Retractile Proof Nets of the Purely Multiplicative and Additive Fragment of Linear Logic

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Abstract. Proof nets are a parallel syntax for sequential proofs of linear logic, firstly introduced by Girard in 1987. Here we present and intrinsic (geometrical) characterization of proof nets, that is a correctness criterion (an algorithm) for checking those proof structures which correspond to proofs of the purely multiplicative and additive fragment of linear logic. This criterion is formulated in terms of simple graph rewriting rules and it extends an initial idea of a retraction correctness criterion for proof nets of the purely multiplicative fragment of linear logic presented by Danos in his Thesis in 1990.

1 Introduction

Proof nets are a parallel syntax (a graphical presentation) for sequential proofs of linear logic (LL), firstly introduced by Girard in [3]. An interesting challenge is to find intrinsic (geometrical) characterizations of proof nets, that is correctness criteria (naively, algorithms) for checking those proof structures which correspond to LL proofs; this is particularly true for proof nets of the pure multiplicative and additive fragment of linear logic (MALL).

Our starting idea is that correctness for MALL proof nets should be formulated as simple as possible, following the spirit of correctness for proof nets of the pure multiplicative fragment of linear logic (MLL, see [3] and [1]). In our work correctness is formulated by an algorithm which implements simple graph rewriting rules. In particular, we extend an initial idea of a retraction correctness criterion for MLL proof nets presented in Danos's Thesis ([2]) and subsequently reformulated as a parsing criterion for MELL proof nets by Guerrini and Masini ([6]). Naively, retractility is a way to simulate sequentialization steps: each retracted (sub)graph corresponds to a correct (sequentializable) (sub)proof structure. Compared with other existing syntaxes for MALL proof nets, like that

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one due to Girard ([4]) or Hughes-van Glabbeek ([7]), our retractile correctness criterion does not rely on any notion of *additive box, slice* or *jump*. This effort should simplify the complexity of checking correctness. However here we do not discuss complexity aspects of our criterion; moreover, for simplicity reasons, we restrict to consider only cut-free proof nets.

After recalling, in next sub-section, some basic notions of the MALL fragment we introduce, in Section 2, a notion of (abstract) proof structure; then, in Section 3, we characterize correctness in terms of a rewriting algorithm which is shown confluent, correct (sequentializable) and complete (de-sequentializable) w.r.t. MALL sequent calculus. Finally, in Section 4, we discuss some directions in the way we could extend our criterion to proof nets with cuts.

1.1 The MALL Fragment of Linear Logic

MALL formulas $A, B, \ldots$ are built from literals (propositional variables $P, Q, \ldots$ and their negations $P^\perp, Q^\perp, \ldots$) by the binary connectives $\otimes$ (*tensor*), $\parr$ (*par*), $\&$ (*with*) and $\oplus$ (*plus*). Negation $(.)^\perp$ extends to arbitrary formulas by the de Morgan laws: $(A \otimes B)^\perp = (A^\perp \parr B^\perp)$, $(A \parr B)^\perp = (A^\perp \otimes B^\perp)$, $(A \& B)^\perp = (A^\perp \oplus B^\perp)$, and $(A \oplus B)^\perp = (A^\perp \& B^\perp)$. A MALL sequent Γ is a non empty set of formula occurrences $A_1, \ldots, A_n$. We omit turnstiles ($\vdash$) since all sequents are right-sided. Sequents are proved using the following rules:

$$\frac{}{A, A^\perp}\,\text{ax} \qquad \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\,\text{cut} \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\,\& \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2$$

2 Proof Structures

Definition 1 (proof structure). *A (cut-free)* proof structure*, shortly* PS*, of MALL is an oriented graph s.t. each edge is labelled by a MALL formula and built on the set of nodes (or vertices) following the typing constraints of Figure 1. Pending edges are called* conclusions*. The orientation is from top to bottom; fixed a node, an entering edge is called* premise *while its unique emergent edge is called* conclusion*. We call* link *the graph made by a node together with its premise(s) and conclusion.*

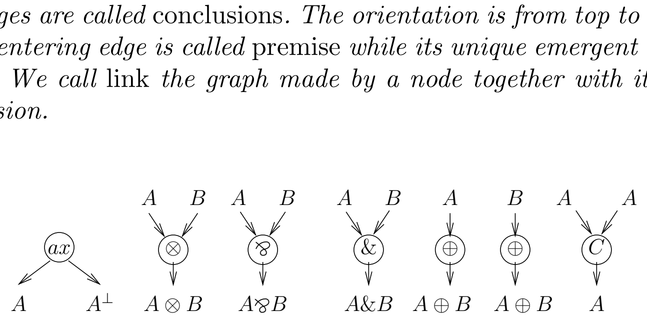

Fig. 1. MALL links

Definition 2 (abstract proof structure). *An* abstract structure *(*AS*) is a non oriented graph G equipped with a set $\mathcal{C}(G)$ of pairwise disjoint pairs of coincident edges (two edges are* coincident *if they share at least a vertex). We call simply* pair *a pair of edges of $\mathcal{C}(G)$ and* base *of a pair (possibly one of) its common vertex(es). A pair is graphically denoted by a crossing* arc *close to the base.*

