Jump from Parallel to Sequential Proofs: Exponentials

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Abstract

In previous works, by importing ideas from game semantics (notably Faggian-Maurel-Curien's *ludics nets*), we defined a new class of multiplicative/additive polarized proof nets, called *J-proof nets*. The distinctive feature of J-proof nets with respect to other proof net syntaxes, is the possibility of representing proof nets which are partially sequentialized, by using *jumps* (that is, untyped extra edges) as sequentiality constraints. Starting from this result, in the present work we extend J-proof nets to the multiplicative/exponential fragment, in order to take into account structural rules: more precisely, we replace the familiar linear logic notion of exponential box with a less restricting one (called *cone*) defined by means of jumps. As a consequence, we get a syntax for polarized nets where, instead of a structure of boxes nested one into the other, we have one of cones which can be *partially overlapping*. Moreover, we define cut-elimination for exponential J-proof nets, proving, by a variant of Gandy's method, that even in case of "superposed" cones, reduction enjoys confluence and strong normalization.

Introduction

Since its inception in 1987 [Gir87], Linear Logic has proved to be a useful tool to enlighten and deepen the relation between *proofs* and *programs*, in the framework of Curry-Howard isomorphism.

Born from a fine semantical analysis of intuitionistic logic, Linear Logic (briefly LL) provides a logical status to the structural rules (*weakening* and *contraction*) of sequent calculus (due to the introduction of the exponential connectives, ! and ?) and splits the usual propositional connectives ("*and*", "*or*") in two classes (the additives $\&, \oplus$, and the multiplicatives $\otimes, \parr$).

The most relevant byproducts of such a refinement are a logical characterization of resource-bounded computation, and the introduction of a graph-theoretical syntax for LL, (the *proof nets*), first introducing parallelism in proofs representation.

Due to its huge expressive power (full second-order LL being as powerful as system F [Gir72]), Linear Logic has been a central topic of research over the last two decades, for different aims and purposes. From one side, a lot of work has been done to analyze the syntactical and semantical structure of LL itself (for a detailed survey, see [Gir06],[Gir07]). From the other side, a variety of subsystems and systems derived from LL have been considered, in order to characterize specific properties (as for example polytime bounded computation, see [Gir98], [Laf04]): among them, a remarkable place is held by *polarized systems*, which have been extensively studied by Olivier Laurent (see [Lau02]).

The cornerstone of these systems is the distinction (defined by Girard in [Gir91b], following the work of Andreoli [And92]) inside LL between *negative* and *positive* formulas, according to two dual syntactical properties: *reversibility* and *focalization*, respectively. A polarized system is a system restricted only to polarized formulas.

The discovery of the positive/negative duality has then contributed in an essential way to the achievement of many important results inside the *proofs-as-programs* paradigm, notably:

- the exploitation of the computational content of classical proofs, in particular its relation with the $\lambda\mu$-calculus (see [Gir91b], [LR03], [Lau03]);
- the development of game models for linear logic, with proofs of injectivity and full completeness (see [Lau04],[Lau05]);

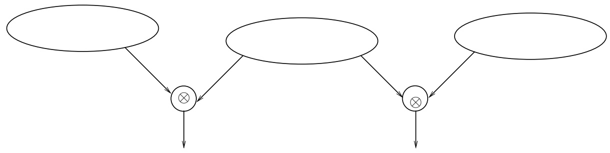


Figure 1: Two "parallel" terminal $\otimes$ rules

- the reduction of non-determinism in linear logic proof search, establishing a new paradigm for linear logic programming (see [And02],[FM08], [Mil95]);
- the birth of *ludics*, a pre-logical framework giving an *interactive* account of logic (see [Gir01], [FH02], [Ter08]);
- the advances in the search for a logical characterization of concurrency, by the interpretation of π-calculus in polarized systems (see [EL10],[HL10],[FP07]).

Nonetheless, the benefits of polarized systems have a cost: *the loss of parallelism.*

As a matter of fact, the restriction to polarized formulas imposes a strict alternation in proofs between a "positive" phase (introducing positive formulas) and a "negative" one (introducing negative formulas), with the exponentials being in charge of switching between polarities; eventually, such a discipline makes polarized systems *sequential* in a strong sense.

As a further evidence, the usual proof net syntax (which represents the ! rule by means of a *box*, the correspondent of a sequent in proof nets syntax), in the framework of polarized systems, is no longer able to represent two positive rules in parallel (as in Fig. 1)[1] , such a configuration being the "core" of the parallelism induced by proof nets in standard LL.

A lot of work has been done recently (mostly from the semantical side, see [AM99, HS02, Abr03, Mel04, Mel05, MM07]) to try to free polarities from such a sequential framework. In [DG08, DGF06, DG09], taking L-nets of Faggian-Maurel-Curien (see [FM05, CF05]) as a model, we proposed a framework for polarized prof nets of the multiplicative and multiplicative additive fragment (called *J-proof nets*), where partially sequentialized nets are allowed; our principal tool relies on the notion of *jump*, that is untyped edges expressing sequentiality constraints, introduced by Girard in [Gir91a].

In the present work we extend the framework of J-proof nets to the multiplicative exponential fragment: our principal result is the replacement of the familiar linear logic notion of exponential box with a less restricting one (called *cone*) defined by means of jumps.

The main difference is that while exponential boxes satisfy a *nesting condition* (that is, any two exponential boxes are either disjoint or included one into the other), cones *can overlap* (that is, the intersection of two cones may not be empty, while neither of them is included in the other); in such a way we recover the possibility of representing the configuration given in Fig. 1.

Moreover, cones are *computationally meaningful*; that is, with respect to cut-elimination, they behave exactly like boxes, allowing to isolate the part of the net to be erased or duplicated during structural reductions.

We stress that replacing boxes with less "sequential" structures (i.e. cones) is quite a novelty, since exponential boxes are commonly believed to be the last, impregnable, stronghold of sequent calculus inside the proof nets syntax; the fact that such an operation naturally arises in a framework (the polarized one) usually considered strongly sequential represents another, unexpected surprise.

Related and future works

In the present paper we replace, in the setting of polarized linear logic, the *explicit* notion of exponential box with the *implicit* notion of cone, which is retrieved by the introduction of jumps. A similar approach is used also by Accattoli and Guerrini in [AG09], with the introduction of Λ-nets, a graph syntax for

[1]This happens because, w.r.t. the configuration given in Fig. 1, in polarized proof nets above the left premise of the right hand $\otimes$ (resp. the right premise of the left hand $\otimes$) eventually there will be a ! link a (resp. b); but then by the usual ! box condition of proof nets, either the left hand $\otimes$ is included in the box associated with a, or the right hand $\otimes$ is included in the box associated with b, so that the two $\otimes$ cannot be at the same "level" See [Lau02] for more details.

λ-terms, where jumps are used to represent sub terms which have a non-linear behavior (i.e. *boxes*). The main difference between the present work and the one of Accattoli-Guerrini is the role of the nesting condition: we introduce cones in order to generalize exponential boxes, relaxing the nesting condition; Accattoli and Guerrini use jumps to reconstruct standard exponential boxes, accepting nesting condition as it is.

The notion of cone seems to be linked to other traditional notions coming from proof nets, like the ones of *empire* and *kingdom* (see [BVDW95] for definitions), as we pointed out in [GF08]; such connections deserve to be properly investigated. In this context, several interesting observations about jumps, boxes and kingdoms in a polarized setting are contained in Accattoli's PhD thesis (see [Acc11]).

We are confident that the semantical analysis of J-proof nets (which we postpone to future work), both *static* (the family of *coherent spaces* based models) and *dynamic* (games, and especially the recent advances on *exponential ludics*, see [BF09]), will shed new light on the nature of cones and its computational meaning. A good tool to perform such analysis may be the notion of *thick subtree*, introduced by Pierre Boudes in [Bou09] to relate static and dynamic semantics of polarized proof nets.

Outline of the paper

The paper is divided in the following six sections:

Section 1: we provide the reader with basic notions concerning polarized systems, graphs and rewriting.

Section 2: we present the syntax of J-proof nets, and the fundamental notion of *cone*, analogous to the one of exponential box in our setting. Basically, we will define cones as upward-closed subgraphs of a net, retrieved from the sequentialization order induced by jumps.

Section 3: we define a *correctness criterion* and prove *sequentialization* for J-proof nets: the criterion will take into account the presence of cones (as the correctness criterion for multiplicative/exponential proof nets takes into account the presence of boxes).

Section 4: we define cut-elimination on J-proof nets, and prove some properties of reduction, namely *weak normalization* and *local confluence*. The portion of a J-proof net to be erased or duplicated during reduction will be determined using cones.

Section 5: using a variation of Gandy's method, we prove *strong normalization* and *confluence* of reduction on J-proof nets.

Section 6: the final section is dedicated to concluding remarks and observations; we will discuss about axioms, the role of the Mix rule for the confluence result, and the relation between J-proof nets and polarized proof nets.

1 Preliminaries

First we present the system $MELLP$ (*multiplicative exponential polarized linear logic*) of Laurent (see [Lau02]); then we modify it to get another system (called *multiplicative exponential hypersequentialized calculus*, briefly $MEHS$), based on the hypersequentialized calculus of Girard (see [Gir00]), which will serve better our purpose. The rest of the section is a reminder of some basic notions of graph and rewriting theory.

1.1 Polarization

A multiplicative/exponential *polarized formula* is a formula obtained by the following grammar (where $n \in \mathbb{N}$ and X range over an enumerable set of propositional variables):

$$\begin{array}{lcccl} N & ::= & X^\perp & \mid & \parr_{i=1}^n(?P_i) \\ P & ::= & X & \mid & \otimes_{i=1}^n(!N_i) \end{array}$$

X and $X^\perp$ will be called *atoms*; if $n = 1$, then we denote $\otimes_{i=1}^n(!N_i)$ by $!N$ (resp. $\parr_{i=1}^n(?P_i)$ by $?P$); if $n = 0$, then we denote $\otimes_{i=1}^n(!N_i)$ by 1 (resp. $\parr_{i=1}^n(?P_i)$ by $\perp$).

Duality is defined as follows:

$$P^{\perp\perp} = P$$

$$\otimes_{i=1}^{n}(!N_i)^{\perp} = ⅋_{i=1}^{n}(?(N_i^{\perp}))$$

$$⅋_{i=1}^{n}(?P_i)^{\perp} = \otimes_{i=1}^{n}(!(P_i^{\perp}))$$

Remark 1 *Our definition of polarized formulas relies on the notion of* synthetic connective *([Gir99]): that is, given a multiset of negative formulas* $\{N_1, \ldots, N_n\}$ *(resp. of positive formulas* $\{P_1, \ldots, P_n\}$*) by* $\otimes_{i=1}^{n}(!N_i)$ *(resp.* $⅋_{i=1}^{n}(?P_i)$*) we indicate the formula which corresponds to all possible combinations of the formulas* $\{!N_1, \ldots, !N_n\}$ *(resp.* $\{?P_1, \ldots, ?P_n\}$*) by the usual binary* $\otimes$ *(resp.* ⅋*) connective of linear logic, equivalent modulo associativity of* $\otimes$ *(resp.* ⅋*) and neutrality of the multiplicative constant* 1 *(resp.* $\perp$*) w.r.t.* $\otimes$ *(resp.* ⅋*).*

Given a polarized formula A , we call *immediate* positive (resp. negative) subformulas of A:

- A, if A is a positive (resp. negative) formula;
- P_i (resp. N_i) if $A = ⅋_{i=1}^{n}(?P_i)$ (resp. $\otimes_{i=1}^{n}(!N_i)$).

Let Γ be a multiset of polarized formulas: by $⅋(\Gamma)$ we denote the formula $⅋_{i=1}^{n}(?P_i)$ where $\{P_1, \ldots, P_n\}$ is the multiset containing, for each formula $A \in \Gamma$, all the immediate positive subformulas of A.

By $\mathcal{N}$ (resp. $\mathcal{P}$) we denote a multiset of negative (resp. positive) formulas.

By $?\mathcal{P}$ we denote the multiset containing $?P$ for every formula $P \in \mathcal{P}$.

The more delicate issue concerning polarities is the polarization of *atoms*, since, while non-atomic formulas are naturally polarized, the polarity assigned to atoms is *arbitrary.* For the sake of simplicity then, in the rest of the paper *we will always consider polarized formulas which do not contain atoms*; we will consider the wider picture, including also atoms, in Section 6.1.

1.1.1 Multiplicative Exponential Polarized Linear Logic ($MELLP$)

The sequent calculus of the multiplicative and exponential fragment of polarized linear logic is obtained by restricting LL to polarized formulas, and allowing structural rules on *negative formulas*:

$$\frac{\vdash \Gamma, ?P_1, \ldots, ?P_n}{\vdash \Gamma, ⅋_{i=1}^{n}(?P_i)}\,⅋ \qquad \frac{\vdash \Gamma_1, !N_1 \qquad \vdash \Gamma_n, !N_n}{\vdash \Gamma_1, \ldots, \Gamma_n, \otimes_{i=1}^{n}(!N_i)}\,\otimes$$

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\,! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\,d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, N}\,w \qquad \frac{\vdash \Gamma, N, \ldots, N}{\vdash \Gamma, N}\,c$$

$$\frac{\vdash \Gamma,\ P \qquad \vdash \Delta,\ P^{\perp}}{\vdash \Gamma,\ \Delta}(Cut)$$

The structure of the calculus verifies the following property (see [Lau02]):

Proposition 1 *Every provable sequent in $MELLP$ contains at most one positive formula.*

Remark 2 *The* 0*-ary cases of the* ⅋ *(resp.* $\otimes$*) rules correspond to the usual rules for the multiplicative constants of Linear Logic, as depicted below:*

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \frac{}{\vdash 1}\,1$$

1.1.2 Multiplicative Exponential Hypersequentialized Logic (*MEHS*)

In order to better enlighten the hidden sequential structure induced by polarities, we switch to another polarized sequent calculus , based on the the *hypersequentialized* calculus, introduced by Girard in [Gir00]. Such a calculus is obtained from the previous one, by clustering together the rules introducing formulas of the same polarity: $\otimes$ and promotion rules are clustered into a unique *positive rule*, while $\parr$, dereliction and structural rules are clustered into a single *negative* rule. In this way we obtain a calculus with only two, strictly alternating, "logical" rules : the positive and the negative one.

