Cut Elimination for Monomial MALL Proof Nets

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joint work with

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Proof Theory & Linear Logic

- Since its inception linear logic (LL, Girard 1987) has changed the proof theoretical way of dealing with cut elimination.
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MALL Sequent Calculus (The Multiplicative-Additive fragment of LL)

- *Formulas* $A, B, \ldots$ are built from *literals* by the binary connectives $\otimes$ (*tensor*), $\parr$ (*par*), $\&$ (*with*) and $\oplus$ (*plus*).
- *Negation* $(.)^\perp$ extends to any formula by de Morgan laws:

$$(A \otimes B)^\perp = (B^\perp \parr A^\perp) \qquad (A \parr B)^\perp = (B^\perp \otimes A^\perp)$$
$$(A \& B)^\perp = (B^\perp \oplus A^\perp) \qquad (A \oplus B)^\perp = (B^\perp \& A^\perp)$$

- *Sequents* Γ, Δ are sets of formula occurrences $A_1, \ldots, A_{n\geq 1}$, proved using the following rules:
  - identity: $$\frac{}{A, A^\perp}\,ax \qquad \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\,cut$$
  - multiplicatives: $$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr$$
  - additives: $$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\,\& \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2$$

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Cut-elimination with the SC is problematic

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it may reduce to:

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Proof Nets (PNs): a possible solution

- PNs are parallel presentations of sequential proofs
- they quotient classes of equivalent proofs, modulo permutations of derivation rules.

MLL: The Multiplicative Fragment of LL is the perfect setting:

1. a PN is a canonical representative of a proof of the SC;
2. the (strong) cut elimination procedure is purely **local**: reducing a cut consists in to modifying only the nodes connected to it.

MALL: A lot of work has been done in order to extend (1) and (2) [Girard'96, Hughes-Van Glabbeek'03...]

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Proof Nets of MALL

In 1996, Girard proposed a new syntax for MALL PNs:

- without additive boxes (sequentiality)
- allowing graph super-positions (weights, slices)

But Girard's proposal was not as good as for MLL:

1. no canonicity: there exist proofs which de-sequentialize into two possible PNs with no way to discriminate them.
   This problem has been solved by Hughes-Van Glabbeek (LICS2003)
2. no full cut elimination: only the *logical* (ready) cuts are reduced in a non-local way

Our Goal

To provide a **New Syntax for Monomial MALL PNs** *with a* **(local) full strong cut elimination**.

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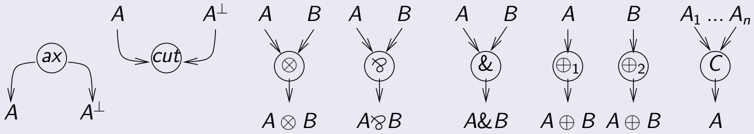
Definition (MALL Pre-Proof Structure)

A **PPS** π is an oriented graph built on the following *links*:



- entering (*premisses*) and exiting (*conclusions*) edges are labelled by MALL formulas;
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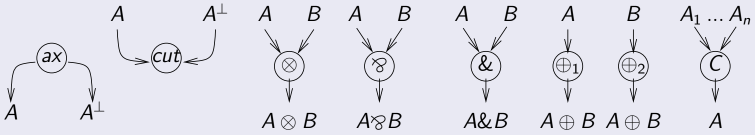

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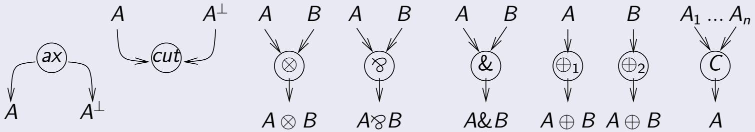

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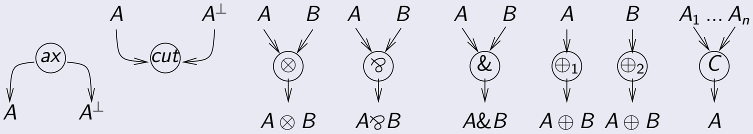

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Definition (Weights)

a **monomial weight** $w, v, \ldots$ is a product "." (conjunction) of Boolean variables or negations of Boolean variables $p, \bar{p}, q, \bar{q}, \ldots$