An abstract proof structure *(*APS*) is an AS such that:*
– each edge is labelled by a MALL formula;
– each pair is denoted by an arc labelled by $\parr$, & or C;
– it is build by iterating the rules of Figure 2 (a mapping *from PS to APS).*

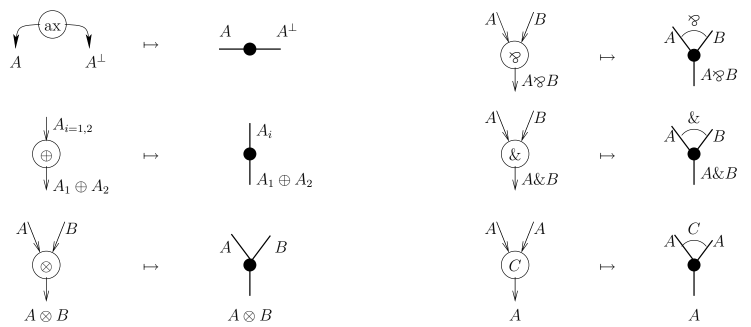

Fig. 2. Mapping PS in to APS

Notation: if π is a PS then π^* denotes its corresponding APS; variables $e_1,\ e_2,...$ denote edges and $v_1,\ v_2,...$ denote vertices of an APS; a dotted edge incident to a vertex v and (eventually) labelled by variables $a, b, ...$, is a compact representation of possibly several edges incident to v; finally, $\delta(v)$ states for the *degree* (the number of incident edges) of a vertex v.

Definition 3 (multiplicative retraction). *A* multiplicative retraction *of an APS π is a rewriting of π into π' (denoted $\pi \rightsquigarrow \pi'$) by means of an instance of the following rules:*

R_1 *(on the left hand side of Figure 3), with the conditions that in π:*
- *vertices v_1 and v_2 are distinct;*
- *the retracted edge e_1 does not belong to any pair of $\mathcal{C}(\pi)$.*

R_2 *(on the right hand side of Figure 3), with the conditions that in π:*
- *vertices v_1 and v_2 are distinct;*
- *the two retracted edges e_1 and e_2 belong to the same $\parr$-pair.*

Definition 4 (additive retraction)
An additive retraction *of an APS π is a rewriting of π into π' by means of an instance of the following rules:*

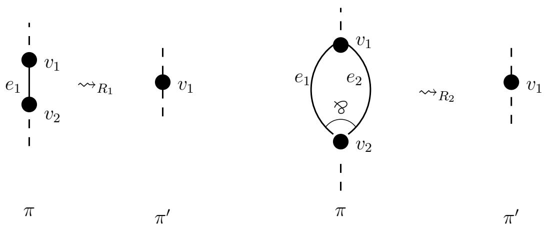

Fig. 3. Multiplicative retraction rules R_1 and R_2

R_3 *(on the left hand side of Figure 4), with the conditions that in* π*:*
- *each vertex* v_i*,* $1 \leq i \leq 4$*, is distinct;*
- *the two retracted edges* e_1 *and* e_2 *belong to the same* C*-pair.*

R_4 *(on the right hand side of Figure 4), with the conditions that in* π*:*
- *each vertex* v_i*,* $1 \leq i \leq 3$*, is distinct;*
- *the two retracted edges* e_1 *and* e_2 *belong to the same* &*-pair.*
- $\delta(v_2) = 1$ *and* $\delta(v_3) = 1$*.*

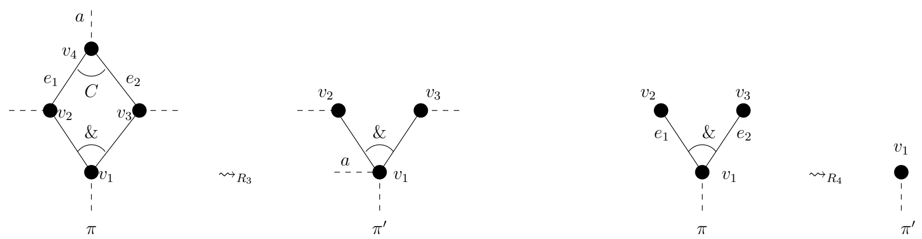

Fig. 4. Additive retraction rules R_3 and R_4

Definition 5 (distributive retraction). *A* distributive retraction *rule*[1] *of an APS* π *is a rewriting of* π *into* π' *by means of an instance of the rule* R_5 *of Figure 5, with the conditions that in* π*:*