The sequent calculus of the multiplicative and exponential fragment of hypersequentialized logic (briefly $MEHS$) is depicted below; such a calculus has the general constraint that *each sequent can contain at most one negative formula*.

$$\frac{\vdash \Gamma_1,\ N_1 \ldots \vdash \Gamma_n,\ N_n}{\vdash \Gamma_1, \ldots, \Gamma_n,\ \otimes_{i=1}^{n}(!N_i)}(+) \qquad \frac{\vdash \Gamma,\ P_1^1, .., P_1^{k_1}, \ldots, P_n^1, .., P_n^{k_n}}{\vdash\ \Gamma,\ \parr_{i=1}^{n}(?P_i)}(-)$$

$$\frac{\vdash \Gamma,\ P \qquad \vdash \Delta,\ P^{\perp}}{\vdash \Gamma,\ \Delta}(Cut)$$

We stress that in the $-$ rule:

- $n, k_1, \ldots, k_n, \in \mathbb{N}$;
- $P_i^j = P_i^{j'}$, for $i \leq n$ and $j, j' \leq k_i$.

Notice that in case $k_i = 0$, $?P_i$ is a weakened formula; in case $k_i = 1$, $?P_i$ is a derelicted formula; in case $k_i > 1$, $?P_i$ are contracted formulas.

Due to the clusterization of rules, the constraint on sequents in $MEHS$ (at most one negative formula) is reversed with respect to $MELLP$ (see Proposition 1). Nevertheless, provability in $MELLP$ and in $MEHS$ are equivalent (modulo translations which allow to switch from the constraint of $MELLP$ to the one of $MEHS$, and vice versa):

Proposition 2 *For every proof π of a sequent $\vdash \Gamma$ in $MELLP$ there exists a proof π' of $\vdash \parr(\Gamma)$ in $MEHS$.*

Proof. The proof is by induction on the height of π; since cut-elimination holds for $MELLP$, we can suppose π cut-free (so by induction π' is cut-free too). We have different cases depending on the last rule r of π: we show only the case where r is a ! rule, the others being either trivial or similar to this one. If r is a ! rule, then we apply the induction hypothesis to the proof π_0 of its premise $\vdash \mathcal{N}, N$ obtaining a cut free proof π_0' of $\vdash \parr(\mathcal{N}, N)$ in $MEHS$. The last rule of such a proof must be a $-$ rule having as premise the proof π_1' of $\vdash \mathcal{P}_1, \ldots, \mathcal{P}_m, P_1^1, \ldots, P_1^{k_1}, \ldots, P_n^1, \ldots, P_n^{k_n}$, where each $P_i^{k_j}$ (resp. each $\mathcal{P}_i$) is an immediate positive subformula of N (resp. a multiset of immediate positive subformulas of a formula in $\mathcal{N}$). To get π' we apply to the conclusion of π_1' the following sequence of rules (in this order): first a $-$ rule having as conclusion $\vdash \mathcal{P}_1, \ldots, \mathcal{P}_m, N$, then a $+$ rule with conclusion $\vdash \mathcal{P}_1, \ldots, \mathcal{P}_m, !N$ and finally a $-$ rule with conclusion $\vdash \parr(\mathcal{N}, !N)$.

$\square$

Proposition 3 *For every proof π of a sequent $\vdash \mathcal{P}, N$ in $MEHS$ (where N is the unique negative formula of the sequent, if it exists), there exists a proof π' of $\vdash ?\mathcal{P}, N$ in $MELLP$.*

Proof. The proof is a simple induction on the height of π, and we leave its verification to the reader. $\square$

We must remark that in $MEHS$ in general cut-elimination does not hold: this is not a real problem, since the only reason behind this limitation lies in the clusterization of the structural rules into the negative one.

A simple way to restore cut elimination is to restrict the focus on *closed* proofs. By closed proof we mean all proofs of $MEHS$ whose final sequent does not contain positive formulas; actually this is not a true restriction, since it is straightforward that every proof π of a sequent $\vdash \Gamma$ of $MEHS$ can be turned into a closed proof π' of $\vdash \otimes(\Gamma)$ by properly adding a final $-$ rule to π (or by properly modifying the final $-$ rule of π, if this is the case), so nothing is lost in term of provability.

Combining the closure of proofs with cut-elimination we can state the following:

Proposition 4 *For every proof π of a sequent $\vdash \Gamma$ in $MEHS$, there exists a cut-free proof π' of $\vdash \otimes(\Gamma)$ in $MEHS$.*

Proof. We will prove that in Section 4.3.2. □

We extend $MEHS$ with the following rule, called *Mix*[2] :

$$\frac{\vdash \mathcal{P}_1 \qquad \ldots \vdash \mathcal{P}_n}{\vdash \mathcal{P}_1, \ldots, \mathcal{P}_n} \, Mix$$

The 0-ary case of the Mix rule corresponds to the introduction of the empty sequent; in this case the following rule becomes derivable:

$$\frac{}{\vdash N} \, Dai$$

We shall make clearer in Section 6.2 the reason behind the introduction of the Mix rule.

1.2 Basic notions on graphs

A *directed graph* G is an ordered pair (V, E), where V is a finite set whose elements are called *nodes*, and E is a set of ordered pairs of nodes called *edges*; if $\langle a, b\rangle$ belongs to E, we say that there is an edge going from the node a to the node b in G.

We say that an edge from a to b is *emergent* from a and *incident* on b; b is called the *target* of x and a is called the *source*. Two nodes a, b *share* an edge x when x is emergent from a and incident on b (or vice versa).

Given a directed graph $G = (V, E)$ and a subset V' of V the *restriction* of G to V is the directed graph (V', E'), where E' is the subset of E containing only elements of V'.

Given a directed graph G a *path* (resp. *directed path*) r from a node b to a node c is a sequence $\langle a_1, \ldots, a_n\rangle$ of nodes such that $b = a_1$, $c = a_n$, and for each a_i, a_{i+1}, there is an edge x either from a_i to a_{i+1}, or from a_{i+1} to a_i (resp. from a_i to a_{i+1}); in this case, x is said to be *used* by r; moreover, we require that all nodes in a path from a node b to a node c are distinct, with the possible exception of b and c.

A graph G is *connected* if for any pair of nodes a, b of G there exists a path from a to b.

A *cycle* (resp. *directed cycle*) is a path (resp. directed path) $\langle a_1, \ldots, a_n\rangle$ such that $a_1 = a_n$.

A *directed acyclic graph* (d.a.g.) is a directed graph without directed cycles.

When drawing a d.a.g we will represent edges oriented up-down so that we may speak of moving downwardly or upwardly in the graph; in the same spirit we will say that a node is just above (resp. hereditary above) or below (resp. hereditary below) another node.

A *d.a.g. with pending edges* is a d.a.g. G where edges with a source but without a target are allowed.

We call *module* a d.a.g with pending edges G, where edges with a target but without a source are allowed.

We call *typed d.a.g.* a d.a.g. whose edges are possibly labelled with formulas (called *types*); we call such edges *typed*.

We call *typed d.a.g. with ports* a typed d.a.g. where for each node b the typed edges incident on b are partitioned into subsets called *ports*, in such a way that if two edges belong to the same port of b, they have the same type.

When drawing a typed d.a.g. with ports, we will denote ports by black spots, unless (for simplicity's sake) when a port contains a single edge.

We recall that the transitive closure of a d.a.g. G induces a strict partial order $<_G$ on the nodes of G, defined as follows: $a <_G b$ iff there is and edge from b to a in the transitive closure of G.

[2] Admitting such a rule corresponds, in proof net syntax, to discarding connectedness from correction graphs (see [Gir96],[TdF00]).

We call *immediate predecessor* of a node b, a node a such that, in the order $<_G$ associated with G, $a <_G b$, and there is no c such that $a <_G c$ and $c <_G b$. Similarly we can define the notion of predecessor, immediate successor and successor.

A strict order on a set is *arborescent* when each element has at most one immediate predecessor.

1.3 Preliminaries on rewriting theory

Let $\xrightarrow{x}$ be a binary relation on a set A; by $\xrightarrow{x}_*$ we denote the reflexive/ transitive closure of $\xrightarrow{x}$. We say that an element R of A is *in normal form* for $\xrightarrow{x}$, whenever there is no $R' \in A$ with $R \xrightarrow{x} R'$; R is *weakly normalizable* for $\xrightarrow{x}$, whenever there is an $R_1 \in A$ such that $R \xrightarrow{x}_* R_1$, and R_1 is in normal form for $\xrightarrow{x}$; R is *strongly normalizable* for $\xrightarrow{x}$ whenever there is no infinite sequence $(R_i)_{i \in N} \in A$ such that $R_0 = R$ and $R_i \xrightarrow{x} R_{i+1}$. We denote by WN^x and SN^x the elements of A which are respectively weakly normalizable and strongly normalizable for $\xrightarrow{x}$. Given a set $B \subseteq A$ if $B \subseteq WN^x$ (resp. $B \subseteq SN^x$) we say that $\xrightarrow{x}$ is *weakly normalizing* (resp. *strongly normalizing*) for B. We say that the relation $\xrightarrow{x}$ is *locally confluent* for $B \subseteq A$ if for every $R, R_1, R_2 \in B$ such that $R_1 \xleftarrow{x} R \xrightarrow{x} R_2$ there exists an $R_3 \in B$ such that $R_1 \xrightarrow{x}_* R_{3*} \xleftarrow{x} R_2$; we say that $\xrightarrow{x}$ is *confluent* for $B \subseteq A$ if for every $R, R_1, R_2 \in B$ such that $R_{1*} \xleftarrow{x} R \xrightarrow{x}_* R_2$ there exists an $R_3 \in B$ such that $R_1 \xrightarrow{x}_* R_{3*} \xleftarrow{x} R_2$. We say that the relation $\xrightarrow{x}$ is *increasing* for $B \subseteq A$ if there is a mapping $|-|$ from B to integers such that for all $R_1, R_2 \in B$, if $R_1 \xrightarrow{x} R_2$ then $|R_1| < |R_2|$.

Proposition 5 (Gandy) *If a relation $\xrightarrow{x}$ is increasing, confluent and weakly normalizing on a set A, then it is strongly normalizing on A.*

Proof. See [Gan80]. □

Proposition 6 (Bezem-Klop) *If a relation $\xrightarrow{x}$ is increasing, locally confluent and weakly normalizing on a set A, then it is strongly normalizing on A.*

Proof. See [Ter03]. □

Lemma 1 (Newman) *If a relation $\xrightarrow{x}$ is strongly normalizing and locally confluent on a set A, then it is confluent on A.*

Proof. See [Ter03]. □

2 J-nets

Definition 1 (J-net) *A* J-net *is a typed d.a.g. with ports and with pending edges, whose edges are possibly typed by polarized formulas and whose nodes (also called* links*) are labelled by one of the symbols $+, -, cut$. An edge typed by a positive (resp. negative) formula will be called positive (resp. negative) edge.*

The typed edges incident on a link are called premises *and the typed edges emergent from a link are called* conclusions *of the link; a pending edge is called a* conclusion *of the proof structure and its source is called a* terminal link.

The label of a link imposes some constraints on both the number and the types of its premises and conclusions:

- *the cut-link has no conclusions and two premises labelled by dual formulas, each of them belonging to a distinct port;*
- *the* negative link *(or $-$ link) has $m \geq 0$* ordered *ports and one conclusion. If the type of the edges belonging to the i-th port is P_i then the conclusion is labelled by $\parr_{i=1}^{n}(?P_i)$, for $n \geq m$;*
- *the* positive link *(or $+$ link) has $n \geq 0$ ordered* ports*, each containing a unique premise, and one conclusion. If the i-th premise is labelled by the formula N_i then the conclusion is labelled by $\otimes_{i=1}^{n}(!N_i)$.*

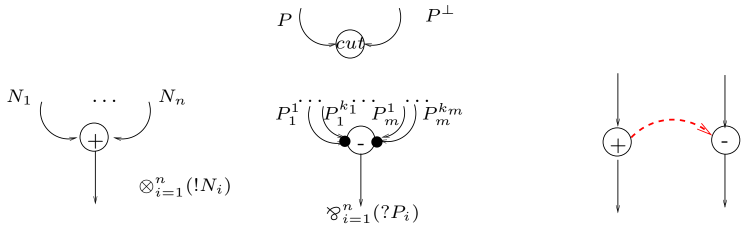


Figure 2: MEHS links

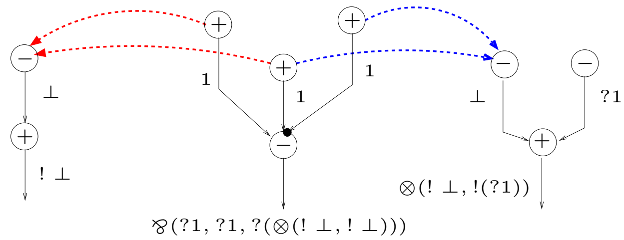


Figure 3: an example of a J-net

Moreover:

1. *we allow untyped edges, called* jumps*, oriented from positive to negative links (we will usually draw them as dashed lines);*

2. *we impose the constraint that a J-net has* at most *one negative conclusion.*

2.1 Order associated with a J-net

The role of jumps in J-net is to express a sequentiality constraint: if a positive link a jumps on a negative link b, this means that a "follows" b, so (bottom-up) we cannot access a unless we have accessed b first. Together with the natural notion of sequentiality induced by the premise/conclusion structure of links, this allows to retrieve an *order* (denoted by $\prec_R$) between links of a J-net R. In case R is cut free, such an order (denoted by $\prec_R$) coincides with the order $<_R$ associated to R as a d.a.g.. In presence of cut links, in order to be able to express sequentiality constraints on them, we identify any cut link c with the positive link just above it [3] (see Fig. 4).

More formally, we define the notion of *order associated with a J-net* R ($\prec_R$) in the following way:

- we take the order $<_R$ associated to R as a d.a.g. ;
- starting from it we define a pre-order $\leq_R$ in the following way:
  1. if b is a cut link and b shares a positive edge with a link a, then $a \leq_R b$ and $b \geq_R a$;
  2. for all other links a, b if $a <_R b$ then $a \leq_R b$;
- the order $\prec_R$ associated to the J-net R is the quotient of the pre-order $\leq_R$ so obtained.

[3] This coincides with the standard (in the setting of linear logic proof nets) identification of cut links with $\otimes$ links during sequentialization, combined with the clusterization of $\otimes$'s in *MEHS*.

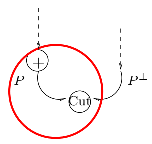


Figure 4: Identification of cut links

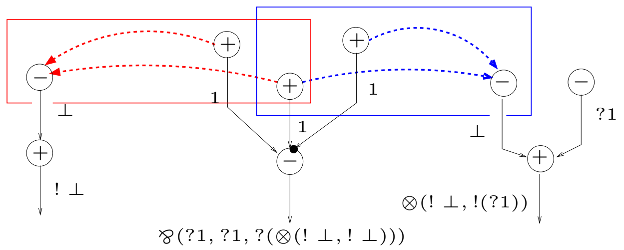


Figure 5: Two cones of a J-net

2.2 Cones

Now we are in the position to introduce the notion of *cone*, which replaces, in the setting of J-nets, the familiar notion of linear logic *exponential box*.

