi$, *is the un-directed graph built on the nodes and edges of* $\varphi(\pi)$ *with the modification that for each* $\parr$*-node we take only one premise and we cut the remaining one (it is called,* left/right $\parr$-switching*);*
- *an* additive switching (induced by $\varphi$) *of* $\pi$ *(or simply a* switching*), denoted* $S_a(\pi)$ *or simply* $S(\pi)$, *is a multiplicative switching where for each* $\&$*-node* $L$ *we cut the (unique) premise in* $S_m(\pi)$ *and we add an directed edge, called* jump*, from* $L$ *to a node* $L'$ *whose weight depends on the eigen weight of the* $\&$*-node* $L$.

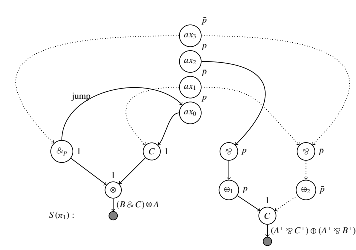
**Definition 4 (proof-net).** *A MALL PS* $\pi$ *is* correct*, so it is a* MALL proof net *(PN), if every (additive) switching* $S(\pi)$ *is an acyclic and connected graph (*ACC*).*

*Example 1.* The PPS $\pi$ on the l.h.s. below is a PS while the PPS $\pi'$ on the r.h.s. is not so since there exist a node whose weight $w' = \overline{p}$ (resp., $w'' = p$) depends on a $\&_p$-node, whose weight is $w = q$, but $\overline{p} \nleq q$ (resp., $p \nleq q$), contradicting Definition 2(4). Observe that jumps are necessary for the correctness criterion (Definition 4), otherwise proof structures that are not image of any sequent proof of MALL would be correct. Consider e.g. the PS $\pi_1$ on the l.h.s. below (bottom). Actually $\pi_1$ is "correct" if we reason only by multiplicative slices although its conclusions $(B \& C) \otimes A, (A^\perp \parr C^\perp) \oplus (A^\perp \parr B^\perp)$ are not a provable sequent in MALL. Actually, fixed a valuation $\varphi$ s.t. $\varphi(p) = 1$, there exists an additive switching $S(\pi_1)$ (induced by $\varphi$) that is not ACC as the one in the right hand side (note that $S(\pi)$ consists of the sub-graph with solid edges).







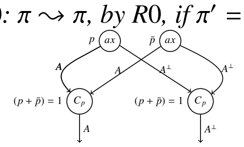
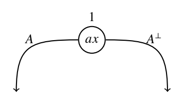


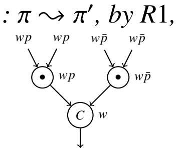
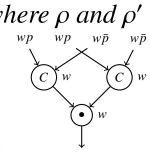
**Definition 5 (substitution, restriction, empire, spreading).** *Let $\pi$ be a PS, $p$ and $q$ eigen weights and $w$ a weight in $\pi$, then:*

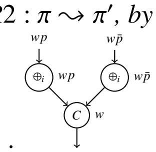
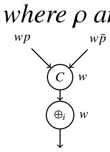
- *the* substitution *of $p$ by $q$ in $\pi$, denoted with $\pi[q/p]$ is the graph $\pi'$ obtained from $\pi$ by replacing each occurrence of $p$ (resp., $\bar{p}$) by $q$ (resp., $\bar{q}$);*
- *the* restriction *of $\pi$ w.r.t. $p$ (resp., of $\pi$ w.r.t. $\bar{p}$), denoted $\pi \downharpoonright^{p}$ (resp., $\pi \downharpoonright^{\bar{p}}$), is what remains of $\pi$ when we replace $p$ with $1$ and $\bar{p}$ with $0$ (resp., we replace $p$ with $0$ and $\bar{p}$ with $1$) and keep only those vertexes and edges whose weight is still non null;*
- *the* empire *(or the* dependency graph*) of the eigen weight $p$ w.r.t. $\pi$, denoted $\mathcal{E}_p$, is the (possibly disconnected) subgraph of $\pi$ made by all links depending on $p$;*
- *the* spreading *of $w$ over $\pi$, denoted by $w.[\pi]$, is the product of $w$ for $\pi$, i.e., $\pi$ in which we replace each weight $v$ with the product of weights $vw$;*
- *$\pi[\rho'/\rho]$ denotes the* substitution[3] *in $\pi$ of a sub-graph (or* module*) $\rho$ with a graph $\rho'$.*

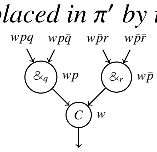
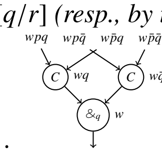
**Definition 6 (canonical form).** *Let $\pi$ be a PS, $\rho$ a sub-graph of $\pi$ and $\pi'$ and $\rho'$ two graphs. We say that $\pi$* commutes to *$\pi'$, denoted $\pi \rightsquigarrow \pi'$, if $\pi'$ is obtained from $\pi$ by replacing $\rho$ with $\rho'$, i.e. $\pi' = \pi[\rho'/\rho]$, by one of the following* commutation rules*:*

- *R0: $\pi \rightsquigarrow \pi$, by R0, if $\pi' = \pi[\rho'/\rho]$ where $\rho$ and $\rho'$ are the two modules below:*

  $\rho$ :  $\rho'$ : 
- *R1: $\pi \rightsquigarrow \pi'$, by R1, if $\pi' = \pi[\rho'/\rho]$ where $\rho$ and $\rho'$ are as below and $\bullet \in \{\otimes, \invamp\}$:*

  $\rho$ :  $\rho'$ : 
- *R2 : $\pi \rightsquigarrow \pi'$, by R2, if $\pi' = \pi[\rho'/\rho]$ where $\rho$ and $\rho'$ are as below, for $i = 1, 2$:*

  $\rho$ :  $\rho'$ : 
- *$R_3$: $\pi \rightsquigarrow \pi'$, by R3, if $\pi' = \pi[\rho'/\rho]$ where $\rho$ and $\rho'$ are as below with the condition that every weight $w$ in $\pi$ containing an occurrence of $r$ (resp., of $\bar{r}$) has been replaced in $\pi'$ by the weight $w' = w[q/r]$ (resp., by the weight $w' = w[\bar{q}/\bar{r}]$).*

  $\rho$ :  $\rho'$ : 

*In every rule, $\rho$ is called a* redex *of $\pi$ (resp., a* reductum *of $\pi'$) moreover, the (unique) contraction node that occurs in the redex is called the contraction (node) in* commutation condition*. We say that a PS $\pi$ is in* canonical form *(CPS) iff it does not contain any redex $\rho$ (or, equivalently, it has no contraction in commutation condition).*

*A proof net $\pi$ is in canonical form (it is a* **canonical proof net***, CPN) iff $\pi$ is a CPS.*

[3] Observe that in general substitutions may not preserve the property of being a proof structure.

