the “dark side” of multiplicatives

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the three main ingredients for PNs and the quest of modularity

1. a syntax: a representations of proof structures (PSs);
2. a semantics: an intrinsic correctness criterion;
3. a sequentialization: mapping correct PSs into sequent proofs (and vice-versa).

- since the *debut* of LL, we knew that: if the second ingredient (the CC) is truly geometrical (i.e., "modular") then it may have "disruptive effects" on the other two ingredients (mainly, on sequentialization).
- concretely, there exist "strange objects" that are correct (e.g. by Danos-Regnier'89 or Girard'87) but not sequentializable; indeed, these "objects" are not even representable in MLL (as e.g. a decomposition by means of $\parr$ and $\otimes$ connectives).
- … a quarter of a century later, we still ignore the "dark side" of multiplicatives.

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correctness criteria vs syntactical presentations of PSs

Both the Danos-Regnier's correctness criterion ("ACC graphs") and the Girard's criterion ("long trips") are "truly geometrical criteria" in the following sense:

> *they are "modular", i.e., there exists a way (a function) to encode the "abstract behavior" of a module (i.e. a piece of proof structure) over its border (or frontier) in such a way to allow the gluing (composition) of this module together with any other module with "orthogonal behavior".*

This fact has an important consequence on the syntax chosen for presenting proof structures.

In particular these criteria are so powerful to allow a "fully abstract presentation" that goes beyond what we normally know: consider e.g. the representation of modules by means of sets of partitions (or permutations) over (the set of points of) the border.

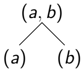
For this kind of syntax (links as partitions or permutations), the standard sequentialization (possibly, via binary connectives decomposition) is inadequate.

partitions and orthogonality

- Let p and p^* be two **partitions of a finite set of points** X.
- We define the **induced graph of p and p^***, denoted by $G(p, p^*)$, as the unoriented graph which has for vertices the classes of p and p^*, two of them being linked iff they share a point.
- We say that p and p^* are **orthogonal** (we write $p \perp p^*$) iff the induced graph $G(p, p^*)$ is an **acyclic and connected graph**.

  Example:

  $$G(p = \{(a, b)\}, p^* = \{(a), (b)\}):$$

- If P and Q are sets of partitions, we say that P and Q are orthogonal ($P \perp Q$) if they are pointwise orthogonal (i.e., $\forall p \in P, \forall q \in Q, p \perp q$).
- We write $P^\perp$ for the set of partitions that are orthogonal to all the elements of P.

generalized connectives

An n-**ary multiplicative connective** consists of two sets of partitions P and Q over $\{1, \ldots, n\}$ s.t. $P \perp Q$ and $P^{\perp} \perp Q^{\perp}$.

Examples:

1. $P = \{\{(a), (b)\}\}, \quad Q = \{\{(a, b)\}\} \Rightarrow P \perp Q, P^{\perp} \perp Q^{\perp}.$

2. $P = \{\{(a, c), (b)\}\}, \quad Q = \{\{(a, b), (c)\}\} \Rightarrow P \perp Q, P^{\perp} \not\perp Q^{\perp}$:

   $P^{\perp} = \{\{(a, b), (c)\}, \{(c, b), (a)\}\},$

   $Q^{\perp} = \{\{(a, c), (b)\}, \{(b, c), (a)\}\}.$

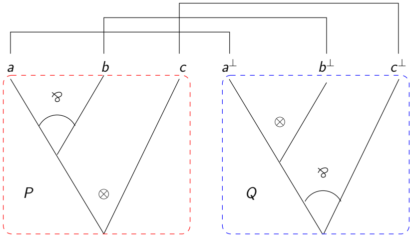
3. $P = \{\{(a, c), (b)\}, \{(b, c), (a)\}\}, \quad Q = \{\{(a, b), (c)\}\} \Rightarrow P \perp Q, P^{\perp} \perp Q^{\perp}.$

decomposable connectives and the "package" sequentialization

- If T_F is the syntactical tree of a formula F built by the binary multiplicative connectives $\parr$ and $\otimes$, then the **(pre-)type** of (the module) T_F the set of partitions induced by all the possible **Danos-Regnier switchings** on its border.
- A connective is **decomposable** iff its first set P is the type of a formula-tree built with binary connectives and its second set Q is the type of the dual tree.

Example: a **decomposable** connective

$$P = \{\{(a,c),(b)\},\{(b,c),(a)\}\} \qquad Q = \{\{(a^\perp, b^\perp),(c^\perp)\}\}$$

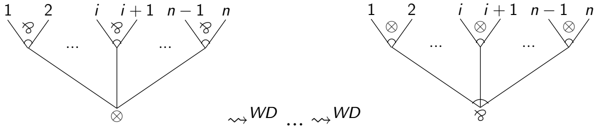


Neither P nor Q is **sequentializable** as a "**package**" (you must decompose it!)