- ϵ_p, for a variable p or its negation $\overline{p}$;
- 1, for the empty product;
- 0, for a product where both p and $\bar{p}$ appear;
- two weights, v and w, are **disjoint** when $v.w = 0$.
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a **monomial weight** $w, v, \ldots$ is a product "." (conjunction) of Boolean variables or negations of Boolean variables $p, \bar{p}, q, \bar{q}, \ldots$

- ϵ_p, for a variable p or its negation $\overline{p}$;
- 1, for the empty product;
- 0, for a product where both p and $\bar{p}$ appear;
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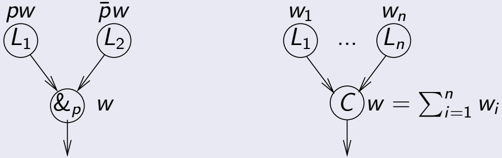

ϵ_p does not occur in w $\qquad$ $\forall i \forall j, w_i w_j = 0 \; (1 \leq i, j \leq n)$

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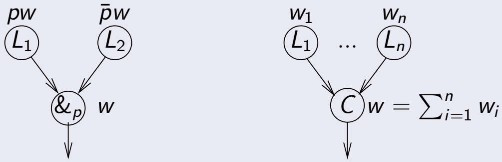

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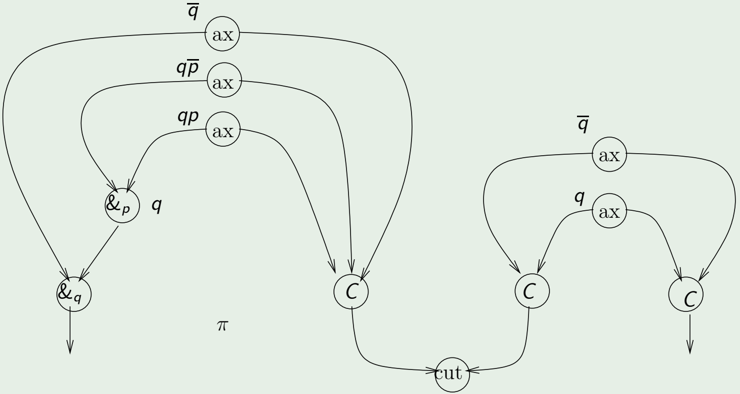
4. *dependency condition*: if v depends on p and w is the weight of the $\&_p$ node, then $v \leq w$.

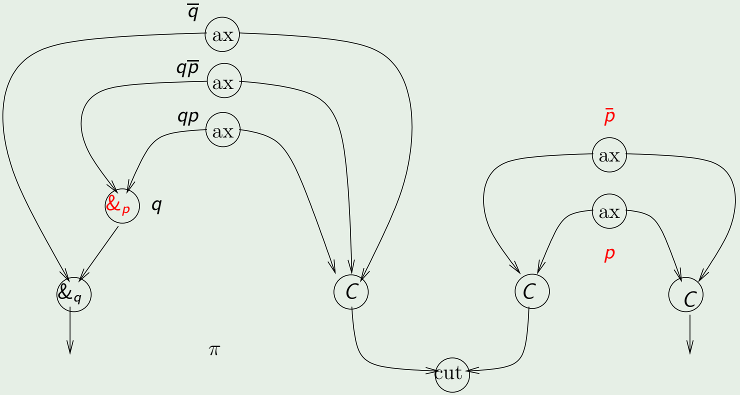
Example (1)

This is a GPS:



Example (2)

This is not a GPS: it violates the *dependency condition*, $p, \bar{p} \not\leq q$

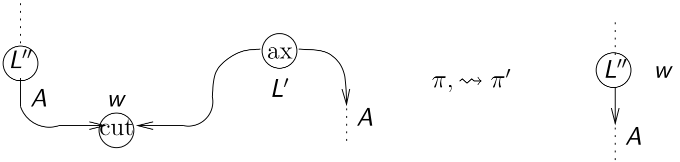


Cut Elimination

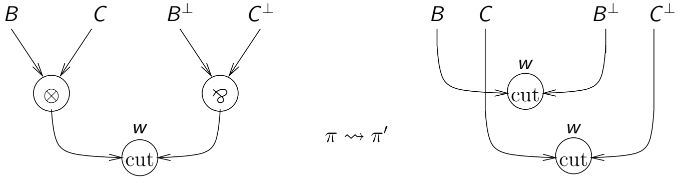
The original Girard's cut elimination is only lazy !