**Proposition 1 (canonical form).** *Let $\pi$ be a PS, $R_i$ $(0 \le i \le 3)$ be one of the commutation rules of Definition 6 and $\pi'$ be the graph s.t. $\pi \rightsquigarrow \pi'$ by $R_i$. If $\pi$ is a PN then:*

(1) *$\pi'$ is a PN too with the same conclusions of $\pi$;*

(2) *there exists a CPN $\pi^c$, with the same conclusions of $\pi$, s.t. it can be obtained from $\pi$ by applying a finite number of instances of commutation rules, i.e. $\pi \rightsquigarrow^* \pi^c$.*

*Proof.* (1) Commutation rules $R0, R1$ and $R2$, trivially preserve the property of being a PN. In case of an instance of $R3$ we need first to ensure that the dependence condition 4 of Definition 2 is preserved. We show that if $L'$ is a node of $\pi'$ whose weight $w'$ depends on an eigen variable $q$ then $w' \le w$ where $w$ is the weight of the $\&_q$-link in $\pi'$.

$[P1]$: observe that since $\pi$ is correct by hypothesis then, neither $\epsilon_r$ nor $\epsilon_q$ nor $\epsilon_r$ may occur in $w$ and, for similar reasons, if $v$ is the weight of the $\&_p$ then, neither $\epsilon_r$ nor $\epsilon_q$ nor $\epsilon_p$ may occur in the weight $v$ of $\&_p$ neither in $\pi$ nor in $\pi'$.

There are two possible cases for $w'$:

1. either $w'$ is not effected by the substitution $[q/r]$, i.e. $w' = w'[q/r]$ and in this case since $w' \le wp$ (by hypothesis is in $\pi$) then $w' \le w$ also in $\pi'$ (by $[P1]$);
2. or $w'$ is effected by the substitution $[q/r]$ that is, $w' = w''[q/r]$ where $w''$ was the weight of $L'$ in $\pi$ before the substitution of the commutation rule $R3$. Since $w''$ is the weight of $L'$ in $\pi$ depending on the weight of $\&_r$, we know that $w'' \le w\bar{p}$ so $w''$ has one of the following forms, $w'' = v'rw\bar{p}$ or $w'' = v'\bar{r}w\bar{p}$. Assume $w'' = v'rw\bar{p}$, then $w'' = v'rw\bar{p} \le w\bar{p}$ and so, by transitivity, $w'' = v'rw\bar{p} \le w$. Now, since $\epsilon_r$ does not occur in $w$ (by $[P1]$), the substitution $v'rw\bar{p}[q/r] = w''[q/r] = w'$ thus $w' \le w$.

It is not difficult to show that if $L'$ is a node of $\pi$ depending on an eigen weight $s \ne r$ then $w'' = w'[q/r] \le v$ where $w''$ and $v$ are, resp., the weights of $L'$ and $\&_s$ in $\pi'$. We only show the case when $s = p$ and we omit the rest, that is, we show that if $w'$ is the weight of a node $L'$ depending on the eigen weight $p$ of the node $\&_p$, then $w' \le v$ where $v$ is the weight of $\&_p$ in $\pi'$. Observe that since $\pi$ is correct then $v$ cannot depend neither on $q$ nor on $r$ in $\pi$ and so $v$ cannot depend on $q$ (after eventually the substitution) in $\pi'$ (otherwise we can easily find a switching in $\pi$ containing a cycle). Now, if $w'$ in $\pi$ has the following form $w' = v'\epsilon_r\epsilon_p$, since by correctness of $\pi$ neither $\epsilon_r$ nor $\epsilon_q$ nor $\epsilon_p$ may occur in $v$, from $w' = v'\epsilon_r\epsilon_p \le v$ in $\pi$ we conclude $w'[q/r] \le v$.

Finally, in order to show that an instance of commutation rule $R3$ also preserves correctness we reason by contradiction. Assume by absurdum that $\pi'$ is not correct, let us say that there exists a switching $S_\varphi(\pi')$ with a cycle or a disconnected component. Assume e.g. that $S_\varphi(\pi')$ (with e.g. $\varphi(\bar{q}) = 1$) contains a cycle, then this cycle must contain at least a node $L'$ whose weight $w'$ contains an occurrence of $\epsilon_q$ that replaced an occurrence of $\epsilon_r$ in $w''$ where $w''$ is the weight of $L'$ in $\pi'$ before the substitution (i.e., $w' = w''[q/r]$ or equivalently, $w'' = v.\epsilon_r$ and $w' = v.\epsilon_q$). Since $w''$ depends on $\&_r$ in $\pi$ then $w''$ has the form $w'' = vw\bar{p}\epsilon_r$ and since $w' = w''[\epsilon_q/\epsilon_r]$, then $w''$ must have the form $w'' = v.w\bar{p}\epsilon_r$ then it is easy to find a switching $S_\varphi(\pi)$ containing a cycle as in the two leftmost graphs of Fig.2, contradicting the assumption that $\pi$ is correct. We reason in a similar way in case $S_\varphi(\pi')$ contains a disconnected component.

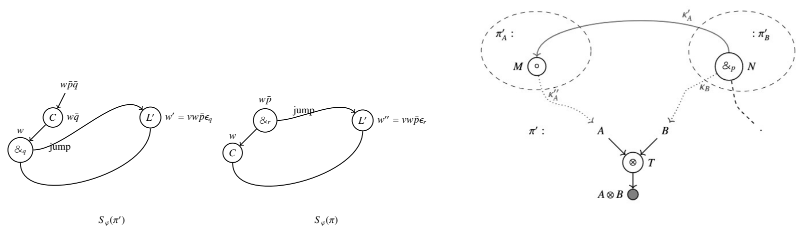
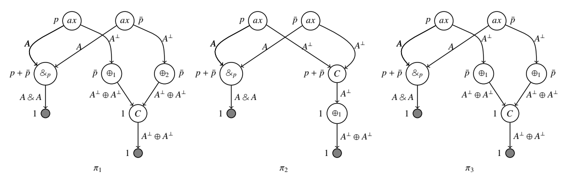


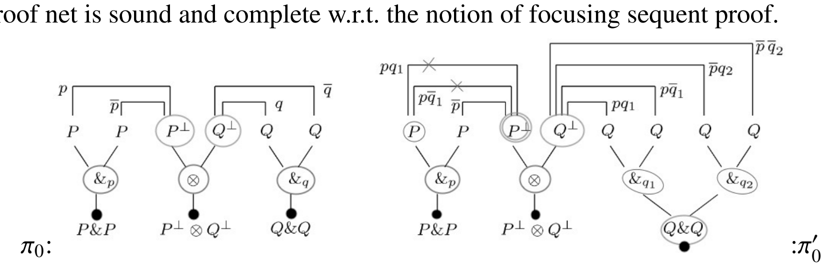
**Fig. 2.** cycles inside switchings (l.h.s.) and a blocking path for $T$ (rightmost h.s.)