Question: are there non-decomposable connectives? if "yes", how to find them?

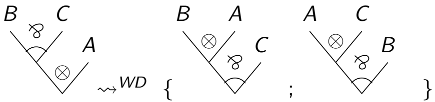
weak-distributivity and type generators [Maieli-Puite]

- a **type generator** (resp., an anti-generator or terminal element) is a bipolar formula with border $\{1, \ldots, n\}$ like below:



generator terminal element

- **weak-distributive law** (WD):



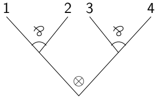
$$A \otimes (B \parr C) \vdash (A \otimes B) \parr C \quad \wedge \quad A \otimes (B \parr C) \vdash (A \otimes C) \parr B$$

- a generator F enjoys the following properties: if $type(F) = P$, then
  1. $P^{\perp\perp} = \bigcup type(F')$ $s.t.$, $F \leadsto^*_{WD} F' \;\wedge\; F'$ is terminal;
  2. $P^{\perp}$ is a singleton (a set with a single partition).
- if $T = T^{\perp\perp}$ then the type T is said **dense**.

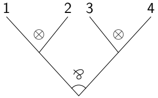
Examples of generators with border $\{1, 2, 3, 4\}$ and dense types

$P_1 = \{\{(1,3),(2),(4)\},\{(1,4),(2),(3)\},\{(2,3),(1),(4)\},\{(2,4),(1),(3)\}\}$
$Q_1 = \{\{(1,2),(3,4)\}\}$

P_1:

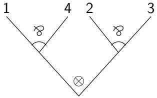


Q_1:

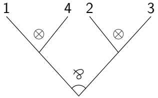


$P_2 = \{\{(1,2),(4),(3)\},\{(1,3),(4),(2)\},\{(4,2),(1),(3)\},\{(4,3),(1),(2)\}\}$
$Q_2 = \{\{(1,4),(2,3)\}\}$

P_2:

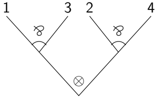


Q_2:

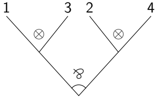


$P_3 = \{\{(1,2),(4),(3)\},\{(1,4),(2),(3)\},\{(3,2),(4),(1)\},\{(3,4),(2),(1)\}\}$
$Q_3 = \{\{(1,3),(2,4)\}\}$

P_3:



Q_3:



non-decomposable connectives with border $\{1,2,3,4\}$

$P_1 = \{\{(1,3),(2),(4)\},\{(1,4),(2),(3)\},\{(2,3),(1),(4)\},\{(2,4),(1),(3)\}\}$
$Q_1 = \{\{(1,2),(3,4)\}\}$

$P_2 = \{\{(1,2),(4),(3)\},\{(1,3),(4),(2)\},\{(4,2),(1),(3)\},\{(4,3),(1),(2)\}\}$
$Q_2 = \{\{(1,4),(2,3)\}\}$

$P_3 = \{\{(1,2),(4),(3)\},\{(1,4),(2),(3)\},\{(3,2),(4),(1)\},\{(3,4),(2),(1)\}\}$
$Q_3 = \{\{(1,3),(2,4)\}\}$

Fact $\sharp 0$: $P_1 \cap P_2 \cap P_3 = \emptyset$.

Fact $\sharp 1$: $\forall i \in \{1,2,3\}, P_i = P_i^{\perp\perp}$.

Fact $\sharp 2$: the following, C_1, C_2 and C_3 are multiplicative connectives, since

$$\forall i \in \{1,2,3\}, (C_i \perp C_i^*) \ \wedge \ (C_i^{\perp} \perp C_i^{*\perp})$$