- *each vertex* v_i*,* $1 \leq i \leq 8$*, is distinct;*
- *the two retracted edges* e_1 *and* e_2 *belong to the same* C*-pair;*
- *the two retracted edges* e_3 *and* e_4 *belong to the same* $\parr$*-pair;*
- $\delta(v_4) = 2$*,* $\delta(v_5) = 2$*,* $\delta(v_6) = 3$ *and* $\delta(v_7) = 3$*.*

Fixed a retraction rule R_i, $1 \leq i \leq 5$, the subgraph of π (resp., of π') depicted on the left (resp., on the right) hand side of the $\rightsquigarrow_{R_i}$ map is called the *retraction* (resp., *retracted*) *graph* of R_i.

We say that two instances R_i and R_j, with $1 \leq i, j \leq 5$, *overlap* (resp., *separate*) when the intersection of their retraction graphs is not empty (resp.,

[1] This rule reflects the *distributive linear law* $(b \parr c) \& (b \parr d) \vdash b \parr (c \& d)$.

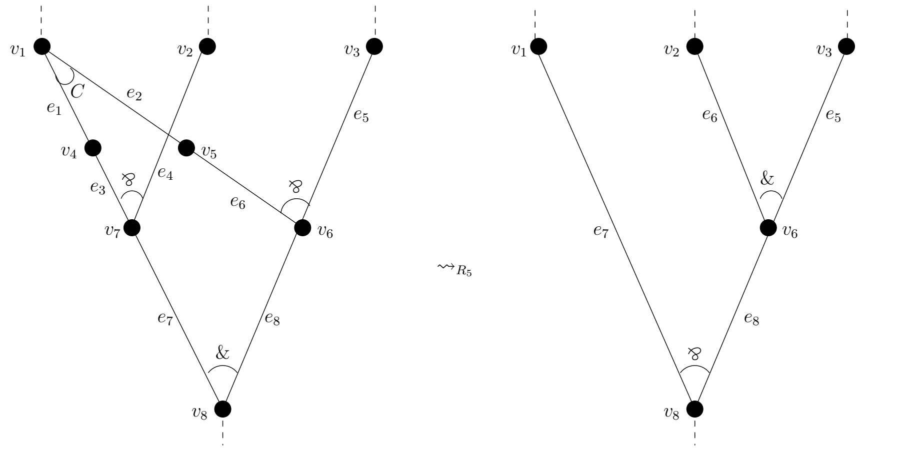

Fig. 5. Distributive retraction rule R_5

empty). R_i and R_j are said *independent* when they can be applied in any order, i.e. R_i immediately before R_j or R_j immediately before R_i.

An APS π with conclusions $A_1, ..., A_n$ is *retractile* when there exists a sequence of retraction instances starting with π and terminating with a single node ($\bullet$) with n incident edges labelled by $A_1, ..., A_n$.

Definition 6 (proof net). *A PS π with conclusions $A_1, ..., A_n$, with $n \geq 1$, is* correct *(i.e., it is a* proof net*) if its corresponding APS π^* is retractile.*

Theorem 1 (confluence). *If π is a retractile APS then any sequence of retraction instances starting from π terminates with a single node with the (possibly) several incident edges labelled by the conclusions of π.*

Proof. Assume Σ is a retractions sequence $\pi \rightsquigarrow \pi_1 \rightsquigarrow ... \rightsquigarrow \pi_n = \bullet$ with R_i as first retraction (i.e. $\pi \rightsquigarrow_{R_i} \pi_1$) and assume there exists a σ such that $\pi \rightsquigarrow_{R_j} \sigma$. We show that σ is retractile too. We reason by induction on the length l of Σ, where l is the number of retraction instances of Σ.

Assume $l = 1$, then $\pi \rightsquigarrow_{R_i} \pi_1 = \bullet$ and so R_i and R_j must be the same instance with $\sigma = \pi_1$. This follows from the definition of the retraction rules (if R_i and R_j are two different instances then the retraction graphs of R_i and R_j can be disjoint or partially overlapping but not included each other).

Assume $l > 1$, then we split our reasoning in two sub-cases.