(2) By induction on the sum of the logical degrees[4] of the formulas that are conclusions of the uppermost contraction nodes that are in commutation condition in $\pi$. $\square$

Instances of CPNs are given below: indeed, only $\pi_1$ and $\pi_2$ are canonical (proof nets).



Actually, the notion of canonical form allows us to exclude redundant structures from the realm of MALL proof nets. Consider e.g. the two instances of monomial PS, $\pi_0$ and $\pi'_0$ below, with same conclusions, given in the Appendix 2 of [6]: only $\pi_0$ is a CPN; indeed, $\pi_0$ can be obtained from $\pi'_0$ by iterating the commutation rules of Definition 6: the only allowed (canonical) contractions are the blue ones while the two red ones, with conclusions resp., $P$ and $P^\perp$ (in $\pi'_0$), are not allowed because they contract two identical axioms (thus, we can apply $R0$); moreover, the rightmost contraction with conclusion $Q \,\&\, Q$ is in commutation condition thus, by rule $R3$, it can be permuted with the two $\&$-nodes above, $\&_{q_1}$ and $\&_{q_2}$. In the following we show that the notion of cut-free canonical proof net is sound and complete w.r.t. the notion of focusing sequent proof.

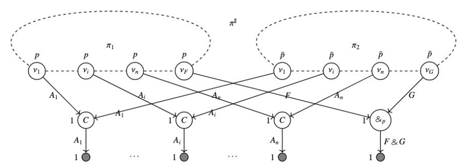



**Theorem 2 (de-sequentialization).** *A proof $\Pi$ of a sequent $\Gamma$ in $\Sigma_1$ can be mapped (ie.,* de-sequentialized*) into a CPN with same conclusions $\Gamma$.*

[4] The *logical degree* of a formula $F$, denoted $\partial(F)$, is defined by induction on the height of $F$: if $F$ is atomic then $\partial(F) = 0$, else $F$ has the form $F_1 \circ F_2$, with $\circ \in \{\invamp, \otimes, \&, \oplus\}$, and $\partial(F) = \partial(F_1) + \partial(F_2) + 1$.

*Proof.* By induction of the height of $\Pi$ via Proposition 1. All cases are easy except the case when last rule of $\Pi$ is an instance of the $\&$-rule, in this case we need to apply Proposition 1 in order to get a canonical proof net. Assume last rule of $\Pi$ is a $\&$-rule. By hypothesis of induction, $\Pi_1$ (resp., $\Pi_2$) de-sequentializes into the canonical proof net $\pi_1$ (resp., $\pi_2$) with same conclusions. We may then link together $\pi_1$ and $\pi_2$ by a $\&$-link between conclusions $F$ and $G$ and by adding $n$ a contraction $C$-links, one contraction for each pair of identical conclusions $A_i, A_i$ coming from resp., $\pi_1$ and $\pi_2$. Thus we build the proof net $\pi^\sharp$, as in the picture below, on which we may finally apply Proposition 1(3) in order to get the canonical proof net $\pi$ which $\Pi$ de-sequentializes to.

$$\Pi : \frac{\begin{array}{c}\Pi_1 \\ \vdash A_1, ..., A_i, ..., A_n, F\end{array} \quad \begin{array}{c}\Pi_2 \\ \vdash A_1, ..., A_i, ..., A_n, G\end{array}}{\vdash A_1, ..., A_i, ..., A_n, F \& G} \&$$



□

**Definition 7 (ready and splitting links).** *Let $\pi$ be a CPN. A link $L$ of $\pi$ is* ready (to sequentialization) *whenever deleting everything of $L$ except its premise(s) produces one or more sub-proof nets having among their conclusions the premise(s) of $L$. A conclusion of $\pi$ is ready if it is the conclusion of a ready link.*

*If $L$ is a terminal $\otimes$-link of $\pi$ of type $\frac{A \quad B}{A \otimes B}$, we say that $L$ is* splitting *for $\pi$ when removing $L$ from $\pi$ (we erase everything of $L$ except its premises) splits $\pi$ in two sub-proof nets: $\pi_A$, having $A$ among its conclusions and $\pi_B$, having $B$ among its conclusions.* **Split**$(\pi)$ *denotes the set of terminal tensor links that are splitting for $\pi$.*

*We say that $\pi$ with at least a terminal tensor link is in* splitting condition *iff it does not contain neither an asynchronous conclusion nor a ready conclusion of type $\oplus$.*

**Fact 2 (terminal links of type $\oplus_i$ or $⅋$ are ready)** *Assume $\pi$ is a CPN with conclusions $\Gamma, F$. If $L$ is a terminal link of type $\frac{A_i}{A_1 \oplus A_2} \oplus_i$ with $F = A_1 \oplus A_2$ (resp., of type $\frac{A \quad B}{A ⅋ B}$ with $F = A ⅋ B$) then, $L$ is a ready link and removing $L$ as in Definition 7 produces a sub-proof net $\pi_{A_i}$ (resp., $\pi_{A,B}$) with conclusions $\Gamma, A_i$ (resp., $\Gamma, A, B$).*

Note that a terminal tensor link of a proof net $\pi$ may be not ready to sequentialiation (it may be "non splitting" for $\pi$). A contraction link is never ready "alone": its "readiness" is subordinate to that one of the $\&$-link which this contraction depends on. In the following, we adopt some notions of [7] adapted to the case of MALL cut-free CPNs.

**Definition 8 (switching, descending and blocking paths).**

– *Given a CPS $\pi$, a* **jump graph** *for $\pi$ (or a* jumped PS $\pi$*), denoted $J(\pi)$, is the graph obtained by adding to $\pi$ some (possibly none) jumps; we allow in $J(\pi)$ jumps from an $\&_p$-node to a $C_p$ node depending on $p$*[5]*.*

[5] Note that a $J(\pi)$ differs from a switching $S(\pi)$ for the following facts: *(i)* we do not consider slices, *(ii)* we do not mutilate premises and *(iii)* there can be multiple (possibly, none) jumps exiting from a $\&_p$-node and going to different nodes depending on $p$ or $C_p$ nodes.