Definition 2 *Given a J-net R, the* cone *of a negative edge a (denoted by C_R^a) conclusion of a node w is the restriction of R to the set $\{b \in R; w \prec_R b\} \cup \{w\}$; an edge $x \neq a$ is said to be on the* border *of C_R^a iff x is emergent from a node b such that $b \in C_R^a$ and either x is a conclusion of R, or x is incident on a node c in R s.t. $c \notin C_R^a$.*

Proposition 7 *Given a negative edge a of R, if an edge belongs to the border of C_R^a, then its source is a positive link; in particular all typed edges in the border of C_R^a are labelled by* positive *formulas.*

PROOF. Suppose that there is an edge in the border of C_R^a whose source is a negative link: then it is the conclusion of a negative link n. The only way for n to be in C_R^a is to be above a positive link which belongs to C_R^a (since n is negative and only positive links can jump); in this case n cannot be in the border of C_R^a. $\square$

Given an edge x (resp. y) emergent from a positive link b (resp. c) we denote by $x \parallel y$ the fact that x, y are both in the border of C_R^a for some negative edge a; by $b \parallel c$ we will denote the fact that b, c are sources of edges belonging to the border of the same cone.

Given a J-net R, the inclusion relation on the set of cones in R, induced by $\prec_R$, is obviously a partial order; moreover:

Remark 3 *If the order $\prec_R$ associated with a J-net R is arborescent, then given any two negative edges a, b of R, either C_R^b and C_R^a are included one into the other, either they are disjoint.*

3 Correctness and sequentialization

3.1 J-nets and sequent calculus

Given an $MEHS$ proof π and a J-net R, we say that R *can be associated with* π, if R can be inductively decomposed in such a way that each step of decomposition of R corresponds to the writing down of a rule of π. If a J-net R can be associated with a proof π of $MEHS$, we say that π is a *sequentialization* of R.

Not all J-nets can be associated with proofs; to formally define the J-nets corresponding to $MEHS$ proofs, we introduce the notion of *sequentializable* J-net. The content of the following definitions is straightforwardly adapted from [Lau99]:

Definition 3 (Sequentialization of a J-net) *We define the relation* "L sequentializes R in ε", *where R is a J-net, L is a terminal link of R and ε is a set of J-nets , in the following way, depending from L:*

- *If L is a positive or a negative* 0*-ary link, and is the only link of R, then L sequentializes R into $\emptyset$;*
- *if L is a cut link, and if it is possible to split the graph obtained by erasing L into two J-nets R_1, R_2, then L sequentializes R into $\{R_1, R_2\}$;*
- *if L is a positive link with n premises, and if it is it is possible to split the graph obtained by erasing L into n J-nets $R_1, \dots, R_n$, then L sequentializes R into $\{R_1, \dots, R_n\}$ J-nets;*
- *if L is a negative link and when we erase L we obtain a J-net R_0 then L sequentializes R in $\{R_0\}$.*

Definition 4 (Sequentializable J-net) *A J-net R is* sequentializable *if one of the following holds:*

- *R is composed by a single connected component, and at least one of its link sequentializes R into a set of sequentializable J-nets or into the empty set;*
- *R is composed by more than one connected component and each component is a sequentializable J-net.*

Proposition 8 *If a J-net R is sequentializable, there exists a proof π of $MEHS$, such that π is the sequentialization of R.*

Proof.

The proof is an easy induction on the number of links of R:

1. $n = 1$: the only node in R is either a positive 0-ary, to which we associate the proof $\dfrac{}{\vdash 1}(+)$ or a 0-ary negative link of conclusion N, to which we associate the proof $\dfrac{}{\vdash N}(Dai)$.
2. $n > 1$: suppose R contains one terminal negative link n with conclusion $\parr_{i=1}^{n}(?P_i)$; then by definition of sequentializable J-net, n sequentializes R into a J-net R_0 with conclusions $\Gamma, P_1^1, \dots P_1^{k_1}, \dots, P_n^1 \dots P_n^{k_n}$; by induction hypothesis there exists a proof π_0 with conclusion $\Gamma, P_1^1, \dots P_1^{k_1}, \dots, P_n^1 \dots P_n^{k_n}$ such that π_0 is the sequentialization of R_0. We obtain the proof π which is the sequentialization of R, by applying a $(-)$ rule with conclusion $\Gamma, \parr_{i=1}^{n}(?P_i)$ to π_0.

   Otherwise suppose R is composed by a single connected component; since it is sequentializable there exists at least one link L which sequentializes R. Then we reason by cases:

   - L is a cut link whose premises are typed by $P, P^{\perp}$; then L sequentializes R into two J-nets R_1, R_2 with conclusions respectively Γ, P and $\Delta, P^{\perp}$; by induction hypothesis there exists a proof π_1 with conclusion $\vdash \Gamma, P$ (resp. π_2 with conclusion $\vdash \Delta, P^{\perp}$) which is the sequentialization of R_1 (resp. R_2). We obtain the proof π which is the sequentialization of R, by applying to π_1, π_2 a cut rule with conclusion $\vdash \Gamma, \Delta$;
   - L is a positive link with conclusion $\otimes_{i=1}^{n}(!N_i)$; then L sequentializes R into $R_1, \dots, R_n$ J-nets with conclusions respectively $\Gamma_1, N_1 \dots \Gamma_n, N_n$; by induction hypothesis there exist n proofs $\pi_1, \dots, \pi_n$ with conclusion respectively $\vdash \Gamma_1, N_1, \dots \vdash \Gamma_n, N_n$, such that $\pi_1, \dots, \pi_n$ are sequentializations respectively of $R_1, \dots, R_n$. We obtain the proof π which is the sequentialization of R, by applying a $(+)$ rule with conclusion $\vdash \Gamma_1, \dots, \Gamma_n, \otimes_{i=1}^{n}(!N_i)$ to $\pi_1, \dots, \pi_n$.

Otherwise, R is composed by more than one connected component, and every connected component $R_1, \dots, R_n$ is a sequentializable J-net; we conclude by applying induction hypothesis on $R_1, \dots, R_n$, getting $\pi_1, \dots, \pi_n$ proofs. Since R has no negative conclusions, we obtain the proof π which is the sequentialization of R by (a sequence of) application of the Mix rule on $\pi_1, \dots, \pi_n$.

$\square$

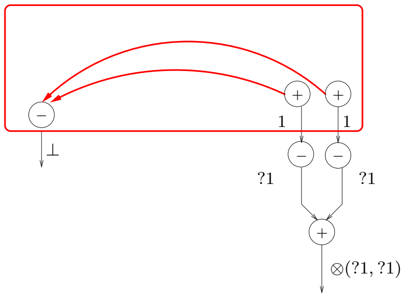


Figure 6: an example of a incorrect J-net

3.2 Correctness

Our purpose now is to define a *correctness criterion*, that is to isolate a geometrical property allowing to characterize (with a non-inductive definition) all J-nets which are *logically* correct (that is, all J-nets which correspond to $MEHS$ proofs). Then we shall prove that the purely geometrical condition we defined characterizes exactly all sequentializable J-nets: this will be called *sequentialization theorem*.

Definition 5 *Given a J-net R, a* flat path Π *from a node a to a node b is a sequence $\langle a_1, \ldots, a_n \rangle$ of nodes s.t. $a_1 = a, a_n = b$ and for each a_i, a_{i+1} one of the two following holds:*

- *a_i and a_{i+i} share an edge; we call such an edge* the edge shared by a_i, a_{i+1} in Π.
- *if a_i and a_{i+i} do not share an edge, then a_{i-1} and a_i share an edge $p : a_{i-1} \leftarrow a_i$ in Π and $p \parallel p'$ for an edge p' incident on a_{i+1}; we call p' the* flat edge *shared by a_i, a_{i+1} in Π.*

Moreover, we require all nodes in a flat path from a to b to be distinct (with the possible exception of a, b). The edges used by *a flat path Π are all the edges (resp. flat edges) shared by the elements of Π.*

Informally, in a flat path, if we are going up through an edge which is in the border of some cone C, we can continue the path by going down through any other edge in the border of C, *even if inside C there is no connection between them* (we call such property the *black box principle*).

Definition 6 *A* switching path *is a flat path which never uses two edges incident on the same negative link (called* switching edges*); a* switching cycle *is a switching path $\langle a_1, \ldots, a_n \rangle$ such that $a_1 = a_n$.*

Definition 7 *A J-net R is* acceptable *when it does not contain any switching cycle.*

We point out that our definition of switching path "sees" cones (as it usually happens in standard proof nets syntax with exponential boxes, see [Pag06]); in this way we can detect the switching cycle of Fig. 6. Contrarily to what usually happens with standard polarized proof net, in our setting not all cut-free proof structures are correct: this means that discarding boxes we have managed to *enlarge* our object space.

Remark 4 *The notion of acceptability above is sufficient to characterize all sequentializable J-nets (as we will show later), but not to define cut elimination (for the same reasons why we cannot define cut-elimination for full $MEHS$). Reduction then is defined only for a particular class of acceptable J-nets (as in $MEHS$), that we call* J-proof nets, *defined just below.*

Definition 8 *An acceptable J-net R is saturated, when for every negative link n and for every positive link p of R adding a jump between n and p either creates a switching cycle or does not increase the order $\prec_R$.*

Remark 5 *We stress that any acceptable J-net R can be turned into a saturated J-net by properly adding jumps on it. In fact, if no jumps could be added to R (either because they do not preserve acceptability of R or because they do not increase the order), R would already be saturated, by definition.*

Definition 9 *A J-net is* closed *when it has exactly one conclusion, which is negative.*

Definition 10 *A J-proof net R is a J-net which is acceptable and closed.*

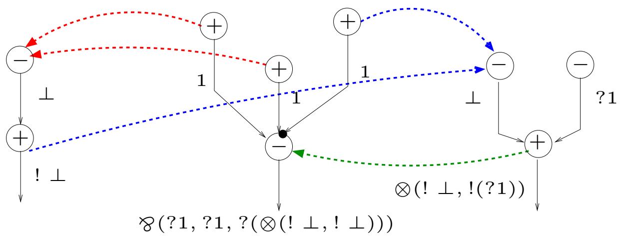


Figure 7: an example of a saturated J-net

3.3 Arborization

To prove sequentialization of acceptable J-nets we extend the method already used for the multiplicative fragment: for a detailed explication and more examples on sequentialization by incremental insertion of jumps on a proof net, we refer to [GF08].

The general design of the proof follows these steps:

- given an acceptable J-net R, we can obtain a saturated J-net R' by adding jumps on R (Remark 5);
- *the order associated with a saturated J-net is arborescent* (Lemma 5);
- if the order associated with R' is arborescent, R' is trivially sequentializable (Proposition 9);
- if R' is sequentializable, then R is sequentializable (Theorem 1).

The most delicate point is the emphasized one, which corresponds to the key *arborization Lemma*.

Due to the role cones play in the definition of acceptability (namely the black box principle), in order to prove the arborization Lemma we need some preliminary lemmas; this is the main difference with respect to the proof of sequentialization for the purely multiplicative case.

Lemma 2 *Given an acceptable J-net R and two different premises a, b of a positive link c in R, $C^a_R \cap C^b_R = \emptyset$.*

Proof. The result follows from the simple observation that if the cones were not disjoint, there would be a node d with two different directed paths from d to c, yielding a switching cycle, contradicting acceptability. □

Lemma 3 *Given an acceptable J-net R, a node $b \in C^a_R$ for some negative edge a of R and a link c which is source of an edge p in the border of C^a_R, there cannot be any switching path Π which starts from b, and ends by entering c using p.*

Proof. If Π enters c from p, then starting from b, at some moment Π must exit C^a_R by crossing an edge p' incident on some node b'. We take the sub path Π' of Π starting from b' and entering in c using p. If $p' = a$, then we trivially get a switching cycle (since, as c is in C^a_R there is directed path from c to b' in R).

Otherwise $p \parallel p'$, and then by definition of switching path we can extend Π' using p' as a flat edge to get back to b'; but then we have a switching cycle. □

Lemma 4 *Let R be an acceptable J-net, and a (resp. b) be a positive (resp. negative) link of R s.t. a, b are incomparable w.r.t. $\prec_R$ and adding a jump from a to b yields a J-net R' which is not acceptable. Then there is a switching path Π in R which starts down from b and ends either:*

1. *with a;*
2. *or by entering from below into a positive link a' in the border of a cone C such that $a \in C$ and $b \notin C$.*

PROOF.

Obviously in R' there is a switching path Π' as the one we are searching; the switching cycle in R' must use the jump $a \to b$, so either it enters into a, or it uses the jump $a \to b$ as a flat edge (by entering into a cone C such that $a \in C$ and $a \to b$ is in the border of C in R'; in this case obviously b must not belong to C) . We have to prove then that a path Π similar to Π' exists also in R. First of all we observe that for every premise c of a, by adding the jump $a \to b$ the edges in the border of C_R^c becomes edges in the border of $C_{R'}^b$ (resp. in the border of all cones C containing b in R); no other borders of cones are modified. Now let us consider the first flat edge $p : a_i \to a_{i+1}$ used by Π' starting from b; since p is a flat edge, $p \parallel p'$ in R', for the edge $p' : a_{i-1} \leftarrow a_i$ used by Π' before p. Moreover, since p is the first flat edge used by Π', the sub path Π'' of Π' ending with p' is a switching path also in R. Suppose that $p \nparallel p'$ in R; we have two different cases:

1. in R, p' is in the border of C_R^c for some premise c of a: then if we extend Π'', entering into C_R^c through p' and then going down from c to a, we find Π;

2. in R, p' is in the border of C_R^b (or in the border of a cone C containing b in R): but then by Lemma 3 we get a switching cycle in R, contradicting the fact that R is acceptable.

□

Lemma 5 (Arborization) *Given an acceptable J-net R, if R is saturated then $\prec_R$ is arborescent.*

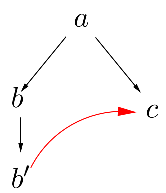
PROOF.

We reason by contraposition, showing that if $\prec_R$ is not arborescent, then R is not saturated (so there exists a negative link c and a positive link b s.t. adding a jump between b and c doesn't create switching cycles and makes the order increase).

If $\prec_R$ is not arborescent, then in $\prec_R$ there exists a link a with two immediate predecessors b and c (they are incomparable).

We distinguish two cases:

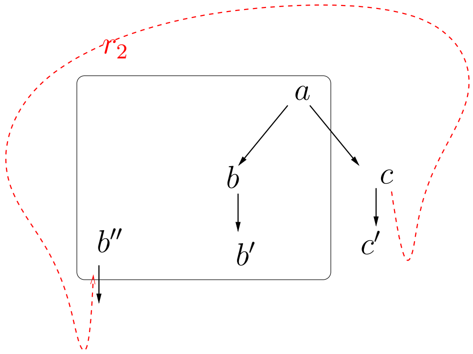
1. either b or c is terminal in R. Let us assume that c is terminal; then b cannot be terminal (by definition of J-net), and there is a positive link b' which immediately precedes b. If we add a jump between b' and c, this doesn't create switching cycles and makes the order increases, so R is not saturated.