1. R_i and R_j are independent. Since, by assumption π_1 is retractile in $n-1$ steps, then by hypothesis of induction applied to π_1 we conclude that any π_1' obtained by $\pi_1 \rightsquigarrow_{R_j} \pi_1'$ must be retractile. This means that σ is retractile since $\sigma \rightsquigarrow_{R_i} \pi_1'$ and R_i and R_j are independent (see Figure 6).
2. R_i and R_j are not independent; this means that R_i and R_j must be two overlapping instances of the R_5 rule like in left hand side of Figure 7. Again,

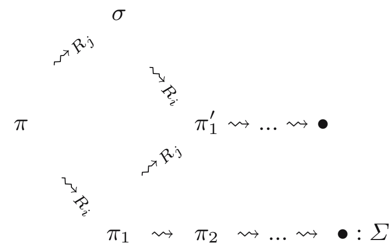

Fig. 6. Confluence of independent retractions

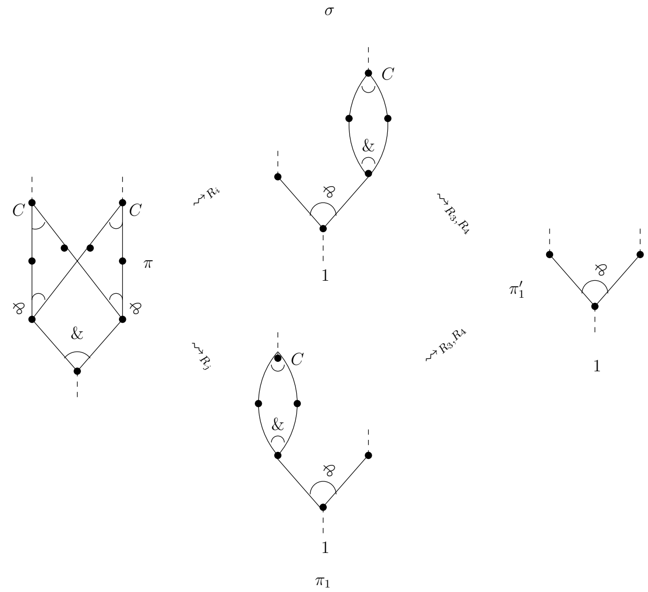

Fig. 7. Confluence of non independent retractions

we reason like in the previous case. Since, by assumption π_1 is retractile in $n-1$ steps, then we can apply the hypothesis of induction to π_1 and conclude that π_1' is also retractile since $\pi_1 \rightsquigarrow_{R_{3,4}} \pi_1'$. This means that σ will be retractile too, since $\sigma \rightsquigarrow_{R_{3,4}} \pi_1'$ (see Figure 6).

3 (De-)Sequentialization

In this section we show that any sequent proof of can be de-sequentialized into a proof net with the same conclusions (Theorem 2) and vice-versa (Theorem 5).

Theorem 2 (de-sequentialization). *A proof π^- of the sequent $\Gamma = A_1, ..., A_{n\geq 1}$ can be desequentialized in to a proof net π with conclusion Γ.*

Proof. By induction on the *height*[2] of the given sequential proof π^-. We only consider the case when last rule of π is a &-rule (the other cases are very simple and we omit them). Assume π^- like in the left hand side of Figure 8, then by hypothesis of induction π_1^- and π_2^- desequentialize respectively into two retractile APS π_1^* and π_2^*, like in the middle side of Figure 8. Clearly the resulting APS π^* (see the right hand side of Figure 8) will be retracted to $\bullet$ by applying (iteratively) rule R_3 and (an instance of) rule R_4.

$$\pi^- : \dfrac{\begin{array}{cc} \pi_1^- & \pi_2^- \\ \Gamma, A & \Gamma, B \end{array}}{\Gamma, A\&B}\,\&$$

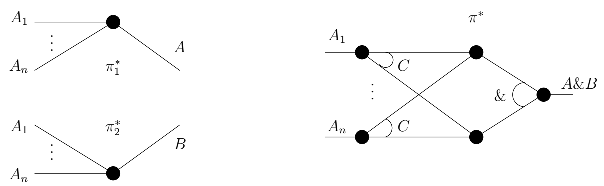

Fig. 8. De-sequentialization of the &-rule

In the following we give an indirect proof of the sequentialization: first we show that any proof net can be weighted in such a way of becoming a *proof net à la Girard* (Section 3.1), then sequentialization follows as a consequence of Girard's one (Section 3.2).

3.1 Girard's Proof Nets

In this section we recall the basic notions of Girard's proof net; for simplification reasons we adopt the syntax of [9].

Definition 7 (Girard's proof structure). *A* proof structure à la Girard *(*GPS*) is a PS with* weights *associated as follows (*weights assignment*):*

1. *first we associate a* boolean variable*, called* eigen weight p*, to each* &*-node (eigen weights are supposed to be different);*
2. *then we associate a* weight*, a product of (negation of) boolean variables* $(p, \overline{p}, q, \overline{q}...)$ *to each node, with the constraint that two nodes have the same weight if they have a common edge, except when the edge is the premise of a & or* C*-node, in these cases we do like in Figure 9:*
3. *a* conclusion node *has weight* 1*;*
4. *if* w *is the weight of a* &*-node, with eigen weight* p*, and* w' *is a weight depending on* p *and appearing in the proof structure then* $w' \leq w$ *(we say that a weight* w depends on p *when* p *or* $\overline{p}$ *occurs in* w*).*

[2] The height, $h(\pi^-)$, of π^- is defined inductively as usual. We consider last rule R of π^-: if $R = ax$ then $h(\pi^-) = 1$ otherwise if R is an unary rule, $\parr$ or $\oplus$, (resp., a binary rule, $\otimes$ or &) then $h(\pi^-) = h(\pi_1^-)+1$ (resp., $h(\pi^-) = max(h(\pi_1^-), h(\pi_2^-))+1$) where π_1^- (resp., π_1^- and π_2^-) is the immediate sub-proof (resp., are the immediate sub-proofs) of π^-.