- *Fixed a $J(\pi)$ for $\pi$, a* **switching path** *$\gamma$ in $J(\pi)$ is a path that exists in some switching $S(\pi)$ of $\pi$. We say that n switching paths $\gamma_1, ...\gamma_n$ of $J(\pi)$ are* compatible *iff there exists a switching $S(\pi)$ s.t. $\gamma_1, ..., \gamma_n$ are paths of $S(\pi)$.*
- *If e is an edge of $\pi$, its* **descending path** *$\delta(e)$ is the unique directed path starting from e and ending with the premise of a terminal node. If N is a node other than the axiom then, $\delta(N)$ denotes the descending path of the unique conclusion of N. $\delta(N)$ is empty if and only if N is terminal.*
- *Let T be a $\otimes$ node of a proof net $\pi$ and $J(\pi)$ one of its possible jump graphs:*
  *– a* **correctness/blocking node** *for T is a node N of type $\triangledown \in \{\parr, \&\}$ with two disjoint switching paths of $J(\pi)$, $\kappa_0$ and $\kappa_1$, going from T to N and s.t. both paths start with a premise of T and end with a premise of N or a jump of N in case N is a $\&$-node; $\kappa_0$ and $\kappa_1$ are called* **correctness paths** *for T;*
  *– a* **blocking path** *for T is a path $\gamma$ in $J(\pi)$ that goes from one premise to the other of T (without passing through the conclusion of T) and bouncing on both the premises (resp., on one premise and one jump) of a blocking node N of type $\parr$ (resp. of type $\&$); in other words, $\gamma$ starts from one premise of T, it ends with the second premise of T and it also enters one premise of N and immediately exits the other premise of N (or it enter one premise of N and exits with a jump, or the other way round, in case $N = \&$); thus $\gamma = \kappa_A \cdot N \cdot \kappa_B$ appears as in the graph (A) of Fig. 3 where "$\cdot$" denotes the concatenation of switching paths.*

E.g. graph (F) of Fig. 3 is an instance of jumped CPN in which the (unique) $\parr$-node is a blocking node for $\otimes_3$ while the (unique) $\&_p$-node is a blocking node for $\otimes_2$.

Next two Facts and Lemmas 1, 2 and 3 are used to prove Lemma 4 which is necessary for the Ready Lemma 5, the "pivot" lemma of the Focusing Theorem 3.

**Fact 3** *The two correctness paths, $\kappa_A$ and $\kappa_B$ for $T = \frac{A \quad B}{A \otimes B}$, are compatible switching paths and $\kappa \cdot T \cdot \kappa_B$, i.e., the path going from $\kappa_A$ to $\kappa_B$ (or the other way round) and bouncing on the two premises of T (without going through the conclusion of T), is a compatible switching path too.*

**Fact 4** *If N is a node of a CPN other than the ax-node, then $\delta(N)$ is a switching path.*

**Lemma 1** **(blocking contraction).** *Let $\pi$ be a CPN with conclusions $\Gamma = A_1, ..., A_n$ s.t. none of them is conclusion of a terminal $\&$-link.*

(i) - *If some $A_i$ is conclusion of a terminal contraction link $L_i$ and N is the $\&$-node which $L_i$ depends on, then neither $A_i$ (i.e., $L_i$) nor N is ready (N is called the* blocking contraction node *for L) see picture (B) of Fig.3.*

(ii) - *Moreover, there does not exist any switching path exiting the conclusion of N and stopping (downwards) at $L_i$ as e.g. $\gamma'$ and $\gamma''$ in pictures (C) and (D) of Fig. 3.*

*Proof.* (i) - By definition of PS, every contraction link depends on the eigen variable of some $\&$ link of $\pi$, thus in particular any terminal contraction link $L_i$ depends on the eigen weight $p$ of some $\&_p$ link of $\pi$ and since by assumption none terminal link of $\pi$ is of type $\&$, the $\&_p$-node must be above some conclusion $A_j$ of $\pi$ (by correctness of $\pi$, $i \neq j$); thus the $\&_p$-link is not ready to sequentialization yet as in Fig.3(B) of where the terminal link $\circ$ below the $\&_p$-link is such that $\circ \in \{\oplus, \otimes, \parr\}$).

(ii) - it follows by correctness of $\pi$ (see graphs (C) and (D) of Fig.3). $\square$

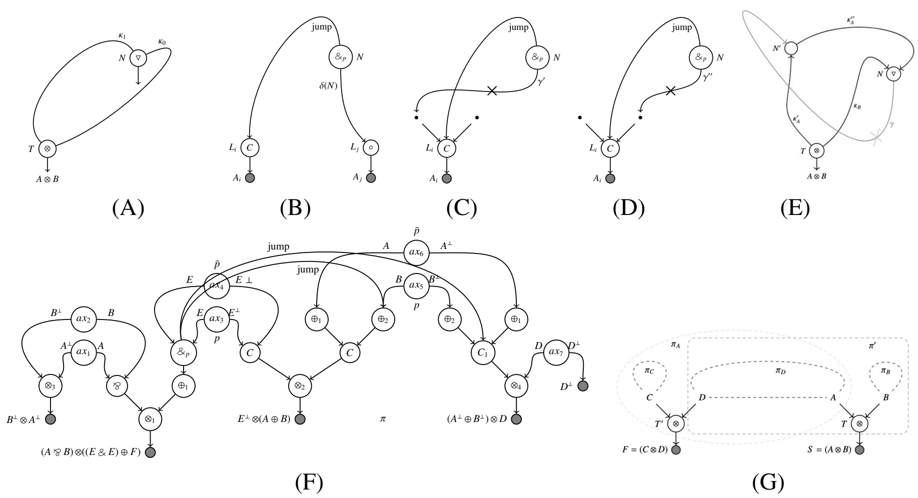


**Fig. 3.** switching, blocking, descendent and correctness paths, and splitting conclusions

E.g., the $\&_p$-node of picture (F) of Fig. 3 is blocking for contraction node $C_1$.

**Lemma 2.** *If $\pi$ is a PN with conclusions $A_1, ..., A_n$ then there exists at least a conclusion $A_i$ that is not conclusion of a contraction link.*

*Proof.* Assume $\pi$ is a PN having only terminal contraction links, $L_1, ..., L_n$. Then, by Lemma 1 if $N$ is the node $\&_p$ which $L_i$ depends on then, $N$ is not ready so it must be above a terminal contraction link $L_j$ with $i \neq j$, by correctness of $\pi$. Since the weight of $L_i$ is 1 (by definition of PS), by the dependency condition, also the weight of $N$ must be 1 but then the $\&_p$-link cannot be above any contraction link $L_j$, a contradiction. □

**Lemma 3 (blocking splitting).** *Let $\pi$ be a PN with a terminal tensor link $T$ that is not splitting for $\pi$ then:* (i) *there exists a blocking node $N$ of type $\triangledown \in \{\parr, \&\}$ for $T$ moreover,* (ii) *every switching path exiting the conclusion of $N$ and compatible with $\kappa_0$ and $\kappa_1$ cannot contain any node of the correctness paths, $\kappa_0$ and $\kappa_1$, for $T$.*