i.e., it only reduces the logical (or ready) cuts

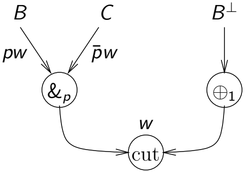
Ready Cut Elimination: *ax*-step



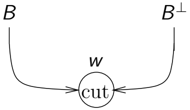
Ready Cut Elimination: $(\otimes/\parr)$-step



Ready Cut Elimination: $(\oplus_i/\&)$-step



$\pi \rightsquigarrow \pi'[p/1]$



π' is obtained by **erasing the $\bar{p}$ slice** in π (i.e., $p = 1$ resp., $\bar{p} = 0$).

Girard's cut elimination stops here!

...in the following we fix this problem

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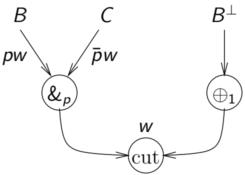

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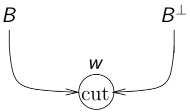

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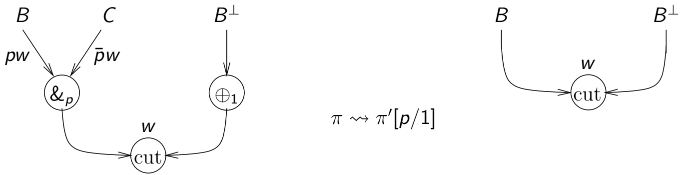

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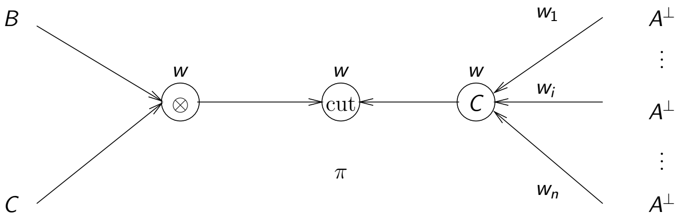
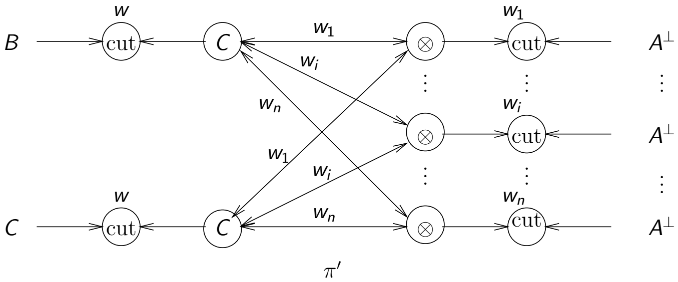
Commutative Cut Elimination: $(\otimes/C)$-step