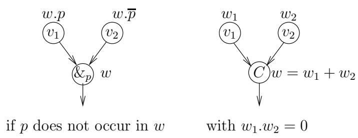

Fig. 9. Weights for GPS

A node L with weight w *depends on* the eigen weight p if w depends on p or L is a C-node and one of the weights just above it depends on p.

Definition 8 (slice and switchings). *A* valuation φ *for a GPS* π *is a function from the set of all weights of* π *into* $\{0,1\}$. *Fixed a valuation* φ *for* π *then:*

- *the* slice $\varphi(\pi)$ *is the graph obtained from* π *by keeping only those nodes with weight* 1 *together its emerging edges;*
- *a* multiplicative switching S *for* π *is the non oriented graph built on the nodes and edges of* $\varphi(\pi)$ *with the modification that for each* $\wp$*-node we take only one premise (*left/right $\wp$-switch*);*
- *an* additive switching *(or simply a* switching*) is a multiplicative switching where for each* $\&$*-node we erase the (unique) premise in* $\varphi(\pi)$ *and we add an oriented edge, called* jump*, from the* $\&$*-node to an* L*-node whose weight depends on the eigen weight of the* $\&$*-node.*

Definition 9 (Girard's proof net). *A GPS* π *is* correct*, so it is a* proof net à la Girard *(*GPN*), if any switching, induced by any valuation of* π*, is acyclic and connected (*ACC*).*

Theorem 3 (sequentialization). *A GPN can be sequentialized into a MALL sequent proof with same conclusions and vice-versa.*

Proof. omitted (see[4]).

3.2 Sequentialization

Definition 10 (GAPS). *A* Girard abstract proof structure *is an APS with weights associated as follows:*

1. *first we associate a distinct eigen weight variable to each* $\&$*-pair (graphically, the arc of each* $\&$*-pair is now labelled by a distinct eigen weight variable);*
2. *then we associate a weight to each node, with the constraint that two nodes have the same weight if they have a common edge, except when the edge occurs in a* $\&$*-pair or* C*-pair, in these cases we do like in Figure 10:*
3. *a* conclusion node *has weight* 1*;*
4. *if* w *is the weight of a node that is the base of an* $\&_p$*-pair and* w' *is a weight depending on* p *and appearing in the GAPS then* $w' \leq w$.

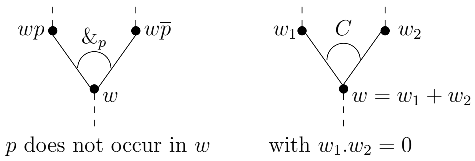

Fig. 10. Weights for GAPS

Remark 1. The notions of valuation and slice are still well defined w.r.t. GAPS. Only the definition of switching needs a slight modification. Fixed a valuation φ for an GAPS π then:

- a multiplicative switching S for π is the (non oriented) graph built on $\varphi(\pi)$ with the modification that for each $\invamp$-pair we take only one edge (*left/right $\invamp$-switch*);
- an additive switching is a multiplicative switching where for each $\&_p$-pair we erase the (unique) edge in $\varphi(\pi)$ and we add a jump from the base of this $\&_p$-pair to an L-node whose weight depends on p.

Lemma 1. *Assume π is an APS such that $\pi \rightsquigarrow_{R_i} \pi'$, with $1 \leq i \leq 5$, and assume there exists a weights assignment making π' a GAPS, then this assignment can be extended in such a way to transform π in to a GAPS.*

Proof. We reason by cases, according to the retraction R_i.

Cases R_1. The weight w associated to node v_1 in the retracted graph of R_1 in Figure 3 is inherited by both nodes v_1 and v_2 of the retraction graph. Since all other weights remain unchanged π is a GAPS. Case R_2 is similar.

Case R_2 is similar.

Case R_3. Assume, in the retracted graph of Figure 4, p is the eigen weight of the &-pair, w is the weight of node v_1, wp is the weight of node v_2 and $w\overline{p}$ is the weight of node v_3. We can easily extend this weight assignment to the corresponding nodes of the retraction graph: it is easy to verify that this assignment preserves the property of being a AGPS w.r.t. π, since all pair of $\mathcal{C}(\pi)$ are pairwise disjoint and all other weights remain unchanged.