*Proof.* By induction on the number $n$ of $\&$-nodes of $\pi$. If $n = 0$ (we are in the MLL case, [7]) then let $T$ be a terminal $\otimes$ node and $S(\pi)$ be a (multiplicative) switching: the removal of $T$ splits $S(\pi)$ into two connected components (by the ACC-correctness). If all $\parr$ nodes are such that both their premises belong to the same connected component, then $T$ is a splitting node for $\pi$ since the removal of $T$ in $\pi$ has two connected components as well (which are ACC-correct). Otherwise there exists a $\parr$ node $N$ with a premise in each connected component of the removal of $T$ in $S(\pi)$. Each of these components contains a premise of $T$ and a premise of $N$ and (by connectivity) a path from the first to the second. The two obtained paths are switching and disjoint. Finally, in the component containing $N$, the obtained path cannot contain the conclusion of $N$

otherwise, one could connect the two paths and obtain a cycle in the correctness graph $S'(\pi)$ obtained from by $S(\pi)$ by changing the choice of the premise of $N$; for similar reasons (since by Fact 4, $\delta(N)$ is a switching path), $\delta(N)$ cannot meet neither $\kappa_A$ nor $\kappa_B$ (see graph (E) of Fig.3).

Otherwise, $n > 0$ then, we assume by absurdum that $\pi$ is the smallest (w.r.t. the graphical size) proof net containing a terminal $\otimes$ node $T$ that is not splitting for $\pi$ and such that there exists no blocking node $N$ for $T$. Le us choose a $\&$ node $N$ in $\pi$ in such a way that it is as low as possible (i.e., the weight of $N$ is 1). It is always possible to find such a node $N$ in a correct proof net, otherwise we can find a switching for $\pi$ containing a cycle, contradicting the correctness assumption. Let $p$ be the eigen weight of $N$ and let $\pi \downharpoonleft^p$ the restriction of $\pi$ w.r.t. $p$. Clearly $\pi \downharpoonleft^p$, after having properly removed the residual unary $\&$ or $C$ nodes left, is a correct proof net, let us say $\pi'$, of smaller size than $\pi$ but with still the same terminal tensor links of $\pi$ (although the premises of these tensors may have been changed). Thus, in $\pi'$ every terminal tensor link is either splitting or there exists a blocking node for such a tensor.

1. if $T$ is not splitting for $\pi'$ then there exists a blocking node $N'$ of type $\triangledown$ for $T$ which is also blocking for $T$ in $\pi$: we can consider a switching $S'_\varphi(\pi)$ that is the switching $S'(\pi)$ induced by a valuation $\varphi$ s.t. $\varphi(p) = 1$ (resp., $\varphi(\bar{p}) = 0$);
2. otherwise, the removal of $T$ splits $\pi'$ into two sub-proof nets, $\pi'_A$ and $\pi'_B$ with $A$ and $B$ the two premises of $T$. Now, if we restore the restriction $\pi \downharpoonleft^p$ then, the removal of $T$ from $\pi \downharpoonleft^p$ induces two graphs, $(\pi \downharpoonleft^p)_A$ $(\pi \downharpoonleft^p)_B$, corresponding resp., to $\pi'_A$ and $\pi'_B$ after removing the residual "unary" node $\&_p$ and all residual "unary" contraction links of type $C_p$. Note that the residual unary node $\&_p$ must occur either in $(\pi \downharpoonleft^p)_A$ or in $(\pi \downharpoonleft^p)_B$; let us say $\&_p$ stays in $(\pi \downharpoonleft^p)_B$. Then, by correctness we know that: $[P1]$: for every switching $S_\varphi(\pi \downharpoonleft^p)_B)$ there is a path from $B$ to the unary node $\&_p$. There are two cases for $(\pi \downharpoonleft^p)_A$:
   (a) either no residual unary $C_p$ node depending on $p$ occurs in $(\pi \downharpoonleft^p)_A$; this means that $\pi'_A$ (the proof net obtained from the restriction $(\pi \downharpoonleft^p)_A$, after the removal of residual unary node, contains none node depending on $p$ therefore it $\pi'_A$ is a sub-proof net of $\pi$ therefore $T$ is splitting for $\pi$; a contradiction;
   (b) or at least a residual unary $C$ node $M$ depending on $p$ occurs in $(\pi \downharpoonleft^p)_A$. Then we can easily build a jump graph for $\pi'$, let us say $J(\pi')$ containing a blocking path for $T$ as in the rightmost h.s. picture of Fig.2, a contradiction:
      i. consider the switching path $\kappa_A$ that starts with the jump from $\&_p$ to $M$ (i.e., $\kappa'_A$) and continues up to the left premise $A$ of $T$ (i.e., $\kappa''_A$), and then consider
      ii. the second switching path $\kappa_B$ starting from the unique premise of $\&_p$ and continuing up to the right premise $B$ of $T$; this path exists by $[P1]$.

      See e.g., also Fig.3(F) with $T = \otimes_2$ and $N = \&_p$.

In order to show *(ii)*, observe that $\delta(N)$ cannot meet $\kappa_A$ (resp., $\kappa_B$) in a node, let us say $M'$ (resp., $M''$) otherwise we could find a switching for $\pi'$ containing a cycle starting from the conclusion of $N = \&_p$ and following the red path in the graph (E) of Fig.3. □

**Lemma 4 (splitting).** *If $\pi$ is a CPN in splitting condition then* **Split**$(\pi) \neq \emptyset$ *(see Def.7).*

*Proof.* Assume $\pi$ is a CPN in splitting condition, then by Definition 7 and by Contraction Lemma 1 none conclusion of $\pi$ that is conclusion of a contraction link is ready (that holds in particular for all synchronous conclusions of type $F \oplus G$). Assume by absurdum that none synchronous conclusion is ready thus none tensor conclusion is splitting for $\pi$. We show that there exists a jump graph $J(\pi)$ for $\pi$ containing a switching path with a cycle, contradicting the assumption $\pi$ is correct. Let $F_i = A_i \otimes B_i$ be the conclusion of a terminal tensor link $T_i$ of $\pi$.