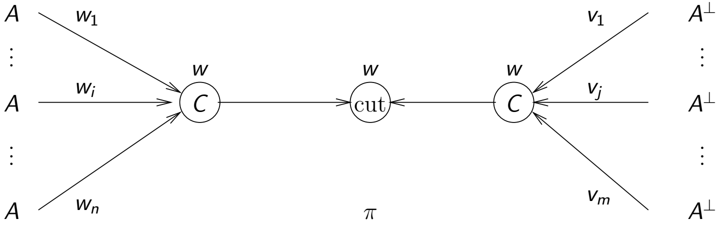
the $(\wp/C)$-step is similar

Commutative Cut Elimination: $(\otimes/C)$-step

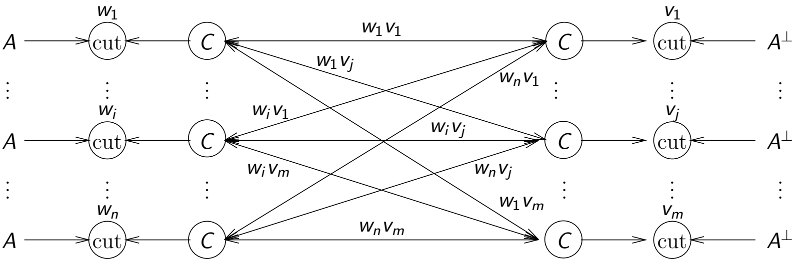


the "↔" edges are axiom links

Commutative Cut Elimination: (C/C)-step

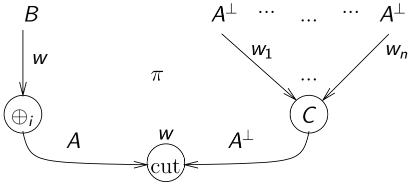


Commutative Cut Elimination: (C/C)-step

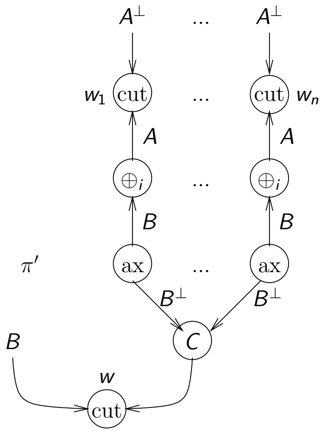


π'

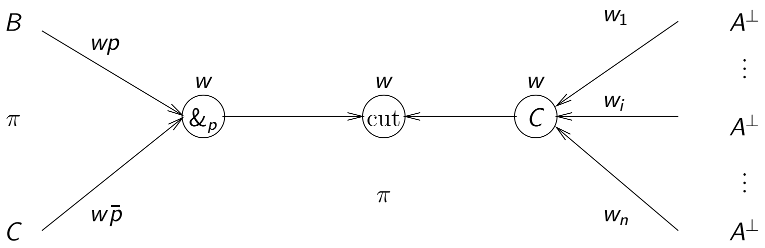
Commutative Cut Elimination: $(\oplus_i / C)$-step



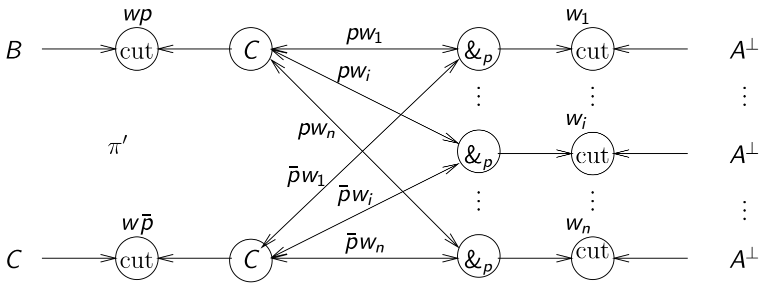
Commutative Cut Elimination: $(\oplus_i/C)$-step



Commutative Cut Elimination: the local ($\&/C$)-step

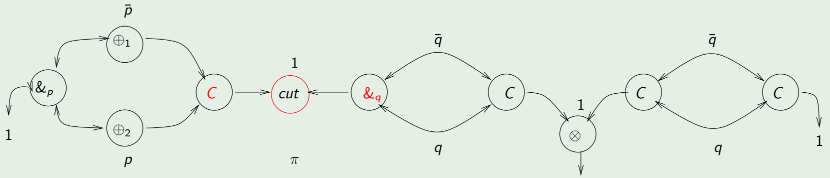


Commutative Cut Elimination: the local $(\&/C)$-step



Problems with the local $(\&/C)$ reduction step

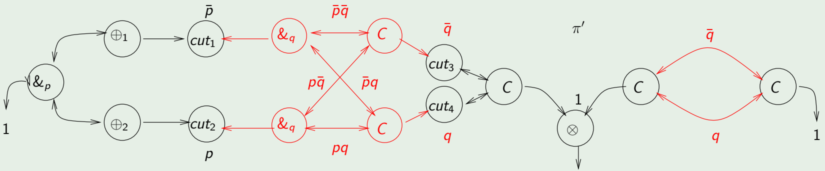
Example (4)



Let us reduce the $(C/\&_q)$ cut of the above GPS π

Problems with the local $(\&/C)$ reduction step

Example (4)

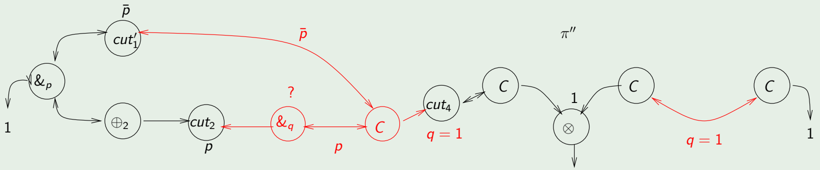


(1) We get a π' that **is not a PS**!