Case R_4. Assume node v_1 has weight w in the retracted graph of Figure 4; then w is trivially inherited by the corresponding node v_1 of the retraction graph. Now, chosen a new eigen weight p for the &-pair, we can assign weights wp and $w\overline{p}$ to nodes, respectively, v_2 and v_3. Now, since all other weights remain unchanged, π is a GAPS.

Case R_5. Assume a weight assignment for the retracted graph of Figure 5 as follows: the $\&_p$-pair together with nodes v_1, v_6 and v_8 have the same weight w, while nodes v_2 and v_3 have weights, respectively, wp and $w\overline{p}$. This assignment can be easily extended to π with a slight modifications: the &-pair with base in v_8 inherits the eigen weight p and v_8 inherits the weight w; then weight wp is assigned to v_7, v_4 and v_2, and weight $w\overline{p}$ to v_6, v_5 and v_3; finally weight $w = wp + w\overline{p}$ is assigned to v_1. It is easy to verify that this new assignment preserves the property of being a AGPS w.r.t. π, since all pair of $\mathcal{C}(\pi)$ are pairwise disjoint and all other weights remain unchanged.

What follows is a well known graph theoretical property (see [5], pages 250-251) we will exploit in the proof of the Lemma 2.

Property 1 (Euler-Poincaré invariance). Given a graph $\mathcal{G}$, then $\sharp CC - \sharp Cy = \sharp V - \sharp E$, where $\sharp CC$, $\sharp Cy$, $\sharp V$ and $\sharp E$ denotes, respectively, the number of connected components, cycles, vertices and edges of $\mathcal{G}$.

We use the predicate $Gir(\pi)$ for saying that a GAPS is correct in the sense of Definition 9, i.e., any switching $S(\pi)$, w.r.t. a fixed valuation $\varphi(\pi)$, is ACC.

Lemma 2. *If π is an GAPS and $\pi \rightsquigarrow_{R_i} \pi'$, $1 \leq i \leq 5$, then $Gir(\pi)$ if $Gir(\pi')$.*

Proof. Let us fix a valuation φ for π. First observe that by Lemma 1 we have only to verify that every switching $S(\pi)$ is ACC. The proof idea, illustrated in Figure 11, relies on the fact that if $\pi \rightsquigarrow_{R_i} \pi'$ then any switching S for π

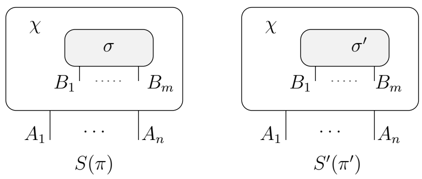

Fig. 11. Recovering $S(\pi)$ from $S'(\pi')$

is nothing else that a switching S' for π' except for the fact we replace the switched retracted graph σ' of R_i by a corresponding switched retraction graph σ. We reason by cases, according to the retraction rule R_i.

1. If $\pi \rightsquigarrow_{R_1} \pi'$ (see the left hand side of Figure 3) then trivially any switching S for π can be recovered from a S' for π' where we replaced the retracted graph σ' with the retraction graph σ of Figure 12. Clearly $S(\pi)$ is ACC.

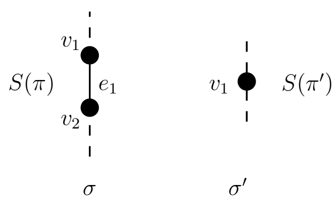

Fig. 12. Recovering $S(\pi)$ from $S'(\pi')$ after a retraction R_1

2. If $\pi \rightsquigarrow_{R_2} \pi'$ (see the right hand side of Figure 3) then any switching S for π can be recovered from a switching S' for π' where we replaced the switched retracted graph σ' by a corresponding switched retraction graph σ (resp., χ) in case we set a left (resp., right) switch for the $⅋$-pair (see Figure 13). Clearly, $S_{⅋_l}(\pi)$ (resp., $S_{⅋_r}(\pi)$) is ACC.

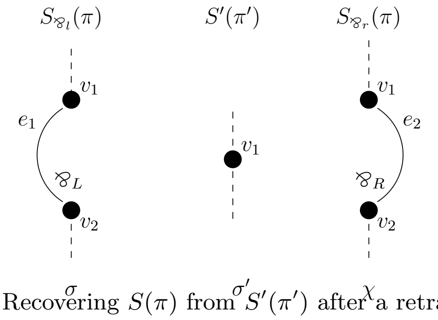

Fig. 13. Recovering $S(\pi)$ from $S'(\pi')$ after a retraction R_2

3. If $\pi \rightsquigarrow_{R_3} \pi'$ (see the left hand side of Figure 4) then any switching S for π can be recovered from a switching S' of π'. We need to consider two cases, according to the jump emerging from the base v_1 of the $\&_p$-pair in S'.
First case: assume S' contains an immediate jump j from the base v_1 of the $\&_p$-pair to its (unique) premise v_2 (see the right hand side of Figure 14). Then S for π can be recovered from S' where we replaced the switched retracted graph σ' by the corresponding switched retraction graph σ on the left hand side of Figure 14. We have to show that S is ACC. First, observe