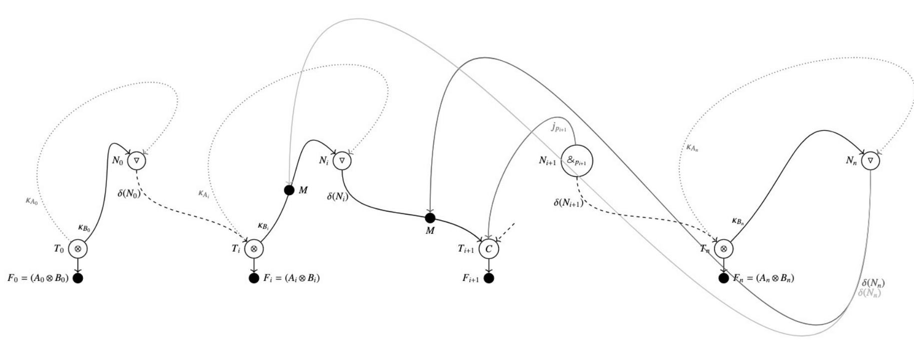
[$P$.1] Since by assumption $T_i$ is not splitting then, by Lemma 3, there exists a blocking node $N_i$ (of type $\triangledown$) for $T_i$ with two correction paths, $\kappa_{A_i}$ and $\kappa_{B_i}$. Starting from $N_i$, we follow $\delta(N_i)$ until we reach a terminal node $T_j$ which, by Lemma 3(ii), must be different from $T_i$. By correctness of $\pi$, $\kappa_{A_i}$, $\kappa_{B_i}$ and $\delta(N_0)$ are compatible switching paths. There are now two cases for $T_j$:

1. either $T_j$ is a terminal tensor link then, we continue as before in [$P$.1] (with $T_j$ at the place of $T_i$) until we reach a new conclusion $T_k$ which must be different from all already visited conclusions, by Lemma 3(ii);
2. or $T_j$ is a terminal contraction link then, we then continue following the jump $j_p$ (taken in the opposite direction) to the $\&_p$-node $N_j$ on which the contraction $T_j$ depends on and then we continue with the descendent path $\delta(N_j)$ until we reach a new conclusion $T_k$ which, by Lemma 1, must be different from all terminal nodes previously visited. Observe that, by correctness of $\pi$, the composition of switching paths, $T_j \cdot j_p \cdot (\&_p = N_j) \cdot \delta(N_j)$, entails a compatible switching path.

Iterating steps 1 and 2 above, we build an infinite sequence

$$\nu = \kappa_{A_i} \cdot N_i \cdot \delta(N_i) \cdots T_j \cdot j_p \cdot N_j \cdot \delta(N_j) \cdots T_k \cdot \kappa_{B_k} \cdot N_k \cdot \delta(N_k) \cdots$$

where $A_i, B_i$ are the premises of a generic terminal tensor link $T_i : \frac{A_i \quad B_i}{A_i \otimes B_i}$ and $j_{p_i}$ denotes a jump (taken in the opposite direction) going from a $\&_p$ node to a terminal contraction node $T_j$ depending on $p$. Since $\pi$ is finite, $\nu$ must visit twice a same node $M$. Observe that $\nu$ exists in a $J(\pi)$ and it is not difficult to show that there exists a switching $S(\pi)$ containing such a $\nu$ (all components of $\nu$ are compatible since steps 1 and 2 preserve compatibility), contradicting the correctness of $\pi$. Next figure illustrates two possibilities for node $M$ depending on whether the descendent path $\delta(N_n)$ meets a descendent path (black option) or a correction path (grey option) already visited by $\nu$; indeed, it does not matter the type of node $M$ we can always find a cycle in a switching path.




□

E.g., Split($\pi$) = $\{\otimes_1, \otimes_4\}$ for the CPN $\pi$ of Fig.3(F) (neither $\otimes_2$ nor $\otimes_3$ is splitting for $\pi$).

**Lemma 5 (ready).** *If $\pi$ is a CPN s.t. it is not an ax-link and it has no asynchronous conclusions, then there exists a terminal link that is a ready $\oplus_i$-link or a splitting $\otimes$-link.*

*Proof.* Let $\pi$ be a CPN s.t. it is not an *ax*-link and it has no asynchronous conclusion. If $\pi$ has no terminal contraction links and, since $\pi$ is not reduced to an axiom link, it must contain at least a terminal synchronous links $L$ ($\oplus_i$ or $\otimes$); if $L$ is an $\oplus_i$-link then it is trivially ready, otherwise $\pi$ has only terminal $\otimes$-links then it is in splitting conditions and so, by the Splitting Lemma 4, there exists a splitting link.

Otherwise, if $\pi$ contains terminal contractions then, by Lemma 2, it cannot contain only terminal contraction links and, since it is not reduced to an axiom link and it has no asynchronous conclusions, it must contain at least a synchronous terminal link $L$; if $L$ is an $\oplus_i$-link then it is trivially ready, otherwise $\pi$ has only terminal links that are contractions (of non asynchronous formulas) or terminal $\otimes$-links then it is in splitting condition and so, by the Splitting Lemma 4, there exists a splitting link. □

**Definition 9 (focusing conclusions).** *Let $\pi$ be a CPN and $F$ be one of its conclusions. $F$ is* focusing *for $\pi$ (we write, $F \in$* **Foc**($\pi$)*) iff one of the following conditions holds:*

1. *$F$ is a positive atom and $\pi$ is reduced to an axiom link.*
2. *$F$ is the conclusion of a terminal $\oplus_i$-link $L$ of type $\frac{A_i}{A_1 \oplus A_2}\oplus_i$ and $A_i$ is asynchronous or a negative atom or $A_i \in Foc(\pi_{A_i})$, for $1 \leq i \leq 2$.*
3. *$F = (A \otimes B) \in Split(\pi)$ and $\pi$ is split at $F$ into two sub-PNs, $\pi_A$ and $\pi_B$, and*
   (a) *$A$ is asynchronous or a negative atom or $A \in Foc(\pi_A)$ and*
   (b) *$B$ is asynchronous or a negative atom or $B \in Foc(\pi_B)$;*

*where $\pi_{A_i}$ (resp., $\pi_A, \pi_B$) is (resp. are) the sub-proof net(s) obtained by removing from $\pi$ the vertex $\oplus_i$ (resp., $\otimes$) of $L$ together with its outgoing edge $A_1 \oplus A_2$ (resp., $A \otimes B$).*

**Proposition 2.** *Let $\pi$ be a CPN with no asynchronous conclusion.*

1. *If $L$ is a terminal $\otimes$-link, $\frac{A \quad B}{A \otimes B}\otimes$, that is splitting for $\pi$ and $\pi_A$ and $\pi_B$ are the two CPNs obtained by splitting $\pi$ at $L$ and $A$ is not a negative atom then $Foc(\pi_A) \setminus \{A\} \subseteq Foc(\pi)$ (and similarly for the $B$ side).*
2. *If $L$ is a terminal $\oplus_i$ link, $\frac{A_i}{A_1 \oplus A_2}\oplus_i$, for $1 \leq i \leq 2$, and $\pi_{A_i}$ is the sub-CPN obtained by removing $L$ and $A_i$ is a non negative atom then $Foc(\pi_{A_i}) \setminus \{A_i\} \subseteq Foc(\pi)$.*

*Proof.* We only discuss case 1 (case 2 is simpler so we omit it). Assume $S = A \otimes B$ is splitting for $\pi$ with no asynchronous conclusion. We reason by induction on the size of $\pi$. We show that if $F \in \mathrm{Foc}(\pi_A) \setminus \{A\}$ then $F \in \mathrm{Foc}(\pi)$. Since $F$ is focusing in $\pi_A$, there are three cases to consider according to Definition 9:

1. $F$ is a positive atom and $\pi_A$ is reduced to an axiom link, with conclusions $F$ and $F^\perp$, one of which being $A$. But, by hypothesis, $A$ is not a negative atom, hence $A \neq F^\perp$; moreover, by hypothesis, $F \in \mathrm{Foc}(\pi_A) \setminus \{A\}$, hence $A \neq F$. Contradiction.

