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Preface

This special issue is devoted to some aspects of the new ideas that recently arose from the work of Thomas Ehrhard on the models of linear logic (LL) and of the λ-calculus. In some sense, the very origin of these ideas dates back to the introduction of LL in the 80s by Jean-Yves Girard. An obvious remark is that LL yielded a first *logical* quantitative account of the use of resources: the *logical* distinction between linear and non-linear formulas through the introduction of the exponential connectives. As explicitly mentioned by Girard in his first paper on the subject, the quantitative approach, to which he refers as 'quantitative semantics,' had a crucial influence on the birth of LL. And even though, at that time, it was given up for lack of 'any logical justification' (quoting the author), it contained rough versions of many concepts that were better understood, precisely introduced and developed much later, like differentiation and Taylor expansion for proofs. Around 2003, and thanks to the developments of LL and of the whole research area between logic and theoretical computer science, Ehrhard could come back to these fundamental intuitions and introduce the structure of finiteness space, allowing to reformulate this quantitative approach in a standard algebraic setting. The interpretation of LL in the category *Fin* of finiteness spaces and finitary relations suggested to Ehrhard and Regnier the differential extensions of LL and of the simply typed λ-calculus: Differential Linear Logic (DiLL) and the differential λ-calculus. The theory of LL proof-nets could be straightforwardly extended to DiLL, and a very natural notion of Taylor expansion of a proof-net (and of a λ-term) was introduced: an element of the Taylor expansion of the proof-net/term α is itself a (differential) proof-net/term and an approximation of α.

I find it interesting to stress the methodological similarities that led to the introduction of LL and of DiLL. The vast area of computer science called denotational semantics aims at giving mathematical counterparts to programming languages; through the Curry–Howard correspondence between proofs and programs, in proof theory this amounts to study the invariants of the cut-elimination process. A nice denotational model usually clarifies the mechanisms at work during the computational process. In some very rare cases, it also reveals some hidden structure of proofs and suggests improvements of the logical system itself. Girard's coherent model of the typed λ-calculus suggested the introduction of the exponential connectives and thus of LL proof-nets, widely acknowledged as one of the main novelties carried by LL. Ehrhard's finiteness spaces suggested the introduction of the co-structural rules and thus the representation of proofs as (possibly infinite) sums of differential nets, which have both a geometric nature (as graphs) and an algebraic one (as elements of the interpretation of proofs).

Using the traditional terminology of logic, one would say that these new proof-theoretical objects (proof-nets and even more differential nets) are in between syntax and semantics. Since there is a growing interest for this kind of object, let us recall here an interesting, rather old, and sometimes a bit neglected precedent in proof theory: Schütte's proof of the completeness theorem for first-order logic. From the computer science point of view, the capacity of proof-nets to abstract from the order in which certain logical rules are applied suggested the possibility to establish a link between logic and concurrency. However, despite a wide use of LL, no deep and really convincing logical insight into concurrency has ever been found yet. Since some kind of non-determinism is present in DiLL's cut-elimination, there is hope that the differential extension of LL can give a new breath to the relationship between proof theory and concurrency.

This volume presents seven contributions, all related to the new perspectives opened by the ideas underlying the discovery of DiLL.

The first paper, 'An introduction to DiLL: proof-nets, models and antiderivatives' by Thomas Ehrhard, will probably become a reference for the researchers interested in differential calculi. The author first presents a semantic-driven approach to DiLL and discusses the deep motivations which led to its discovery. He then introduces (a possible formalization of) differential nets with boxes, for which he gives a categorical interpretation: with every DiLL derivation is associated a morphism, and it is stated that two derivations corresponding to the same differential net have the same interpretation. As pointed out by the author, a direct interpretation of differential nets would be much more satisfying, but though the existing categorical models of DiLL (and LL) usually associate a morphism with a sequent calculus proof, this time the author sets up a framework in which one can prove that such a morphism is independent from the chosen sequentialization of any differential net (and thus in particular of any LL proof-net). In the very general context considered in the paper, the natural question of the existence of antiderivatives is addressed, and the author gives a sufficient condition for the existence of the antiderivative of a morphism in a suitable category. The last part of this work is devoted to apply the constructions introduced in the categorical setting (including antiderivatives) in two important concrete models: the relational model, which is quickly revised, and the finiteness space model, which is carefully described.