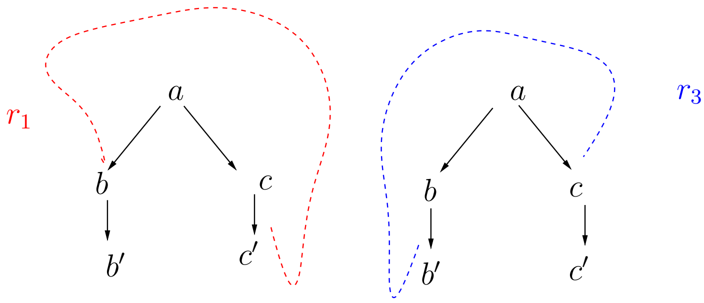
2. Neither b or c are terminal in R. Each of them has an immediate positive predecessor, respectively b' and c'.

   In this case we proceed *ad absurdum* by assuming that R is saturated: then adding a jump, either from b' to c or from c' to b creates a switching cycle.

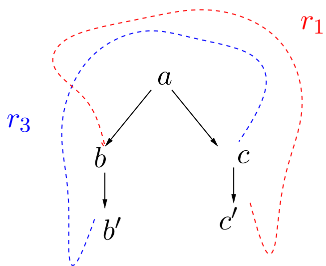
   Since by adding to R the jump $b' \to c$ we break correctness, that means by Lemma 4 that there is in R a switching path $r_1 = \langle c, c' \ldots. b\rangle$, or a switching path $r_2 = \langle c, c' \ldots. b''\rangle$, for a positive link b'' such that b'' is in the border of some cone C of R, $b \in C$ but $c \notin C$, and r_2 enters b'' from below. First we show that the former is the only possible case: If $b' \in C$, then the edge $a \to c$ must be on the border of C (otherwise $c \in C$). But if $a \to c$ is on the border of C, then we can extend r_2 by going down to c using $a \to c$ as a flat edge, and we get a switching cycle in R: this contradicts the acceptability of R. So the only possible case is that there is path $r_1 = \langle c, c' \ldots. b\rangle$ in R.



Similarly, since adding a jump $c' \to b$ breaks correctness then there is a switching path $r_3 = \langle b, b' ... c\rangle$.



Assume that r_1, and r_3 are disjoint: then the path obtained by concatenation of r_1, and r_3 is a switching cycle. This contradicts the fact that R is acceptable.



Assume that r_1 and r_3 are not disjoint. Let x be the first node in r_3 (starting from b) where they meet. Observe that x must be negative (by acceptability of R). Each path uses one switching edge of x, and its conclusion (hence the paths meets also in the node below x). From the fact that x is the first point starting from b where r_1 and r_3 meet it follows that: (i) r_3 enters x using one of its switching edges, and exits from the conclusion; (ii) each path must use a different switching edge of x. Then we distinguish two cases:

- r_1 enters in x using one of its switching edges; we build a switching path contradicting acceptability of R taking the sub path of r_1 ending with x and the sub path of r_3 starting with x;
- r_1 enters x from the conclusion; then we build a switching path contradicting acceptability of R by composing the sub path of r_1 ending with x, the (reversed) sub path of r_3 starting with x, and the path $\langle b, a, c\rangle$.

In each case we obtained a contradiction. Therefore the assumption that R was saturated is false, so R is not saturated.

□

3.4 Sequentialization

Definition 11 (Splitting) *Let R be a J-net, c a positive or a cut link, and $b_1, \ldots, b_n$ the nodes which are immediately above c (the premises of c have the same type as the conclusions of $b_1, \ldots, b_n$). We say that c is* splitting *for R if it is terminal, and removing c there is no more connection (i.e. no sequence of connected edges) between any two of the nodes b_i.*

Lemma 6 (Splitting Lemma) *Let R be an acceptable J-net without negative conclusions, such that R is composed by a single connected component, and $\prec_R$ is arborescent; if c is a terminal positive link (resp. a cut link) which is minimal in $\prec_R$ then c is the unique minimal link in $\prec_R$ and is splitting.*

Proof. For simplicity's sake we just consider the case when c is a terminal positive link; if c is a cut link we can just consider it as a positive link having as premises the negative premise of c and the negative premises of the positive link c' which shares its conclusion with c (since they are identified in $\prec_R$)

By Remark 3 the cones of the premises $b_1, \ldots, b_n$ of c are disjoint. First we want to show that given a b_i the typed edges in the border of $C_R^{b_i}$ are conclusions of R. To prove it, suppose there is a negative link m with conclusion z such that m does not belong to $C_R^{b_i}$ and a premise of m is in the border of $C_R^{b_i}$; then $C_R^{b_i} \cap C_R^z \neq \emptyset$; by Remark 3 either $C_R^z \subset C_R^{b_i}$ (contradicting our hypothesis) or $C_R^{b_i} \subset C_R^z$, but then $c \in C_R^z$, contradicting minimality of c. Moreover, we can prove that there is no connection between the nodes of $C_R^{b_i}, C_R^{b_j}$ for $i \neq j$ once erased c; suppose that a node in $C_R^{b_i}$ is connected with a node in $C_R^{b_j}$, and consider the path Π connecting them; let d be the first (negative) node outside $C_R^{b_i}$ in Π, and d' the last node inside $C_R^{b_i}$ in Π (so that there is an edge $d' \to d$ in R). By minimality of c, d must belong to a cone of a premise of c, but by Proposition 2 it could be only $C_R^{b_i}$ (otherwise d' would be shared by the cones of two different premises of c): contradiction. In the same way we can prove that if there exists another minimal positive (or cut) link a in R, given a negative premise a_k of a, there cannot be any connection between the nodes in $C_R^{b_i}$ and the nodes in $C_R^{a_k}$, once erased a and b; but then there cannot be any connection at all, since a and b are terminal, contradicting the fact that R is connected. So $R = C_R^{b_1} \cup \ldots \cup C_R^{b_n} \cup c$. □

Proposition 9 *A J-net R whose associated order is arborescent is sequentializable.*

Proof.

The proof is by induction on the number of links of R:

$n = 1$: in this case, R is composed by a positive or a negative link without premises, and it is trivially sequentializable;

$n = k + 1$: suppose R has a terminal negative link n; then it is minimal in $\prec_R$. The graph R_0 obtained by removing n is obviously an acceptable J-net whose order associated is arborescent, so by induction hypothesis it is sequentializable; then R is sequentializable.

Otherwise, R does not have any terminal negative link. Now suppose R is composed by more than one connected component; obviously each component $R_1, \ldots, R_n$ is an acceptable J-net whose order associated is arborescent , so by induction hypothesis it is sequentializable; but then, R is sequentializable.

If R is composed by a single connected component, there is a unique positive terminal link (or cut link) c which is minimal in $\prec_R$. By the splitting Lemma c is splitting, so it sequentializes into $R_1, \ldots, R_n$ acceptable and arborescent J-nets: the rest follows by induction hypothesis.

□

Corollary 1 *Given an arborescent J-net R, its sequentialization π is unique (up to permutation of Mix rules).*

Proof. Trivial from the fact that the order associated with R (which correspond to the order of the $(+)$ and $(-)$ rules of π) is arborescent, and from the proof of Theorem 9. □

Theorem 1 (Sequentialization) *Any acceptable J-net is sequentializable.*

Proof. It is easy to see that given any J-net R, if a saturation of R is sequentializable into a proof π, then also R is sequentializable into π (all splitting links in the saturation of R are splitting also in R). $\square$

4 J-proof nets and cut elimination

In this section we present the dynamic behavior of J-proof nets, by defining the procedure of *cut-elimination* on J-nets, and proving that the correctness criterion is stable under reduction. Then, after giving some relevant examples of reductions, we prove two basic properties of such rewriting: *local confluence* and *weak normalization* for J-proof nets.

4.1 Cut elimination

Given two J-nets R_1, R_2 and a *cut* link c of R_1, we define the relation $\xrightarrow{cut}$ on J-nets by saying that $R_1 \xrightarrow{cut} R_2$ ("R_1 reduces to R_2 in one step") whenever R_2 could be obtained from R_1 by replacing a module β , called *redex*, contained in R_1 and which contains c (the "reduced" cut), with a module γ, called *contractum*, following the rule depicted in Fig. 8, called $+/-$ step:

$+/-$ **step** the redex β is composed by :

- the cut c, the positive link a (with n ports) and the negative link b (with $m \leq n$ ports) that share their conclusion with c (we say that a, b are the *active links* of c) ;
- the set of negative links $a_1, \ldots, a_n$ such that the conclusion of a_i belong to the i-th port of a, and the set of positive links $b_1^1, \ldots, b_1^{k_1}, \ldots, b_m^1, \ldots b_m^{k_m}$ such that the conclusions of $b_j^1, \ldots, b_j^{k_j}$ belong to the j-th port of b ;
- the cones $\pi_1, \ldots, \pi_n$ of the premises of a and any negative link n such that for some π_i an edge in the border of π_i is a premise of n;
- any positive link w which jumps on b, and any negative link z such that a jumps on z.

To replace β with γ the following constraints must be respected:

1. w is different from a (resp. z is different from b);
2. the premises of b are not in the border of any of the cones $\pi_1, \ldots, \pi_n$;
3. the typed edges in the border of $\pi_1, \ldots, \pi_n$ are not conclusions of R_1;
4. $\pi_1, \ldots, \pi_n$ are disjoint.

The contractum γ is obtained by :

- erasing c, a and b;
- for all the positive links $b_i^1, \ldots, b_i^{k_i}$ whose conclusion belonged to the i-th port of b in β, we consider the negative link a_i whose conclusion belonged to the i-th port of a in β and we make k_i copies of the corresponding cone π_i; then we connect pairwise each copy of a_i with one of the positive links $b_i^1, \ldots, b_i^{k_i}$ through a new cut link. Moreover, we make k_i copies of each edge p in the border of π_i in β; if p is typed, then we assign all the copies $p_i^1, \ldots, p_i^{k_i}$ of p to the same port of the negative link n which contained p in β;
- we add a jump from w and from all positive links $b_1^1, \ldots, b_1^{k_1}, \ldots, b_m^1, \ldots b_m^{k_m}$ to z;
- for any negative link a_j whose conclusion belonged to the j-th port of a in β, such that $m < j < n$ (so that for a_j there is no corresponding port of b), we erase the corresponding cone π_j; consequently, we erase each typed edge p in the border of π_j from the port of the negative link n that contained it in β.

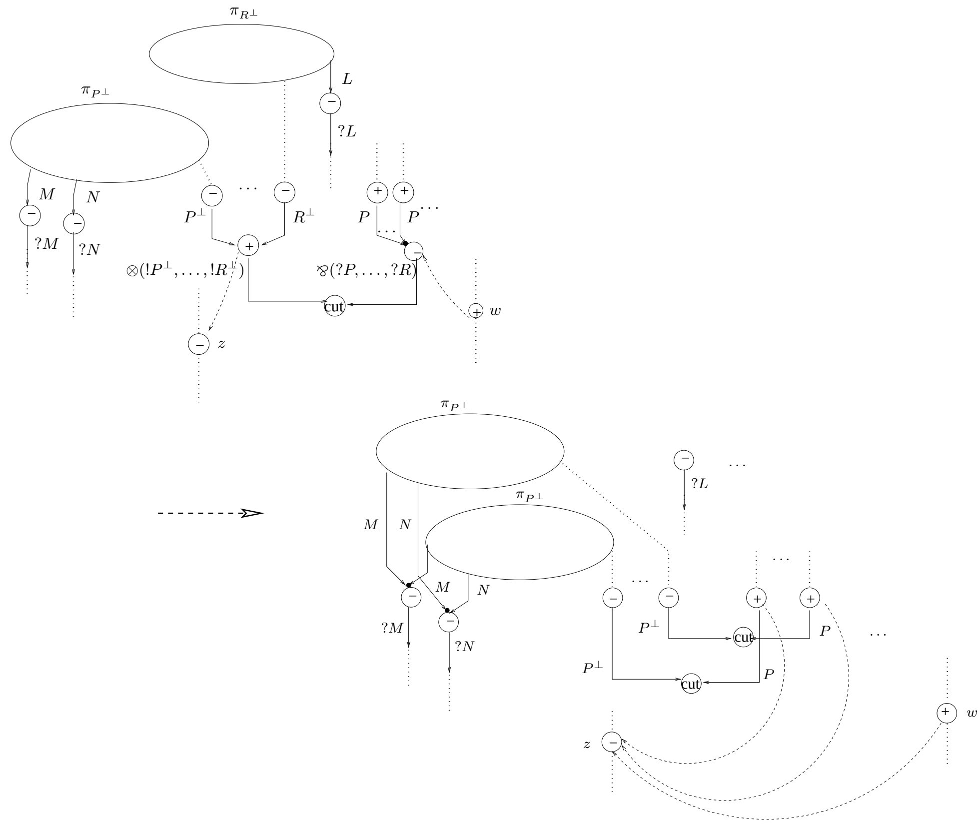


Figure 8: The $+/-$ reduction step

Given a J-net R, and a *cut* link t of R, we denote by $t(R)$ the J- net R' obtained by reducing t in R. When a node a (resp. an edge l) of $t(R)$ comes from a (unique) node (resp. edge) $\overleftarrow{a}$ (resp. $\overleftarrow{l}$) of R we say that $\overleftarrow{a}$ (resp. $\overleftarrow{l}$) is the *ancestor* of a (resp. l) in R and that a (resp l) is a *residue* of $\overleftarrow{a}$ (resp. $\overleftarrow{l}$) in $t(R)$; otherwise we said that a is a *created* node (in particular, all the new cut links introduced by the $+/-$ step are created). We denote sometimes the residues of a node b (resp. an edge r) by $\overrightarrow{b}$ (resp. $\overrightarrow{r}$).

Remark 6 *We stress the fact that reduction preserves the order, in the sense that if $R_1 \xrightarrow{cut} R_2$ reducing a cut c, given two nodes a, b of R_1 such that $a \prec_{R_1} b$ and two residues a', b' respectively of a, b in R_2, then $a' \prec_{R_2} b'$. Nevertheless, the inclusion relation on cones may change during reduction: if a premise p_i of the negative active link of c belongs to the border of a cone C in R_1, then given the corresponding negative premise n_i of the positive active link of c, the cone C' of $\overrightarrow{n_i}$ in R_2 is included into C in R_2 (and the edges in the border of C' are in the border of C too).*

Theorem 2 (Preservation of correctness) *Given a J-proof net R, if $R \xrightarrow{cut} R'$, then R' is a J-proof net.*

Proof.