2. $F = (C \otimes D) \in \mathrm{Foc}(\pi_A)$ and $\pi_A$ is split at $F$ into two sub-CPNs, $\pi_C$ and $\pi_D$ s.t.:
   [$\star$1] $C$ is asynchronous or a negative atom or $C \in \mathrm{Foc}(\pi_C)$;
   [$\star$2] $D$ is asynchronous or a negative atom or $D \in \mathrm{Foc}(\pi_D)$.
   Since $A$ is a conclusion of $\pi_A$ different from $F$ (by hypothesis, $F \in \mathrm{Foc}(\pi_A) \setminus \{A\}$) and $\pi_A$ is split at $F$ into $\pi_C$ and $\pi_D$ then $A$ must be in the conclusions of $\pi_C$ or $\pi_D$. We assume, without loss of generality, that $A$ is a conclusion of $\pi_D$ (other than $D$, obviously). Let $\pi'$ be the PS consisting of $\pi_D$ and $\pi_B$ and the splitting link of $\pi$ at $S$, as in the picture (G) of Fig.3. It is not difficult to see that:
   [$\star$3] $\pi'$ is a CPN split at $S$ into $\pi_D$ and $\pi_B$ and
   [$\star$4] $\pi$ is split at $F$ into $\pi_C$ and $\pi'$.
   In case $D \in \mathrm{Foc}(\pi_D)$, then $D$ must be synchronous otherwise $\pi_D$ must be an axiom and, since $A \neq D$, we would get $A = D^\perp$, contradicting the assumption; now, since $\pi'$ is smaller (in size) than $\pi$, by the induction hypothesis applied to [$\star$3], we infer:
   [$\star$5] $\mathrm{Foc}(\pi_D) \setminus \{A\} \subseteq \mathrm{Foc}(\pi')$ so $D \in \mathrm{Foc}(\pi')$ since $D \in \mathrm{Foc}(\pi_D) \setminus \{A\}$ and $A \neq D$.
   From [$\star$5], [$\star$2] and $D \neq A$, we get:
   [$\star$6] $D$ is asynchronous or a negative atom or $D \in \mathrm{Foc}(\pi')$.
   From [$\star$1], [$\star$6] and [$\star$4], by Definition 9, we conclude that $F \in \mathrm{Foc}(\pi)$.
3. $F = C \oplus D$ is a synchronous formula of $\pi_A$ that is conclusion of a ready terminal link $\oplus_1$ (resp., $\oplus_2$) and $\pi_C$ (resp., $\pi_B$) is the sub-CPN s.t. $C$ (resp., $D$) is asynchronous or a negative atom or $C \in \mathrm{Foc}(\pi_C)$ (resp., $D \in \mathrm{Foc}(\pi_D)$) then, we proceed as in case 2.

$\square$

**Theorem 3 (focusing).** *If $\pi$ is a PN with no asynchronous conclusion then, Foc($\pi$) $\neq \emptyset$.*

*Proof.* We proceed by contradiction. Assume there exists a CPN $\pi$ with no asynchronous conclusion and s.t. $\mathrm{Foc}(\pi) = \emptyset$. We choose $\pi$ of minimal size. There are two cases:

1. Either $\pi$ has no synchronous conclusion then, since it contains neither asynchronous conclusion (by assumption) and since by Lemmas 1 and 2 it cannot contain any contraction link, $\pi$ must be an axiom link. But then, one of the two conclusions must be a positive atom which, by Definition 9, is focusing for $\pi$. Contradiction.
2. Or $\pi$ does contain at least one synchronous conclusion, and since it contains no asynchronous conclusion, by application of the Ready Lemma 5, we know that there exists a synchronous conclusion $F$ of $\pi$ that is either a ready conclusion of type $A \oplus B$ or a splitting conclusion of type $A \otimes B$. We only discuss the latter case (the former case is similar, so omitted). Assume there exists a synchronous conclusion $F = A \otimes B$ of $\pi$ which splits $\pi$ into two sub-proof-nets, $\pi_A$ and $\pi_B$. Suppose that:
   [$\star$1] $A$ is neither asynchronous nor a negative atom.
   By construction, the conclusions of $\pi_A$ other than $A$ are conclusions of $\pi$ hence not asynchronous. Since $A$ itself is not asynchronous by [$\star$1], then none of the conclusions of $\pi_A$ are asynchronous. Since $\pi_A$ is strictly smaller than $\pi$, which is a PN of minimal size without asynchronous nor focusing conclusions, we infer that:
   [$\star$2] $\mathrm{Foc}(\pi_A) \neq \emptyset$.
   Now, $A$ is not a negative atom by [$\star$1], hence by Proposition 2, we have that:
   [$\star$3] $\mathrm{Foc}(\pi_A) \setminus \{A\} \subseteq \mathrm{Foc}(\pi)$.
   Since $\mathrm{Foc}(\pi) = \emptyset$, by [$\star$3], we conclude that $\mathrm{Foc}(\pi_A) \subseteq \{A\}$ and thus $\mathrm{Foc}(\pi_A) = \{A\}$, by [$\star$2]. Hence $A \in \mathrm{Foc}(\pi_A)$. Thus, by discharging hypothesis [$\star$1], we conclude:

[$\star$4] $A$ is asynchronous or a negative atom or $A \in \mathrm{Foc}(\pi_A)$.

Symmetrically, we can equally prove that:

[$\star$5] $B$ is asynchronous or a negative atom or $B \in \mathrm{Foc}(\pi_B)$.

From [$\star$4] and [$\star$5], by Definition 9, we conclude that $F \in \mathrm{Foc}(\pi)$. Contradiction. $\square$

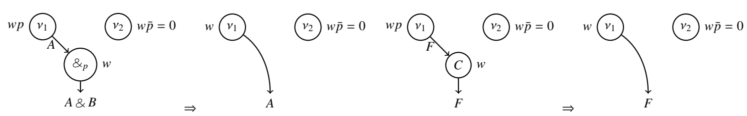
Consider e.g. the CPN $\pi$ of Fig.3(F) then, $\mathrm{Foc}(\pi) = \{(A \parr B) \otimes ((E \,\&\, E) \oplus F)\}$.