e.g. two axioms, with weights q and $\bar{q}$, do not satisfy the GPS *dependency condition* $(q, \bar{q} \not\leq p$, resp., $q, \bar{q} \not\leq \bar{p})$

Problems with the local $(\&/C)$ reduction step

Example (4)



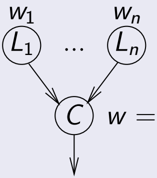
(2) π' may reduce the $(\oplus_1/\&_q)$ cut to π'' that **is not** even **a PPS**!

e.g., erasing the $\bar{q}$ slice induces a "degenerated" $\&_q$-link

Definition (New Monomial MALL Proof Structure)

A MALL **proof structure** (PS), is a pair $\langle \pi, E \rangle$ s.t.:

- $E = \{\epsilon_p.w = 0 \mid w \text{ is a weight } \epsilon_p\text{-free}\}$;
- π is a GPS with these modifications:
  - two & nodes may have the **same** eigen weight p;
  - all weights $v_1(\&_p), ..., v_n(\&_p)$ are **pairwise disjoint** ($v_i.v_j = 0$);
  - the weight of each contraction (C) is taken **modulo** E:



$w = \sum_{i=1}^{n} w_i \mod E$

$\forall i, j,\ w_i w_j = 0 \text{ and } w_i \leq w \ (1 \leq i, j \leq n)$

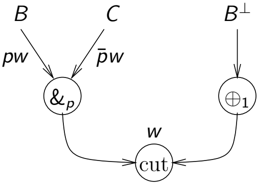
Definition (...continues)

new dependency condition: if w occurs in $\langle \pi, E \rangle$, then

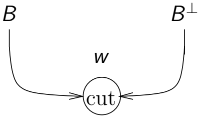
$$w \leq (\sum_{i=1}^{n} v_i) \mod E$$

- each v_i is :
  - either the weight of a node $\&_p$
  - or the suffix of an equation $\epsilon_p.v_i = 0$ of E;
- $\sum_{i=1}^{n} v_i$ is a monomial weight (modulo E);
- all weights $v_1, \dots v_n$ are pairwise disjoint.

Full Cut Elimination: the $(\oplus_i/\&)$-step



$\langle\pi, E\rangle \rightsquigarrow \langle\pi', E'\rangle$



- $E' = E \cup \{\bar{p}.w = 0\}$;
- π' is obtained from π by :
  - erasing the slice $\bar{p}$ rooted at w
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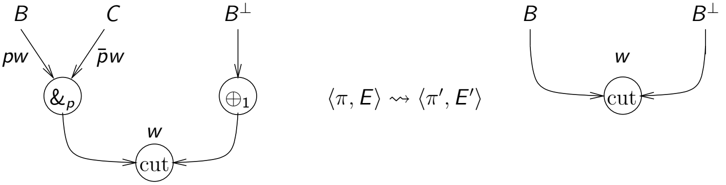

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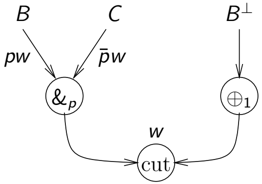
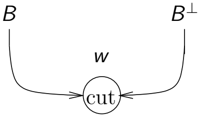


$\langle \pi, E \rangle \rightsquigarrow \langle \pi', E' \rangle$



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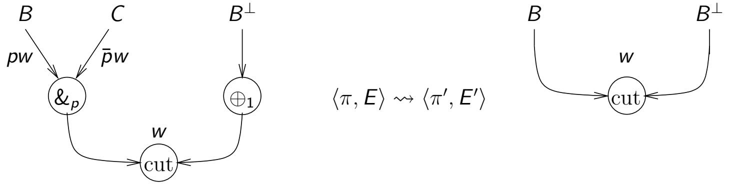

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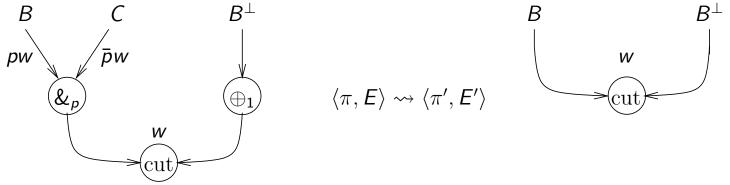