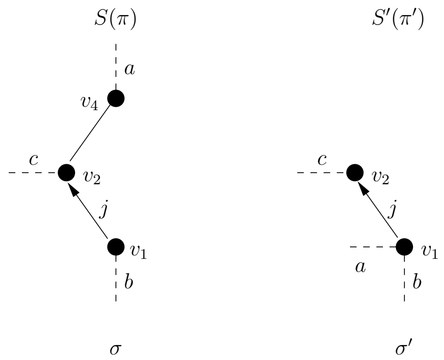

Fig. 14. Recovering $S(\pi)$ from $S'(\pi')$, with an immediate jump, after a retraction R_3

that edges a, b and c are connected in S as well in S', so the number of connected components in S is 1. Moreover in S the difference $\sharp V - \sharp E$ must be the same as that one in S', that is 1, since in S there is only one more edge and one more vertex than in S'. So by the Euler-Poincaré invariance in S we have $\sharp CC - \sharp Cy = 1$, therefore $\sharp Cy$ in S must be 0. So S is ACC.

Second case: S contains a remote jump j from the base v_1 of the $\&_p$ pair to a node v depending on p that is different from the (unique) premise of $\&_p$ (see the right hand side of Figure 15). Clearly S can be recovered from

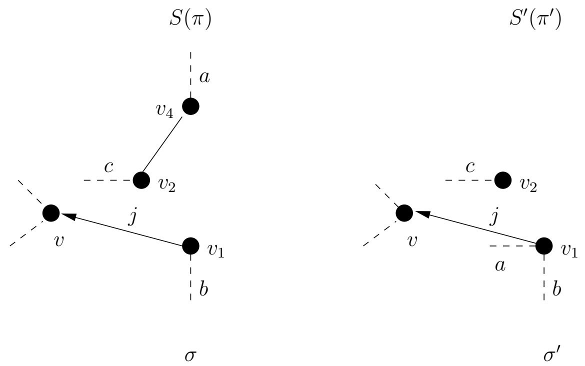

Fig. 15. Recovering $S(\pi)$ from $S'(\pi')$, with a remote jump, after a retraction R_3

a switching S' where we replaced the switched retracted graph σ' by the switched retraction graph σ on the left hand side of Figure 15. We show that S is ACC. Although vertices v_1 and v_2 are connected in S' by assumption, they cannot be connected through a, otherwise we could easily set a switching S'' for π' that is identical to S' except for the immediate jump from v_1 to v_2 and get a cycle, contradicting the assumption $Gir(\pi')$. This means that the retraction step R_3 preserves backwards the connection of S, so $\sharp CC$ of S is 1. Now, observe that the difference $\sharp V - \sharp E$ of S is the same as that one of S' (i.e. 1), so by the Euler-Poincaré invariance we have, in S, $\sharp CC - \sharp Cy = 1$. This means that, in S, $\sharp Cy$ must be 0. So S is ACC.

4. If $\pi \rightsquigarrow_{R_4} \pi'$ (see the right hand side of Figure 4) then a switching S for π can be recovered from a switching S' for π' plus a jump j emerging from the base v_1 of the $\&_p$ pair. Now, observe this jump j in S can only be directed to its (unique) premise, otherwise: (i) either there would exist in π' a node whose weight depends on a variable p that is not an eigen weight of any $\&$-pair or (ii) there would exist in π two $\&$-pairs with the same eigen-weight variable p. Both cases contradict that π is an AGPS (Lemma 1).
5. If $\pi \rightsquigarrow_{R_5} \pi'$ (see Figure 5) then a switching S for π is exactly a switching S' for π' except for the the jump emerging from the $\&_p$-pair and the switch for the $⅋$ pair occurring in the retraction R_5. So, let us fix in S' a left $⅋$-switch and jump from the base v_6 of the $\&_p$-pair (the case with the right $⅋$-switch, is analogous): there are two possible jumps.
First case: assume in $S'(\pi')$ we jump from the base v_6 of the $\&_p$ pair to its (unique) premise. Then a switching S for π is exactly a switching S' for π' except for the fact we replaced the switched retracted graph σ' with σ of Figure 16. Clearly S is connected, so via the Euler-Poincaré invariance, we conclude that S is also acyclic (we reason like in the previous point 3).