The canonical form of PNs (Proposition 1) together with the Focusing Theorem 3 assure a seqentialization strategy (Theorem 4) mapping CPNs in to focusing proofs.

**Theorem 4 (focusing sequentialization).** *A cut-free CPN $\pi$ with conclusions $\Gamma$* sequentializes *into a focusing MALL sequent proof $\pi^\sharp$ with conclusion:*
$\star$ $\vdash \Gamma' \Downarrow F$*, if $\Gamma = \Gamma', F$ does not contain asynchronous conclusion and $F \in Foc(\pi)$;*
$\star$ $\vdash \Gamma' \Uparrow L$ *otherwise, where $\Gamma = \Gamma', L$ and $\Gamma'$ is a multiset of non-asynchronous formulas and $L$ is a list of formulas.*

*Proof.* By induction on the size of $\pi$. If $\pi$ is an (atomic) axiom link with conclusions $A, A^\perp$ then, $\pi^\sharp : \frac{\vdash A^\perp \Downarrow A}{\vdash A^\perp, A \Uparrow}$. Otherwise, if $\pi$ contains an asynchronous conclusion $F$ then since $\pi$ is in canonical form, $F$ is conclusion of a terminal asynchronous link $L$, $\parr$ or $\&$.

1. If $L : \frac{A \quad B}{A \parr B}$ and $\pi$ has conclusions $\Gamma = \Gamma', A \parr B$ then we can remove $L$ (the vertex $\parr$ together with its conclusion edge labelled by $A \parr B$) and get a CPS $\pi_{A,B}$ with conclusions $\Gamma', A, B$ which is trivially correct. By hypothesis of induction $\pi_{A,B}$ sequentializes in to $\pi'^\sharp :\vdash \Gamma'' \Uparrow L, A, B$ from which we conclude by an instance of $\parr$-rule $\pi^\sharp : \frac{\vdash \Gamma'' \Uparrow L, A, B}{\vdash \Gamma'' \Uparrow L, A \parr B}$. Note that some instances of $R \Uparrow$ can be applied in case that $A$ or $B$ were no longer asynchronous formulas.
2. If $L$ is a link of type $\&$, i.e. $L : \frac{A \quad B}{A \,\&\, B}$, with eigen weight $p$ and $\Gamma = \Gamma', A \,\&\, B$, then:
   (a) take the restriction of $\pi$ w.r.t. $p$, $\pi \downharpoonleft^{p}$ (resp., the restriction of $\pi$ w.r.t. $\overline{p}$, $\pi \downharpoonleft^{\overline{p}}$);
   (b) in $\pi \downharpoonleft^{p}$ (resp., $\pi \downharpoonleft^{\overline{p}}$) erase the (unique) vertex labeled by $\&_p$ and merge its emergent edge (its conclusion) together with its unique incident edge (its unique non-null premise) labelled by $A$ (resp., by $B$), as in the figure below;
   (c) in $\pi \downharpoonleft^{p}$ (resp., $\pi \downharpoonleft^{\overline{p}}$) erase every residual (unary) vertex of type $C_p$ and merge its outgoing edge (the conclusion) together with its unique incident edge (the unique non-null premise) labelled by the contracted formula $F$ (as below).




   The resulting graph is a proof net $\pi_A$ (resp., $\pi_B$) with conclusions $\Gamma', A$ (resp., $\Gamma', B$). By hypothesis of induction, $\pi_A$ and $\pi_B$ sequentialize in to $\pi_1^\sharp :\vdash \Gamma'' \Uparrow L, A$ and $\pi_2^\sharp :\vdash \Gamma'' \Uparrow L, B$ thus by an instance of $\&$-rule we conclude $\pi^\sharp : \frac{\vdash \Gamma'' \Uparrow L', A \quad \vdash \Gamma'' \Uparrow L', B}{\vdash \Gamma'' \Uparrow L', A \,\&\, B}$ where $\Gamma' = \Gamma'', L$. Note that some instances of $R \Uparrow$ could be applied upwards in case that $A$ or $B$ were no longer asynchronous formulas.

In case $\pi$ has no asynchronous conclusions, since by hypothesis is not an axiom link, at least one of its conclusions is conclusion of a synchronous link (by the Ready Lemma 5) then, by Focusing Theorem 3, there exists $F \in \mathrm{Foc}(\pi)$. Assume $F = A \otimes B$.

1. If $A \in \mathrm{Foc}(\pi_A)$ and $B \in \mathrm{Foc}(\pi_B)$, we apply the hypothesis of induction on $\pi_A$ and $\pi_B$ and we get two focusing proofs, $\pi_1^\sharp : \Gamma_1' \Downarrow A$ and $\pi_2^\sharp : \Gamma_2' \Downarrow B$, that we can assemble together in to the proof $\pi^\sharp$ as in the l.h.s. below.
2. Otherwise, in the case $A$ (resp., $B$) is a negative atom or an asynchronous formula, we apply the hypothesis of induction on $\pi_A$ (resp., $\pi_B$) and we get a focusing proof $\pi_1^\sharp :\vdash \Gamma_1', A \Uparrow$ (resp., $\pi_2^\sharp :\vdash \Gamma_2' \Uparrow B$) from which we conclude with a proof $\pi^\sharp$ as in the r.h.s. below (where e.g., $A$ is a negative atom and $B$ is an asynchronous formula).

$$\pi^\sharp : \frac{\Gamma_1' \Downarrow A \qquad \Gamma_2' \Downarrow B}{\Gamma' \Downarrow A \otimes B} \otimes \qquad\qquad \pi^\sharp \; \frac{\dfrac{\dfrac{\Gamma_1', A \Uparrow}{\Gamma_1' \Uparrow A} R\Uparrow}{\Gamma_1' \Downarrow A} R\Downarrow \qquad \dfrac{\Gamma_1' \Uparrow B}{\Gamma_2' \Downarrow B} R\Downarrow}{\Gamma' \Downarrow A \otimes B} \otimes$$

□

## 4 Conclusions

We are finally ready to give a proof of **Andreoli's Theorem** 1:

*Proof.* Let $\Pi$ be a proof in $\Sigma_1$ of the sequent $\vdash \Gamma, L$ which, by Theorem 2 and Proposition 1, de-sequentializes in to the canonical proof net $\pi$ of $\Gamma, L$ which finally sequentializes, by Theorem 4, in a proof $\Pi'$ of the sequent $\vdash_{\Sigma_2} \Gamma \Uparrow L$ in $\Sigma_2$ (see the diagram).

$$\begin{array}{ccc} \text{sequent proof : } \Pi & \xrightarrow{de-sequentialization} & \pi \text{ : proof net} \\ \downarrow & & \downarrow \\ \text{focused proof : } \Pi' & \xleftarrow{foc-sequentialization} & \pi' \text{ : canonical proof net} \end{array}$$

□

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