We must show that closedness and acceptability are preserved. The former property is trivial, since cut-elimination does not modify the conclusions of a net; let us focus on the latter. Let t be the cut reduced by the $+/-$ step which replaces a module β with a module γ. We will proceed ad absurdum, by showing that if there was a switching cycle in R', then there would be a switching cycle in R too, contradicting the hypothesis that R is a J-proof net.

First of all we must show that given two edges p, p' in the border of the same cone in R', if $\overleftarrow{p}$ and $\overleftarrow{p'}$ are not in the border of the same cone in R, then there is a switching path connecting them in R; if this was not the case, we could lose some switching paths going back from R' to R.

So let us suppose that $p \parallel p'$ in R' but $\overleftarrow{p} \nparallel \overleftarrow{p'}$ in R; then given $\overleftarrow{p}, \overleftarrow{p'}$ by Remark 6 one of them (say $\overleftarrow{p}$) must be in the border of $C_R^{n_i}$ for a negative premise n_i of the active positive link of t, and the other one (that is $\overleftarrow{p'}$) must be in the border of some other cone C of R, together with a positive premise p_i of the active negative link of t (so $p_i \parallel \overleftarrow{p'}$ in R); but then there is a switching path connecting $\overleftarrow{p}$ with $\overleftarrow{p'}$ in R, since inside the module β, there is a switching path going from $\overleftarrow{p}$ to p_i through t, and $p_i \parallel \overleftarrow{p'}$.

Now we can proceed with the main proof: suppose then that there is a switching cycle in R': if it does not cross the module γ in R', then the switching cycle is also in R and we are done.

Otherwise, it does cross γ: let us call $c_1, \ldots, c_n$ the cut links of R' created by reducing t in R, a_i being the negative premise and b_i being the positive premise of c_i for $i = 1, \ldots, n$; let us call f any negative link and g any positive link in γ in R' such that g jumps on f but $\overleftarrow{g}$ does not jump on $\overleftarrow{f}$ in R.

The switching cycle may cross the module in different ways, using n new cuts and m new jumps created by reducing t in R; we detail some relevant cases, showing that each of them brings to a contradiction.

1. Suppose the cycle crosses exactly one cut link c_i of R' created by reducing t in R and no new jumps (so $n = 1$ and $m = 0$); in this case, the cycle connects a_i with b_i with a path going outside the module γ in R'. Then there is a switching cycle in R, obtained by choosing $\overleftarrow{b_i}$ as a switching edge in R.

2. Suppose that the cycle crosses exactly two new cut links c_i, c_j of R' created by reducing t in R and no new jumps (so $n = 2$ and $m = 0$). Then we have two subcases:

   - Suppose the cycle connects a_j with b_i and b_j with a_i (for $j \neq i$) with a path going outside the module γ in R'; then we reason as in case 1, opportunely choosing a switching edge.
   - Suppose the cycle connects a_i with a_j and b_i with b_j (for $j \neq i$) with a path going outside the module γ in R'. Again we have two subcases:
     - if $\overleftarrow{a_i} \neq \overleftarrow{a_j}$, then it is easy to see that there is a switching cycle in R too, since $\overleftarrow{a_i}$ and $\overleftarrow{a_j}$ are connected inside the module β in R.
     - if $\overleftarrow{a_i} = \overleftarrow{a_j}$, since there is a switching path connecting a_i and a_j outside the module γ in R', such a path must go down on an edge m_i in the border of $C_{R'}^{a_i}$ and go up to an edge m_j in the border of $C_{R'}^{a_j}$ (or the other way round). Now if $\overleftarrow{m_i} = \overleftarrow{m_j}$, by definition of γ, m_i and m_j are incident on the same negative link in R', so they cannot belong to the same switching path, so this cannot be the case. The only possibility left is that $\overleftarrow{m_i} \neq \overleftarrow{m_j}$; but then in R there exists a switching path connecting $\overleftarrow{m_i}$ with $\overleftarrow{m_j}$; since they are two edges of the border of the same cone $C_R^{\overleftarrow{a_i}}$, this means that there is a switching cycle in R.

   If the cycle crosses more than two cut links of R' created by reducing t in R and no new jumps (so $n \geq 2$ and $m = 0$) the argument is completely analogous.

3. suppose the cycle does not cross any of the new cut links of R' created by reducing t in R, but it does cross one new created jump (so $n = 0$ and $m = 1$). In this case the cycle connects f with g outside the module γ in R'. Then there is a switching path connecting $\overleftarrow{g}$ with $\overleftarrow{f}$ outside the module β in R too, and is easy to see that there is a switching cycle in R too (since $\overleftarrow{g}$ and $\overleftarrow{f}$ are connected in the module β in R). If the cycle crosses more than two new jumps created by reducing t in R and no new cut links (so $n = 0$ and $m \geq 1$) the argument is completely analogous.

4. suppose the cycle crosses at least one cut link c_i of R' created by reducing t in R, and one new jump (so $n \geq 1$ and $m = 1$). In this case the cycle connects a_i with f outside the module γ in R'. Then there is a switching path connecting $\overleftarrow{f}$ with $\overleftarrow{a_i}$ outside the module β in R too, and is easy to see that there is a switching cycle in R too (since $\overleftarrow{f}$ and $\overleftarrow{a_i}$ are connected in the module β in R). If the cycle crosses more than two new jumps created by reducing t in R (so $n \geq 1$ and $m \geq 1$) the argument is completely analogous.

In all cases, supposing that R' contains a switching cycle implies that R contains a switching cycle, contradicting the hypothesis that R is a J-proof net. So R' is a J-proof net.

□

Proposition 10 *Let R be a J-proof net. R is in normal form w.r.t. $\xrightarrow{cut}$ iff R is cut-free.*

Proof. The right to left direction is trivial. Concerning the left to right direction, let us suppose that R is in normal form but is not cut-free; we proceed *by absurdum* showing that R is not a J-proof net. If R is not cut-free, then it contains a cut link c such that c cannot be reduced by a $+/-$ step; so it does not respect one or more of the constraints 1-4 given in the definition of the $+/-$ step. It is easy to see that the violation of any of the constraints implies that R is not a J-proof net:

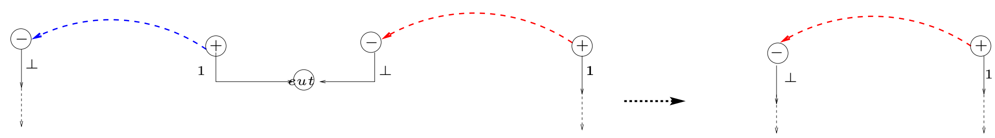
- violation of conditions 1,2 implies that R contains a switching cycle (so R is not acceptable, then is not a J-proof net);
- violation of condition 3 implies that R is not closed (but then R is not a J-proof net);
- violation of condition 4 contradicts the fact that R does not contain switching cycles, since by Lemma 2 the cones of the premises of any positive link in an acceptable J-net must be disjoint; so R is not acceptable and then it is not a J-proof net.

Then we have shown that if R contains an irreducible cut (that is, a cut that cannot be eliminated through a $+/-$ step), R is not a J-proof net. □

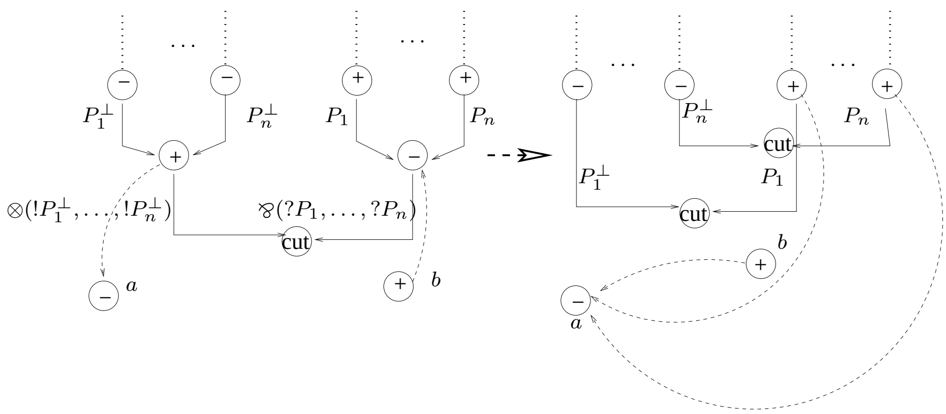
4.2 Some special reduction steps

Concerning the definition of $+/-$ reduction step, in order to clarify the relation between usual proof net reduction and the one we just defined, we depict below some "special cases" of reduction:

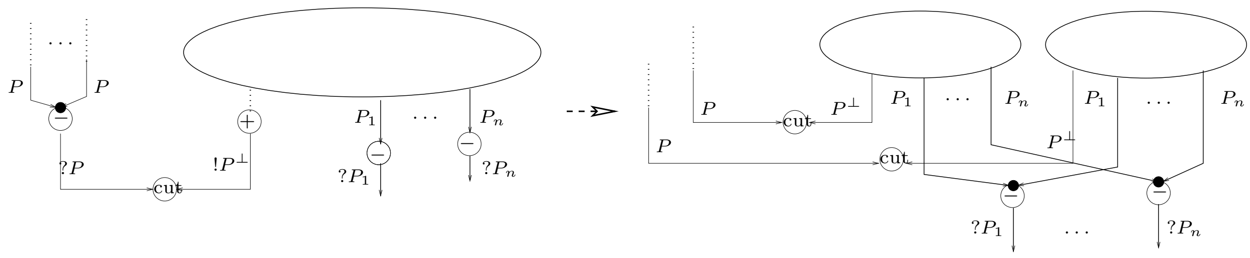
- the "axiom" reduction step, where we reduce a cut between a 0-ary positive link (which jumps on a link z) and a 0-ary negative link (on which a link w jumps) :



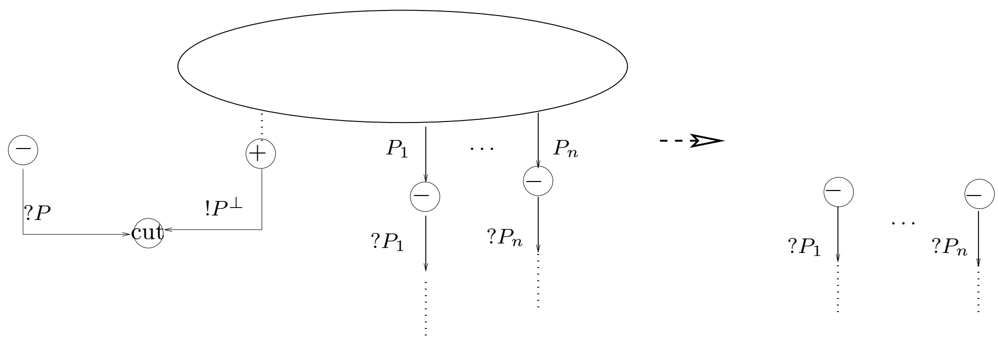
- the "multiplicative" reduction step, where we reduce a cut between a positive and a negative link with the same number n of ports, each containing a unique premise :



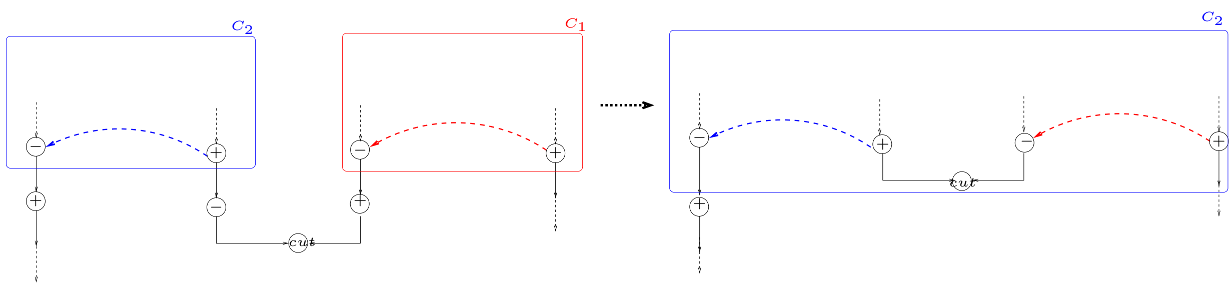
- the "contraction" reduction step, where we reduce a cut between a unary positive link and a negative link with a unique port, containing n premises:



- the "weakening" reduction step, where we reduce a cut between a unary positive link and a 0-ary negative link:



- the "commutative" reduction step, where we reduce a cut between a unary positive link (whose premise has a cone C_1) and a unary negative link (whose premise is in the border of a cone C_2); in this step we make the cut "enters" in C_1 and put the content of C_1 into C_2 (and in all the cones which include C_2):



4.3 Properties of reduction

4.3.1 Local confluence

Given a J-proof net R, the cones associated with a cut c are the cones of the premises of the positive link which shares its conclusion with c; given two cuts c_1, c_2 we say that $c_1 <_{cut} c_2$ if c_2 belongs to one of the cones associated with c_1; this relation is trivially a partial order.

Proposition 11 *The relation* $\xrightarrow{cut}$ *is locally confluent on J-proof nets.*

Proof.

Let c_1, c_2 be two *cut* links of a J-proof net R. We must show that there exists a J-proof net R' such that $c_1(R) \xrightarrow{cut}_* R'$ and $c_2(R) \xrightarrow{cut}_* R'$. We have the following cases:

1. $c_1 <_{cut} c_2$ or $c_2 <_{cut} c_1$;

2. c_1 and c_2 are incomparable with respect to $<_{cut}$, and an edge in the border of a cone associated with c_1 is premise of a negative link whose conclusion is premise of c_2, or vice versa (in this case we say that c_1 and c_2 are in *opposition*, see for example Fig. 9);

3. c_1 and c_2 are incomparable with respect to $<_{cut}$ and one of the cones associated with c_1 and one of the cones associated with c_2 are not disjoint.

If none of the previous case holds, then c_1 and c_2 are incomparable with respect to $<_{cut}$, the cones associated with c_1 and the cones associated with c_2 are disjoint, and c_1, c_2 are not in opposition; but then it is clear that the cuts c_1, c_2 are independent, so the order of reductions does not influence the final result.

In the cases 1. and 2. the proof is a straightforward adaptation of the standard local confluence proof for $MELL$ (see for example [Dan90]).

The only situation which escapes the standard $MELL$ proof, being specific to J-proof nets, is the one described at point 3.; we discuss it now.