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Full Cut Elimination

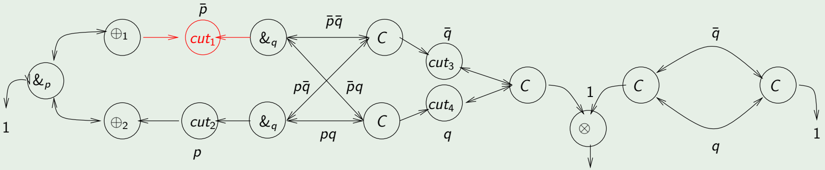
The below reduction steps R are performed like before with GPS

- *axiom*-step
- $(\otimes/\invamp)$-step
- $(\otimes/C)$-step
- $(\invamp/C)$-step
- $(\oplus_i/C)$-step
- $(\&/C)$-step
- (C/C)-step

$$\langle \pi, E \rangle \rightsquigarrow_R \langle \pi', E \rangle$$

– $\pi \rightsquigarrow_R \pi'$ like before with GPS
– E remains unchanged.

Example (5)



Observe: the $\langle \pi, \emptyset \rangle$ above is (now) a PS
(it satisfies the new dependency condition: $q, \bar{q} \leq p + \bar{p}$)

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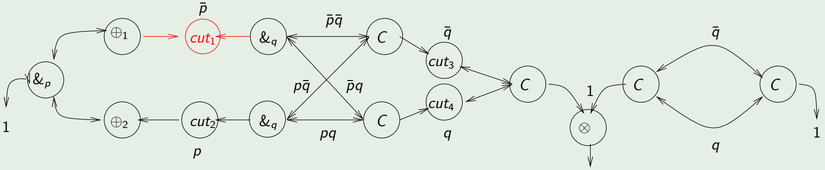
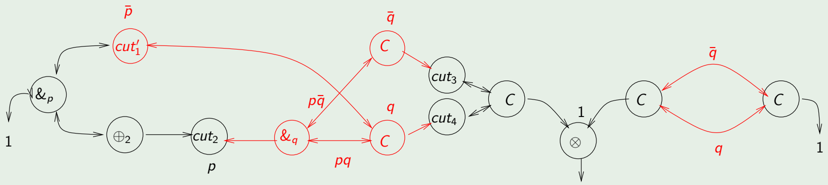


Let us reduce the cut_1 $(\oplus_1/\&_q)$.

Example (5)



We get the PS $\langle \pi', \{\bar{p}\bar{q} = 0\} \rangle$ above.

Observe: modulo $\{\bar{p}\bar{q} = 0\}$, $\bar{q} = p\bar{q}$ and $q = (\bar{p} + pq)$

Definition (Proof Nets)

A PS is correct (it is a **Proof Net**) if all local switchings are ACC.

(the notion of *local switching* is a variant of the Girard's switching)

Theorem (Sequentialization)

A PN with conclusion Γ *can be sequentialized into a sequent proof with same conclusion* Γ *and vice-versa.*

Proof.

- we exploit an expansion procedure which allows us to unfold each PN into a GPN;
- it can be shown that each expansion step preserves the *Girard's sequentialization.*

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Stability under the Cut Elimination

Theorem (Stability of proof structures)

$\langle \pi, E \rangle \rightsquigarrow \langle \pi', E' \rangle$ and $\langle \pi, E \rangle$ is a PS, then $\langle \pi', E' \rangle$ is a PS too.

Theorem (Stability of the correctness criterion)

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Strong Cut Elimination and Confluence

Theorem (Strong Cut Elimination)

We can always reduce a PN $\langle \pi, E \rangle$ into a PN $\langle \pi', E' \rangle$ that is cut-free; this reduction is strongly terminating.

Theorem (local confluence)

Assume a PN $\langle \pi, E \rangle$ s.t.

- $\langle \pi, E \rangle \rightsquigarrow_{cut_1} \langle \pi_1, E_1 \rangle$
- $\langle \pi, E \rangle \rightsquigarrow_{cut_2} \langle \pi_2, E_2 \rangle$,

with $cut_1 \neq cut_2$, then there exists PN $\langle \pi^*, E^* \rangle$ to which $\langle \pi_i, E_i \rangle$, for $1 \leq i \leq 2$, reduces in **at most one cut reduction step**.

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questions ?