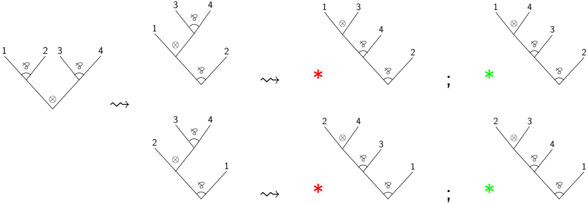
actually

$$\forall i \in \{1,2,3\}, (C_i = C_i^{\perp\perp})$$

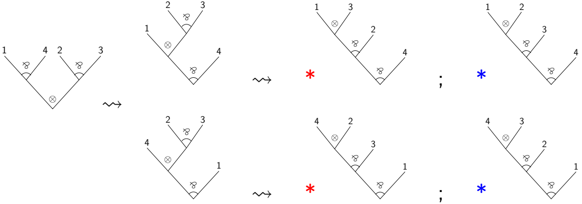
$$\begin{array}{lclclcccc}
C_1 & = & P_1 \cap P_2 & = & \{ & \{(1,3),(2),(4)\}, & \{(2,4),(1),(3)\} & \} \\
C_1^* & = & Q_1 \cup Q_2 & = & \{ & \{(1,2),(3,4)\}, & \{(1,4),(2,3)\} & \} \\
\\
C_2 & = & P_1 \cap P_3 & = & \{ & \{(1,4),(2),(3)\}, & \{(2,3),(1),(4)\} & \} \\
C_2^* & = & Q_1 \cup Q_3 & = & \{ & \{(1,2),(3,4)\}, & \{(1,3),(2,4)\} & \} \\
\\
C_3 & = & P_2 \cap P_3 & = & \{ & \{(1,2),(4),(3)\}, & \{(4,3),(1),(2)\} & \} \\
C_3^* & = & Q_2 \cup Q_3 & = & \{ & \{(1,4),(2,3)\}, & \{(1,3),(2,4)\} & \}
\end{array}$$

C_1, C_2, C_3" are not decomposable (proof via "WD-rewriting $\rightsquigarrow$")

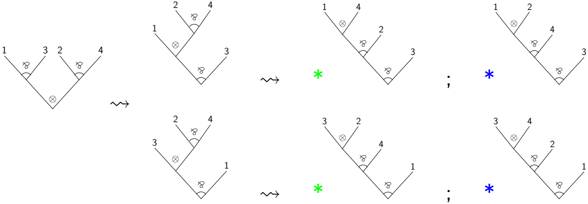
P_1:



P_2:

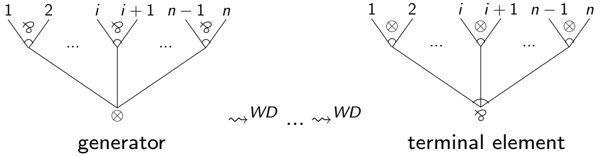


P_3:

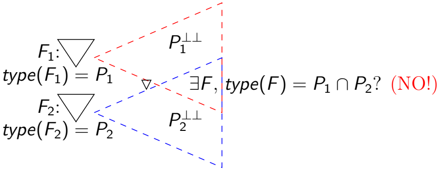


C_1, C_2, C_3 are not decomposable connectives (proof continues)

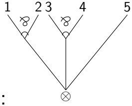
there exists no decomposable formula tree T_F whose terminal tree elements (i.e. the pre-type of F, computed by exhausting application of the week-distributive rewriting steps) are the colored ones (resp, red, blue and green ones).

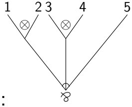


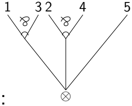
Conjecture [Maieli]:
a non-decomposable connective $C(1, ..., n)$ (resp., $C^*(1, ..., n)$) can be obtained by considering the intersection (resp. the union) of a family of binary generators (resp., terminal elements or axioms) with border any permutation σ over $\{1, ..., n\}$:

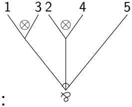


one more non-decomposable connective with border $\{1,2,3,4,5\}$

P_1: 

Q_1: 

P_2: 

Q_2: 

Q_1 : (12) (34)

Q_2 : (13) (24)

$P_1^{\perp\perp}$:

(145)	(3)	(2)	✓
(135)	(4)	(2)	
(235)	(4)	(1)	✓
(245)	(3)	(1)	
(45)	(13)	(2)	
(25)	(13)	(4)	
(25)	(14)	(3)	✓
(35)	(14)	(2)	✓
(15)	(23)	(4)	✓
(15)	(24)	(3)	
(45)	(23)	(1)	✓
(35)	(24)	(1)	

$P_2^{\perp\perp}$:

(145)	(2)	(3)	✓
(125)	(4)	(3)	
(325)	(4)	(1)	✓
(345)	(2)	(1)	
(45)	(12)	(3)	
(35)	(12)	(4)	
(35)	(14)	(2)	✓
(25)	(14)	(3)	✓
(15)	(32)	(4)	✓
(15)	(34)	(2)	
(45)	(32)	(1)	✓
(25)	(34)	(1)	

$C = P_1 \cap P_2$:

(145)	(2)	(3)
(325)	(4)	(1)

$C^* = Q_1 \cup Q_2$:

(12)	(34)	(5)
(13)	(24)	(5)

$$(C \perp C^*) \wedge (C^\perp \perp C^{*\perp})$$

conclusions

"Maybe we witness here the limits of sequential presentations of logic."
V. Danos & L. Regnier, The Structure of Multiplicatives, 1989, p. 202.