Since a finitary relation, which is a morphism of the category *Fin*, is a particular relation, it is very natural to investigate whether and how some of the most distinctive properties of the category *Rel* of sets and relations can be given counterparts in the category *Fin*. In the second paper, 'Transport of finiteness structures and applications' by Christine Tasson and Lionel Vaux, the authors prove a key property of *Fin*: the *transport lemma*. This powerful tool allows them to develop a general theory of *Rel* functors that can be transported into *Fin*, and they apply this theory to show that a wide class of recursive datatypes can be encoded as *Fin* functors, thus generalizing similar results previously obtained by Ehrhard.

The two following papers contribute to the debate on the links between DiLL and concurrency. The aim is to model the so-called *true concurrency* that is to find models of interaction that are not interleaving: parallel actions cannot be rendered as sums of possible sequentializations. In 'The true concurrency of Differential Interaction Nets,' Damiano Mazza questions, in an abstract and general framework, the possibility of representing true concurrency in Differential nets, arguing that, under certain hypothesis, this is impossible. On the other hand, in 'Order algebras: a quantitative model of interaction,' Emmanuel Beffara introduces the structure of *order algebra*, which yields a model of concurrent interaction with quantitative features. Even if order algebras are not (yet) a model of DiLL, they represent non-determinism as linear combinations like differential nets, which is promising in the perspective of building concurrent calculi related to logic in a Curry–Howard sense.

In 'Execution Time of λ-Terms via Denotational Semantics and Intersection Types,' Daniel de Carvalho proposes a quantitative approach to the execution of λ-terms, by establishing a precise link between the number of steps that Krivine's machine uses to execute a λ-term and, on the one hand, the size of type derivations and, on the other hand, the size of types, in a particular intersection types system (system R). This analysis has been later extended to cut elimination in multiplicative exponential LL. All the results are formulated without reference to DiLL, but one could use differential nets: indeed, types are closely related to the points of the interpretation of a λ-term (or a proof-net) in *Rel*, which are themselves closely related to the differential nets occurring in the Taylor expansion of the λ-term (or the proof-net).

In the paper 'Jump from Parallel to Sequential Proofs: exponentials' by Paolo Di Giamberardino, the question at stake is the relation between parallel and sequential proofs, a crucial one since the inception of LL. Di Giamberardino extends to the exponential connectives the use of *jumps* to analyse sequentiality in polarized LL: a jump can be thought as an untyped extra edge adding some sequentiality constraints, and the use of jumps allows then to represent proofs with different levels of sequentiality. A notable feature of this work is the generalization of the notion of exponential box with the one of *cone*: contrary to the order relation induced by boxes in a proof-net, the order relation induced by the cones of a proof-net with jumps is not necessarily arborescent, since cones can overlap. Di Giamberardino proves that this fact does not prevent cut elimination of proof-nets with jumps from enjoying the standard good properties of strong normalization and confluence.

The last paper of the volume, 'An explicit formula for the free exponential modality of LL′ by Paul-André Melliès, Nicolas Tabareau and Christine Tasson, attacks the general question of the categorical counterpart of the exponential connectives of LL. Understanding the exponentials is obviously a crucial question, which hides behind every one of the contributions presented in this volume. This work revisits Lafont's proposal to interpret the !-modality by the free commutative comonoid construction in symmetric monoidal categories and provides a simple description of this construction under a few commutation conditions. The authors study the cases of coherence spaces and Conway games, for which they show that the free !-modality can be obtained applying their method. This *is not* the case for the free !-modality of Ehrhard's finiteness spaces: a new model of LL is then introduced (configuration spaces) and compared to the model of finiteness spaces.

I want to thank all the authors for their contributions and the anonymous referees for their work. A special thank has to be addressed to Giuseppe Longo, the promoter of this special issue. During one of his staying in Roma Tre, by attending the seminars of our research group, I guess he realised that something new and interesting was happening

around the differential extension of LL and the λ-calculus, and he thus proposed me to take care of a special issue on these themes. Even if it took quite some time to go through the whole process, I do hope this volume will be appreciated by the researchers in the field.

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