In absence of erasing or duplication (so in the purely multiplicative case), local confluence of J-proof nets has been proved in [DG08, DGF06]; then we can restrict our analysis to three specific sub-cases, dealing only with erasing and duplication:

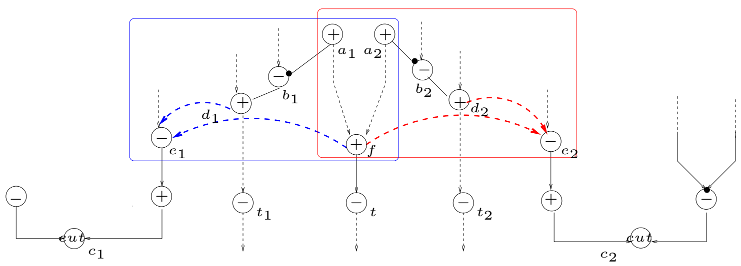
1. c_1 is reduced through a "weakening" step, while c_2 is reduced through a "contraction" step;

2. both c_1 and c_2 are reduced through a "contraction" step;

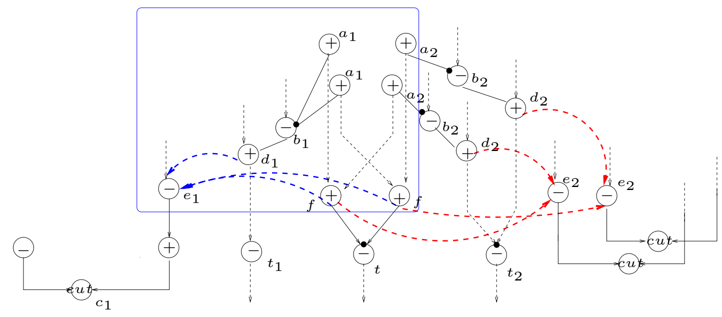
3. both c_1 and c_2 are reduced through a "weakening" step.

Once dealt with these cases, the extension to the general case will follow from the fact that the general $+/-$ step is actually a superposition of multiplicative,"weakening" and "contraction" steps, and that all the cones involved in the $+/-$ step are disjoint (by condition 4 of definition of $+/-$ step).

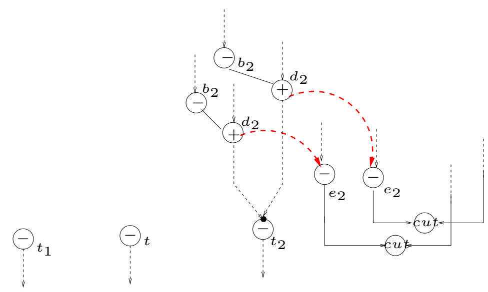
Let us consider case 1). Suppose we have a J-proof net R containing the cuts c_1, c_2, as pictured below (we mark in red the cone C_1 associated with c_1, and in blue the cone C_2 associated with c_2):



Now we reduce c_2 with a "contraction" step, "entering" into the cone C_2 associated with c_2 and duplicating its content (both the part shared with C_1 and the one specific to C_2), obtaining a J-proof net R' :

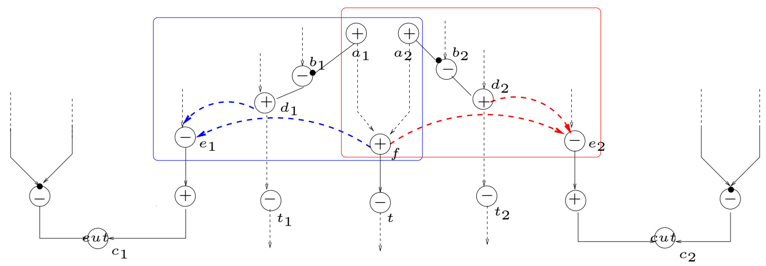


Finally, we reduce the residue of c_1 in R' with a "weakening" step, erasing the content of the cone C_1 associated with c_1, obtaining a J-proof net R'':

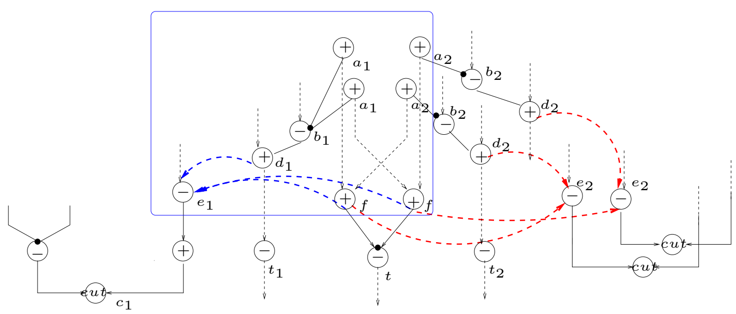


It is easy to check that reducing first c_1 in R and then the residue of c_2 yields R''.

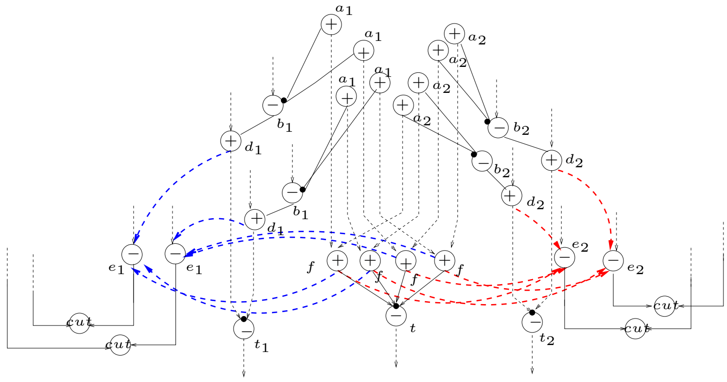
In case 2), R contains two cuts c_1, c_2 as pictured below :



Now we reduce c_2 with a "contraction" step, "entering" into the cone C_2 associated with c_2 and duplicating its content, (both the part shared with C_1 and the one specific to C_2) obtaining a J-proof net R';

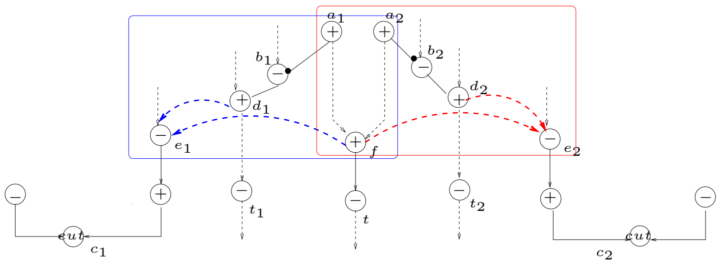


Then we reduce the residue of c_1 in R' with a "contraction" step, "entering" into the cone C_1 associated with c_1 and duplicating its content, obtaining a J-proof net R'';

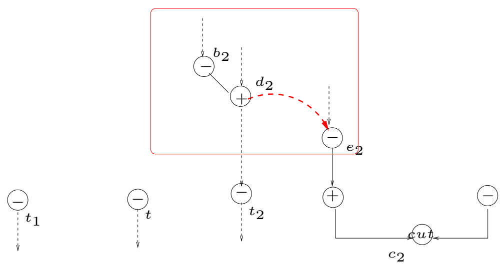


We remark that if a node belongs to $C_1 \cap C_2$ in R, then it has four residues in R''; otherwise, if it belongs to just one of the C_i, it has two residues in R''. Also in this case, one can easily check that reducing first c_1 in R and then the residue of c_2 yields R''.

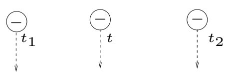
Finally, in case 3) the situation is as below:



we first reduce c_1 with a "weakening" step, erasing the content of the cone C_1 associated with c_1 (both the part shared with C_2 and the one specific to C_1), obtaining a J-proof net R':



Then we reduce the residue of c_2 with a "weakening" step, erasing the content of the cone C_2 associated with c_2, obtaining a J-proof net R'':



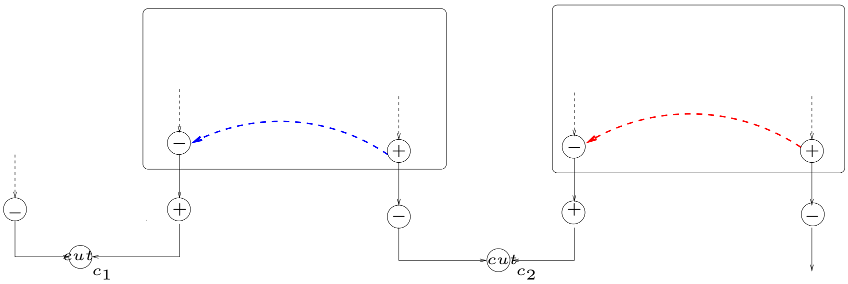


Figure 9: Two cuts incomparable with respect to $<_{cut}$

It is straightforward that reducing first c_2 in R and then the residue of c_1 yields R''.

□

4.3.2 Weak normalization

The following proof of weak normalization is straightforwardly adapted from the one contained in [Lau08] for $MELL$ proof nets.

Sizes The size of a negative edge n conclusion of a negative link a is the number of the typed edges incident on a .

The size $|A|$ of a formula A is one plus the sizes of the immediate subformulas of A.

The size of a cut c is the pair $(|N|, t)$ where N is the type of the negative premise n of c and t is the size of n.

We compare the sizes of the cuts of a J-proof net by considering the lexicographic ordering on $\mathbb{N} \times \mathbb{N}$.

The size of a J-proof net R is the multiset of the sizes of its cuts.

We briefly recall the definition of *multi-set* order: let X be a set, $\ll$ an order relation over X, and let $\mathcal{M}_{fin}(X)$ denote the finite multisets over X. If $M, N \in \mathcal{M}_{fin}(X)$, we write $N <_{1,m} M$ if N is obtained from M by replacing an element by a multi-set of elements which are strictly smaller (w.r.t. $\ll$). The multiset order $<_m$ is the transitive closure of $<_{1,m}$.

We compare the sizes of J-proof nets by considering the multiset order $<_m$ (where $\ll$ is the lexicographic order on the sizes of cuts).

Priority A *priority path* is a path starting down from a negative edge which behaves in the following way:

- if it enters a cut from one of its premises it goes up through the other premise;
- if it enters a positive link from its conclusion it goes up through one of its negative premises;
- if it enters a negative link, from its negative conclusion a, then it goes down through one of the edges in the border of C^a_R;
- if it enters a negative link from above, it goes down through its conclusion;
- if it enters a positive link from its premises it stops.

Proposition 12 *Given a J-proof net R:*

1. *every priority path crosses cuts from the negative to the positive premise;*
2. *every priority path is acyclic.*

Proof.

We prove 1) by induction on the length of the path. The first node the path meets must be a cut, otherwise it would stop; so it must cross a cut from the negative to the positive premise. This positive premise is the conclusion of a positive link; we go up through one of its negative premises a and go down through one of the positive edges in the border of C_R^a; such a positive edge must be premise of a negative link (if it were premise of a cut, the cut itself would be included into C_R^a by definition of $\prec_R$); we apply the induction hypothesis on the sub path starting from the conclusion of the link and conclude. To prove 2) we just observe that a priority path is switching, so that if there were a cycle there would be a switching cycle in R, contradicting the fact that R is a J-proof net.

□

By Proposition 12 we can define another partial order on cut links that we call *priority* order (denoted by $\lessdot$); $c_1 \lessdot c_2$ when there is a priority path starting from the negative premise of c_1 to c_2. A cut link is *priority* if it is maximal for such a partial order.

Weak normalization

Theorem 3 *If R is a J-proof net, then $R \in WN^{cut}$.*

Proof.

We prove that if R has a cut, then we can always reduce a cut in such a way to reduce the size of R, by choosing a cut which is at the same time maximal w.r.t. the order $<_{cut}$ and w.r.t. the priority order (that is $\lessdot$). In the following we will show that if R has a cut then such a maximal cut exists.

Consider a cut c_1 such that the cones associated with c_1 do not contain any other cut of R (that is, c_1 is maximal with respect to $<_{cut}$). Then we search for a priority cut c_2 such that $c_1 \lessdot c_2$. If such a cut c_2 does not exist, then c_1 is also maximal w.r.t. $\lessdot$: we reduce c_1 and this decreases the size of R. In fact, the new cuts created by reducing c_1 are all of smaller size w.r.t. c_1, no cuts are duplicated (since c_1 is maximal with respect to $<_{cut}$) and the sizes of the cuts of R different from c_1 do not change, since there is no cut greater than c_1 in the priority order.

Otherwise, there exists a priority cut c_2 s.t. $c_1 \lessdot c_2$; note that then there is a switching path (the priority path) connecting c_1 with c_2. Now, we iterate the procedure on c_2; we search for a cut c_3 s.t. $c_2 <_{cut} c_3$, and such that c_3 is maximal w.r.t. $<_{cut}$. If such a c_3 does not exist, then c_2 is maximal also w.r.t. $<_{cut}$: then we reduce c_2 and for the same reasons as above the size of R decreases.

Otherwise the cut c_3 exists; we remark that then there is a switching path connecting c_2 with c_3, since c_3 belongs to a cone associated with c_2 (by definition of $<_{cut}$) and such a switching path cannot cross the one from c_1 to c_2 (otherwise there would be a switching cycle in R, contradicting the fact that R is a J-proof net).

Now, either c_3 is a priority cut (and in this case is maximal both w.r.t. $<_{cut}$ and $\lessdot$ and we reduce it, decreasing the size of R), or c_3 is majored in the priority order by another priority cut c_4; in this case we iterate on c_4 the same reasoning made for c_2, and so on. In this way we build a switching path which eventually terminates with a cut link we can reduce (otherwise by finiteness of R we would find a switching cycle, contradicting the fact that R is a J-proof net).