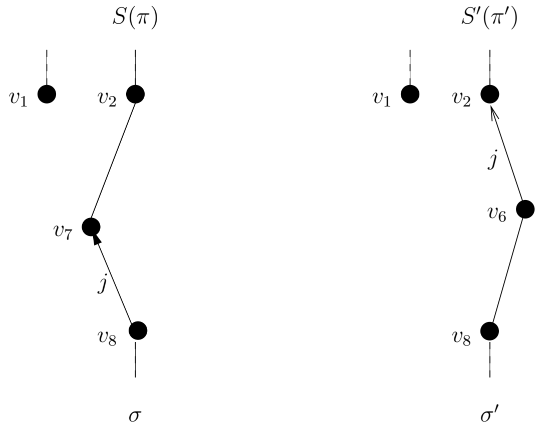

Fig. 16. Recovering $S(\pi)$ form $S'(\pi')$, with immediate jump, after a reduction R_5

Second case: assume in $S'(\pi)$ we jump from the base v_6 of the $\&_p$ pair to a remote node v depending on p. Then a switching S for π can be recovered from a switching S' where we replaced the switched retracted graph σ' with σ of Figure 17. Now observe that σ' does not induce any cycle in S, otherwise

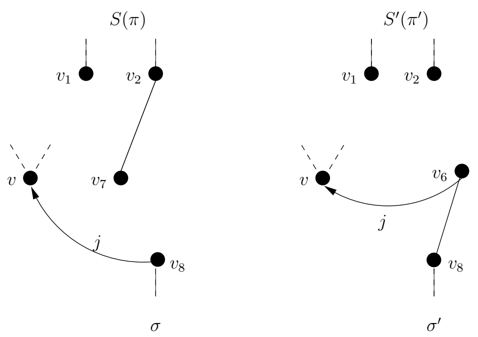

Fig. 17. Recovering $S(\pi)$ form $S'(\pi')$, with a remote jump, after a reduction R_5

this cycle would already occur in S'. Moreover, the number of vertices and edges in S is the same as in S' so, via the Euler-Poincaré invariance, we conclude that S is connected.

Theorem 4 (PN$\mapsto$GPN). *If π is a PN then there exists a weight assignment transforming π in to a GPN.*

Proof. If π is a PN then π^* retracts to a node v with possibly several incident edges labelled by the conclusions of π. Trivially v is an AGPS satisfying the predicate *Gir*, then by iteration of Lemma 2 we conclude that π is a GPN.

Theorem 5 (sequentialization). *If π is a PN with conclusions Γ then the sequent Γ is provable in the MALL sequent calculus.*

Proof. It follows from Girard's sequentialization (see [4]) via Theorem 4.

4 Conclusions and Forthcoming Work

We presented a simple system of graph rewriting rules which can be viewed as a geometrical correction criterion for cut free proof structures of MALL. Each proof structure that is correct in our sense it will be so also in Girard's sense but not vice-versa. In general the other direction of Theorem 4 does not hold: the proof structure π_1, on the left hand side of Figure 18 is not correct for us (it is not retractile); nevertheless we can find a weight assignment transforming π_1 in to a Girard proof net. Actually we only accept correct the proof structure π_2 depicted on the right hand side of Figure 18 which (in our opinion) better embeds the two different sequentializations induced by the permutability of the &-rule w.r.t. the $\otimes$-rule of the sequent calculus.

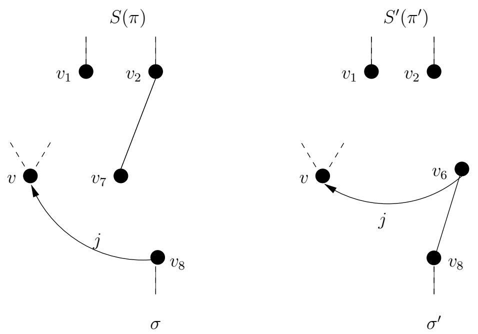

Fig. 18. Examples of PS w.r.t. sequentialization

As future work we aim at comparing the complexity of the retractility correctness criterion w.r.t. Girard and Hughes-van Glabbeek's criteria. Moreover we aim at extending retraction rules in such a way to take in to account proof nets with cuts. At the moment we are investigating some local (commutative) cut reduction steps, following the style of the Interaction Nets ([8]). Our idea is sketched in Figure 19 where the $\star$ symbol states for a binary MALL connective

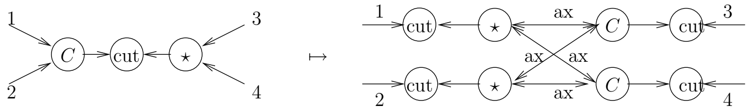

Fig. 19. Commutative cut step reduction

or a contraction C. But in that case the correspondence with monomial GPS is lost: as soon as we replace the $\star$ symbol with the & connective we are suddenly faced to proof structures weighted with polynomials.

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