□

Remark 7 *Proposition 4 follows directly from weak normalization on J-proof nets, modulo sequentialization.*

5 Strong normalization and confluence

Following the method used by Pagani and Tortora de Falco to prove strong normalization for LL in [PTdF10], we prove that WN^{cut} implies SN^{cut} for J-proof nets using a variation of Gandy's method (see [Gan80]) due to Bezem and Klop (see [Ter03]). More precisely:

- we modify the relation $\xrightarrow{cut}$, to get a new relation $\xrightarrow{\neg e}$ which never erases pieces of a J-proof net: we call such a reduction relation *non erasing*, and we show that switching from $\xrightarrow{cut}$ to $\xrightarrow{\neg e}$ preserves weak normalization and local confluence for J-proof nets;

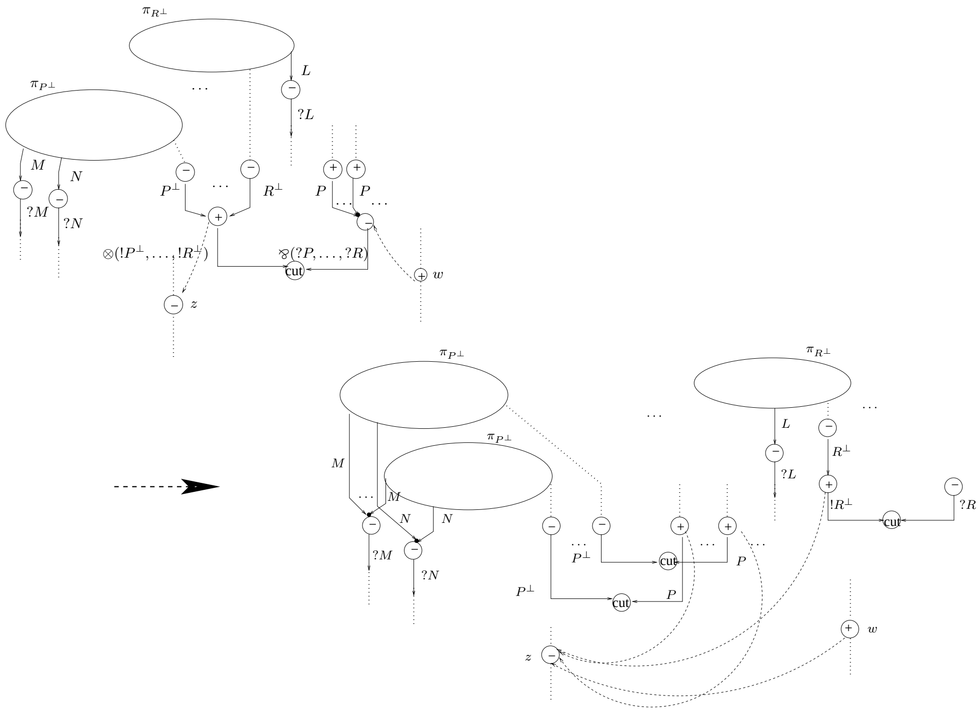


Figure 10: The non erasing reduction step

- we show that $\xrightarrow{\neg e}$ is increasing, proving by Proposition 6 that $\xrightarrow{\neg e}$ is strongly normalizing on J-proof nets;
- we prove that strong normalization of $\xrightarrow{\neg e}$ implies strong normalization of $\xrightarrow{cut}$ for J-proof nets.

The content of the present section is a straightforward adaption of [PTdF10]; the main differences with respect to the method used by Pagani and Tortora de Falco are the followings:

- due to the presence of synthetic connectives, we have a single elementary reduction step, while in [PTdF10] the reduction relation is composed by several different elementary reduction steps;
- we define the relation $\xrightarrow{\neg e}$ as a modification of the relation $\xrightarrow{cut}$, while in [PTdF10] the non-erasing reduction relation is just a restriction of the ordinary reduction relation;
- Pagani and Tortora de Falco consider full linear logic (including additives and second order quantifiers), while we restrict to the multiplicative/exponential fragment.

5.1 Non erasing reduction

We define the reduction $\xrightarrow{\neg e}$ by replacing the step 4) of the definition of $+/-$ step of Section 4.1 with the following one (see Fig 10):

(4*). To any negative link a_j whose conclusion belonged to the j-th port of a in β such that $m < j < n$ (so that for a_j there is no corresponding port of b) , we add a unary positive link to a_j and we connect it through a new cut link with a 0-ary negative link; then we make the newly added positive link jump on z.

We stress that, with respect to the former reduction rule, we simply "freeze" the erasing part of reduction (that is the one corresponding to the "weakening" step); the rest of the reduction step is the same as before.

Theorem 4 *Given a J-proof net R, if $R \xrightarrow{\neg e} R'$, then R' is a J-proof net.*

PROOF. Simple adaptation of the proof of Theorem 2. □

Proposition 13 *The reduction $\xrightarrow{\neg e}$ is locally confluent on J-proof nets.*

PROOF. The proof is just a variation of the proof of local confluence of $\xrightarrow{cut}$ for J-proof nets. □

Proposition 14 *If R is a J-proof net, then $R \in WN^{\neg e}$.*

PROOF. The proof is a straightforward adaptation of the proof of weak normalization of $\xrightarrow{cut}$ for J-proof nets, with some minor modifications (for example, we have to restrict the order $\lessdot$ excluding all cut links which could be reduced only by a "weakening" step). It is easy to verify that at each step the size of the J-proof net (defined in Section 4.3.2) decreases. □

5.2 Labelled reduction and $SN^{\neg e}$

Definition 12 *A labelled J-proof net is a pair (R, l) where R is a J-proof net and l is a function from the set $\{m : m$ is a premise of a positive link or the conclusion of $R\}$ to integers. We define the degree $|l|$ of (R, l) as the sum of the values of l.*

Definition 13 *Let t be a cut on (R, l). The result of the labelled non erasing reduction (denoted by $\xrightarrow{\neg e^l}$) of t is the following labelled J-proof net (R', l'):*

1. *R' is defined following the reduction $\xrightarrow{\neg e}$ of the previous subsection;*
2. *l' is defined in the following way: let p be the positive link of R which shares its conclusion with t; let $t_1, \ldots, t_m$ be the premises of p such that $\overrightarrow{t_i}$ is premise of a cut link created by the reduction of t. We consider the set $M = \{n; n$ is premise of a positive link and $p \in C_R^n\}$. We remark that, by point 3) of the definition of reduction, for all n and for all t_j, $\overrightarrow{t_j} \in C_{R'}^n$; moreover every $n \in M$ has a label $l(n)$.*
   - *Suppose M is not empty; Now let k_i be the number of residues of a given t_i in R': then for all $n' \in M$ which are* minimal *(that is, there is no $n'' \in M$ such that $C_R^{n'} \subset C_R^{n''}$) and for all $t_1, \ldots, t_m$, $l'(\overrightarrow{n'}) = (((l(t_1) + 1) * k_1) + \ldots + ((l(t_m) + 1) * k_m) + l(n'))$. For all other negative edges s $l'(\overrightarrow{s}) = l(s)$;*
   - *If M is empty, then let c be the conclusion of R: then for all $t_1, \ldots, t_m$, $l'(\overrightarrow{c}) = (((l(t_1) + 1) * k_1) + \ldots + ((l(t_m) + 1) * k_m) + l(c))$. For all other negative edges s $l'(\overrightarrow{s}) = l(s)$.*

We depict the labelled version of the reduction step in Fig. 11: we mark by h, k, i (in red) the value of l in the redex.

Proposition 15 *Let (R, l) be a labelled J-proof net; if $(R, l) \xrightarrow{\neg e^l} (R', l')$, then $|l| < |l'|$.*

PROOF. Trivial from the definition of labelled reduction. □

Proposition 16 *The labelled reduction $\xrightarrow{\neg e^l}$ is locally confluent on labelled J-proof nets.*

PROOF.

The proof is just an adaptation of the one for local confluence of $\xrightarrow{\neg e}$ on J-proof nets. We just show one case, in order to clarify how labelling is modified during normalization. Suppose we have a labelled J-proof net (R, l) with two cut c_1, c_2 as pictured below (we denote by h, m, r the labels assigned by l):

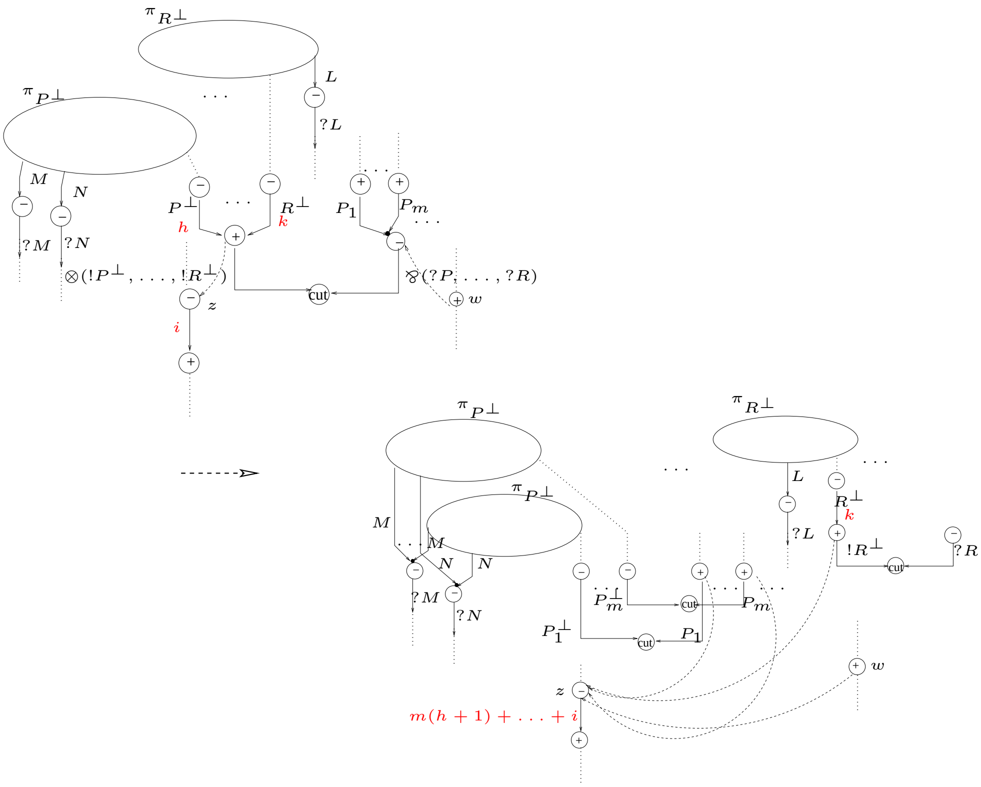
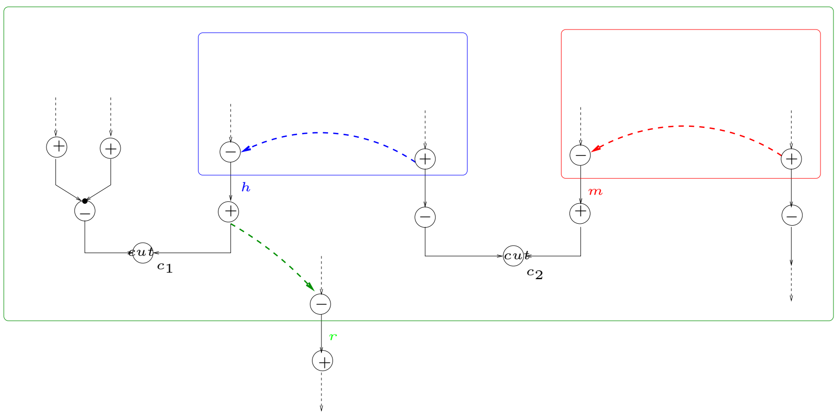
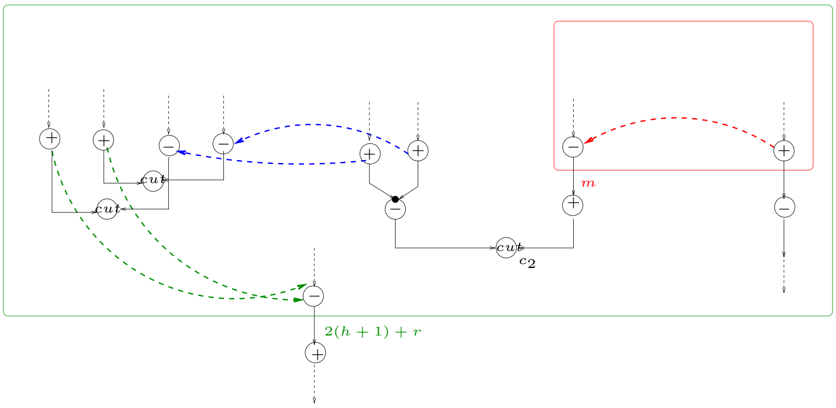


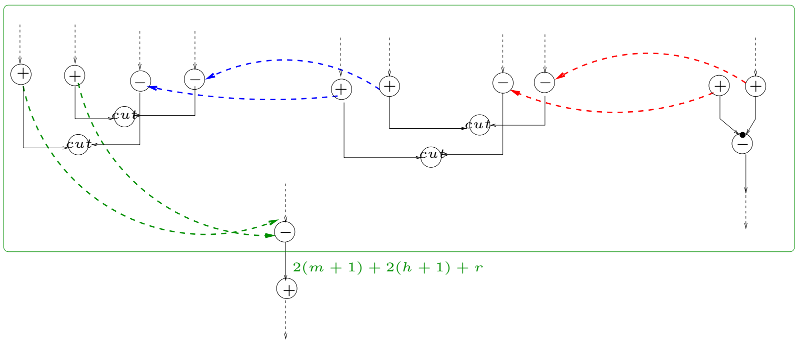
Figure 11: The labelled reduction step



We first reduce c_1: we enter in the cone associated with c_1 and duplicate its content, obtaining a J-proof net R'. We change l to l' by replacing the label r with $2(h+1)+r$ (meaning that w.r.t. l, we have duplicated the cone associated with c_1 and entered in each copy). In this way we get a new labelled J-proof net (R', l').



Then we reduce the residue of c_2 in (R', l'): we enter in the cone associated with it and duplicate its content, obtaining a J-proof net R''. We change l' to l'' by replacing the label $2(h+1)+r$ with $2(m+1)+2(h+1)+r$ (meaning that w.r.t. l', we have duplicated the cone associated with c_2 and entered in each copy). In this way we get a new labelled J-proof net (R'', l'').



We leave to the reader to check that reducing first c_2 in (R, l) and then the residue of c_1 yields the same labelled J-proof net (R'', l'').

□

Proposition 17 *If (R, l) is a labelled J-proof net, then $(R, l) \in WN^{\neg e^l}$.*

PROOF. The proof is a simple consequence of Proposition 14. □

Proposition 18 *Let (R, l) be a labelled J-proof net. If $(R, l) \in WN^{\neg e^l}$, then $(R, l) \in SN^{\neg e^l}$.*

PROOF. Since $\xrightarrow{\neg e^l}$ is increasing, locally confluent and weakly normalizing on J-proof nets, the result follows from Proposition 6. □

Proposition 19 *Let R be a J-proof net. If $(R, l) \in SN^{\neg e^l}$, then $R \in SN^{\neg e}$.*

PROOF. Trivial (we just forget the labelling). □

Proposition 20 *If R is a J-proof net, then R belongs to $SN^{\neg e}$.*

PROOF. From Proposition 17, Proposition 18, and Proposition 19. □

5.3 $SN^{\neg e}$ implies SN^{cut} (and confluence)

Remark 8 *Let R, R' be J-proof nets; if $R \xrightarrow{cut} R'$, through a "weakening" step, then any cut link t' of R' has an ancestor t in R. If t' can be reduced by a non erasing reduction step, then the same holds for t.*

Proposition 21 *If $R \xrightarrow{cut} R_1$ through a "weakening" step, and $R_1 \xrightarrow{\neg e} R_2$, then there exists an R_3 such that $R \xrightarrow{\neg e} R_3$ and $R_3 \xrightarrow{cut} *R_2$ only through "weakening" steps.*

Proof. Let u (resp. t) be the cut reduced in $R \xrightarrow{cut} R_1$ (resp. $R_1 \xrightarrow{\neg e} R_2$). Since $R \xrightarrow{cut} R_1$ through a weakening step and t is a cut link of R_1, by Remark 8, t has an ancestor t' in R which can be reduced with a non erasing step. Let R_3 be the J-proof net obtained by reducing t' in R with a non erasing step; checking all cases, we find that reducing the residues of u in R_3 (which can only be reduced through "weakening" steps) yields R_2.

□

Proposition 22 *If $R \xrightarrow{cut} R'$ by reducing a cut t with a reduction which is not a "weakening" step, then there exists R'' such that $R \xrightarrow{\neg e} R''$ in one step and $R'' \xrightarrow{cut} *R'$ only through "weakening" steps.*

Proof. Since t is not reduced with a "weakening" step, we can reduce it with a non erasing step, obtaining a J-proof net R''; now, either $R'' = R'$, or the non erasing reduction creates $u_1, \dots, u_n$ cut links which can only be reduced through "weakening" steps; by reducing them we get R'. □

Proposition 23 *Let R be a J-proof net. If $R \in SN^{\neg e}$ then $R \in SN^{cut}$.*

Proof. Suppose R does not belong to SN^{cut}, and consider an infinite reduction sequence $r : R \xrightarrow{cut} \dots R_i \xrightarrow{cut} R_{i+1} \dots$ starting from R. Now, by Proposition 22 we can obtain another sequence r' from r by replacing any reduction $R_i \xrightarrow{cut} R_{i+1}$ in r which is not a "weakening" step, with a non erasing reduction $R_i \xrightarrow{\neg e} R_{i'}$ followed by a sequence of "weakening" steps $R_{i'} \xrightarrow{cut} *R_{i+1}$; obviously r' is infinite. Now r' is an infinite sequence of reductions which contains only non erasing and "weakening" steps. We define for any number n a sequence q of non erasing steps of length n, starting from R, contradicting the hypothesis that $R \in SN^{\neg e}$. Let k be the least number s.t. $R_k \xrightarrow{cut} R_{k+1}$ in r' (so it is a "weakening" step). If $k > n$ or does not exist, we take q as the prefix of r' of length n. Otherwise we define q by induction on $n - k$. If $n = k$, q is the prefix of r' of length n. If $k < n$, let m be the least integer such that $m > k$ and $R_m \xrightarrow{\neg e} R_{m+1}$. Such an m must exist, otherwise there would be an infinite suffix of "weakening" steps (so that they erase a portion of the net at each step). Now $R_{m-1} \xrightarrow{cut} R_m$ with a "weakening" step and $R_m \xrightarrow{\neg e} R_{m+1}$, so we can apply Proposition 21: we apply it $m-k$ times, obtaining a sequence of reductions r'' which has a prefix of non erasing steps of length $k+1$. We obtain q by applying the induction hypothesis on r''. □

Proposition 24 *If R is a J-proof net, then R belongs to SN^{cut}.*

Proof. From Proposition 20 and Proposition 23. □

Proposition 25 *The relation $\xrightarrow{cut}$ is confluent on J-proof nets.*

Proof. From Lemma 1, Proposition 24 and Proposition 11. □

6 Observations and remarks

In this final section we deal with some questions we left opened in the previous sections, namely the relation between J-proof nets and polarized proof nets, the inclusion of *atoms*, and the role played by the Mix rule.

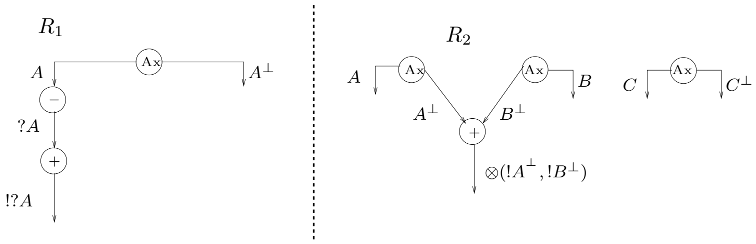


Figure 12: Two not balanced J-nets

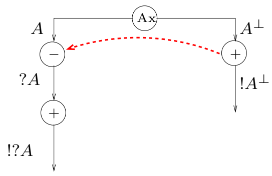


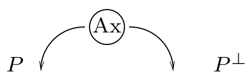
Figure 13: A balanced J-net

6.1 Adding axioms

We add to $MEHS$ rules the axiom rule: [4]

$$\frac{}{\vdash X,\ X^{\perp}}(Ax)$$

extending accordingly the definition of J-net with the corresponding *ax* link, with no premises and two conclusions typed by dual formulas:



The notion of order $\prec_R$ associated with a J-net R is defined as before.

Now, in order to preserve the basic property of cones (namely that the source of any edge in the border of a cone is a positive link, see Proposition 6), we must impose on J-nets a constraint called *balancedness*; with respect to usual polarized proof nets, such a condition corresponds to enclosing all axiom links into a box.

Definition 14 *A* balanced J-net *is a J-net with axioms R such that*

1. *for every $-$ link b and for every ax link a of R, such that a shares an edge with b, there exists a positive link c below a in $\prec_R$ which jumps on b in R.*
2. *if R is composed by more than one connected component, then no negative conclusion of R is the conclusion of an axiom link*[5] *.*

Definition 15 *A J-net with axioms R is* acceptable *when it is balanced and switching acyclic.*

We can properly extend the notion of sequentializable J-net to include axioms (we leave this to the reader), so that we can state the following proposition:

Proposition 26 *An acceptable J-net with axioms is sequentializable.*

[4] In order to maintain our correspondence with $MELLP$, we must add to $MELLP$ the axiom rule $\frac{}{\vdash ?X,\ !X^{\perp}}(Ax)$. Actually, this make our correspondence weaker: we are only able to represent in $MEHS$ the subsystem $MELL_{pol}$ of $MELLP$ (see [Lau02]).

[5] This second condition is required for sequentialization, in order to respect the "positive contexts" constraint of the Mix rule.

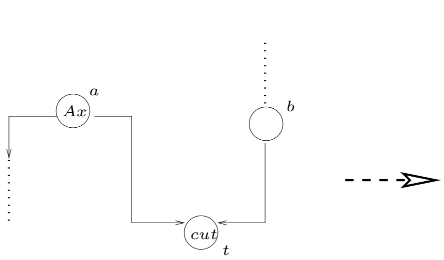


Figure 14: the ax reduction step

Cut elimination with axioms

Definition 16 *A J-proof net with axioms R is a J-net with axioms which is acceptable and closed.*

Now we extend the relation $R \xrightarrow{cut} R'$ adding another reduction rule, the ax step, which replaces a module β containing a *cut* link t in R with a module γ as follows (see Fig. 14):

- β is composed by t, an axiom link a which shares an edge with t and a link b which shares the other premise of t;
- γ is composed just by b.

Theorem 5 (Preservation of correctness) *Given a J-proof net with axioms R, if $R \xrightarrow{cut} R'$, then R' is a J-proof net with axioms.*

Proof. The proof is a simple extension of the one given in Section 4.1. The ax step trivially preserves switching acyclicity and balancedness, so we have to prove only preservation of balancedness for the $+/-$ *step*. Point 2. in the definition of balancedness is preserved due to the the fact that J-proof nets are closed, and that if $R \xrightarrow{cut} R'$, then R and R' have the same conclusions. Concerning point 1. of the definition of balancedness, let t be the cut between the (positive) link a and the (negative) link b reduced by the $+/-$ step. Let us call $c_1, \ldots, c_n$ the cut links of R' created by reducing t in R, a_i being the link whose conclusion is the negative premise of c_i and b_i being the link whose conclusion is the positive premise of c_i for $i = 1, \ldots, n$. Since we have added axiom links, b_i could be either a positive link or an ax link. Now, we want to show that if R is balanced, R' is balanced too. Suppose that in R a jumps on a negative link n s.t. n shares an edge with an axiom link d, and $a \prec_R d$: to prove preservation of balancedness, we must show that there exists a positive link a' in R' such that a' jumps on n in R' and $a \prec_{R'} d$. Now it is easy to verify that one of the following holds:

- one of the b_i is a positive link so that b_i jumps on n in R' (by definition of $+/-$ step) and $b_i \prec_{R'} d$ (by definition of $\prec_{R'}$);
- one of the b_i is an ax link which shares a conclusion with b (which is negative) in R, so that (see Fig. 15):
  1. there is a positive link m which jumps on b in R and $m \prec_R b_i$ (by balancedness of R);
  2. such an m jumps on n in R' and $m \prec_{R'} d$ (by definition of $+/-$ step and $\prec_{R'}$).

In any case, balancedness is preserved. □

We leave to the reader the proof that the relation $\xrightarrow{cut}$ extended with the ax step is still strongly normalizing and confluent on J-proof nets with axioms.

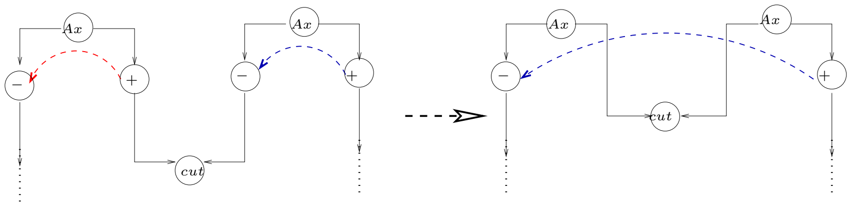


Figure 15: preservation of balancedness

6.2 Mix and confluence

Now we want to try to get rid of the Mix rule: the standard way to deal with it is by imposing *connectedness of the correction graphs*:

Definition 17 (Correction graph) *Given a J-net R, a* switching *s is the choice of an incident edge for every negative link of R; a* correction graph *$s(R)$ is the graph obtained by erasing the edges of R not chosen by s.*

Definition 18 (s-connected) *A J-net R is s-connected iff the followings hold:*

- *there is not maximal (w.r.t. $\prec_R$) negative link*[6]*;*
- *for any possible choice of switching s, the correction graph $s(R)$ is connected.*

Given an arborescent J-proof net R (resp. a cone C of R), we say that a link a of R has *depth n* in R (resp. in C) if it is contained in exactly n cones of R (resp. n cones contained in C).

Proposition 27 *Given an arborescent J-proof net R, R is s-connected iff for every cone C of R there is exactly one positive link with depth 0 in C.*

Proof. Left to right direction: by s-connectedness we can deduce that every cone C of R contains at least one positive link at depth 0 (otherwise there would be a maximal negative link); we must prove that such a link is unique. We proceed *ad absurdum*: suppose then that a cone C of a negative edge c (which is the conclusion of a negative link n) contains two positive links a, b at depth 0: by s-connectedness for every switching s there must exist a path connecting a and b in $s(R)$.

Now, let us consider a path connecting a with b in $s(R)$ starting from a. We have three possible cases:

1. the path connecting a with b in $s(R)$ starts by going down from a and entering n; then it must eventually enter back into C by going up through an edge in the border of C, since any path connecting a with b in $s(R)$ uses a single switching edge of n; but then such a path is a switching path, and by Lemma 3 this contradicts the fact that R is a J-proof net.

2. the path connecting a with b in $s(R)$ starts by going down from a trough an edge which does not enter n; such an edge must be in the border of C (otherwise a would not be at depth 0 with respect to C). Then the path must eventually enter back into C by going up through another edge in the border of C. As above, we get a switching path which contradicts the fact that R is a J-proof net by Lemma 3.

3. the path connecting a with b in $s(R)$ starts by going up through a premise a_i of a. Now by Remark 3 all the cones of the premises of any two links at depth 0 with respect to C are disjoint, so the path connecting a with b in $s(R)$ must exit from $C_R^{a_i}$ by going down through an edge in the border of $C_R^{a_i}$ which is at the same time in the border of C, and eventually it must enter back into C by going up through another edge in the border of C; but then such a path would be a switching path, and by Lemma 3 this would contradict the fact that R is a J-proof net.

[6] Such a condition corresponds in our setting to the standard way to conciliate weakening and connectedness in proof nets, by adding jumps on weakening links (see [Gir96],[TdF00]).

In all cases we obtained a contradiction, so there cannot be two positive links at depth 0 with respect to C.

Right to left direction: it is enough to observe that in this case the order $\prec_R$ can be represented by a tree which branches only on positive nodes, and without maximal negative nodes; to any path in such a tree corresponds a switching path on R. $\square$

Proposition 28 *If R is an arborescent and s-connected J-proof net, then the sequentialization π of R has no occurrence of the rules Mix or* Dai.

Proof. By Proposition 27, for every cone C of R there is exactly one positive link with depth 0 in C; so we can have neither a negative link with no successor (corresponding to a *Dai* rule), nor a negative link with more than one immediate successor (which would correspond to an application of a Mix rule). $\square$

Now we must check that s-connectedness is stable under reduction: we first verify the property for the restricted case of *saturated* J-proof nets:

Proposition 29 *If R is an arborescent and s-connected J-proof net, then R is saturated.*

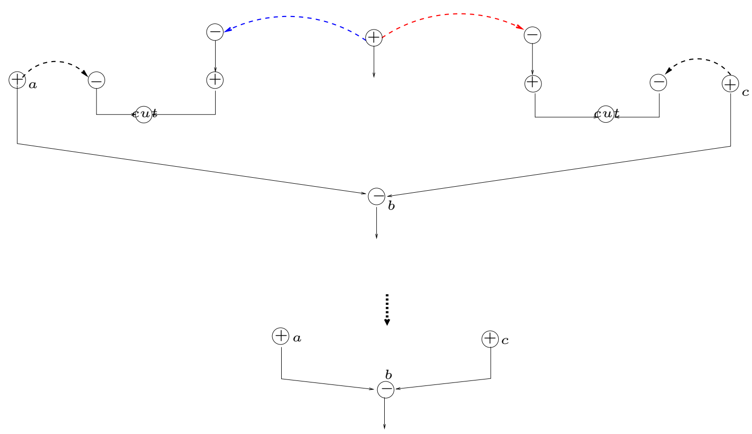
Proof. Let us consider a positive link a and a negative link b of R such that a, b are incomparable with respect to $\prec_R$. By closedness there exists a unique terminal negative link n of R s.t. n is the minimum of $\prec_R$ (by arborescence and s-connectdeness); n is clearly different from b (since n is comparable with all the links of R). By Proposition 27, there exists a unique positive link p such that p is at depth 0 w.r.t. C^n_R; clearly p is different from a (since p is comparable with all the links of R). Since a and b are incomparable, and $p \prec_R a$ (resp. $p \prec_R b$), there exists a directed path from a to p (resp. from b to p). But then adding a jump from a to b creates a switching cycle in R.

$\square$

Proposition 30 *Let R be a saturated and s-connected J-proof net. If $R \xrightarrow{cut} R'$, then R' is saturated and s-connected.*

Proof. Preservation of s-connectedness is a simple consequence of arborescence of R and Remark 6. The only delicate case is if $R \xrightarrow{cut} R'$ with a "weakening step". Let us call n the negative premise of the positive active link of the cut to be reduced in R. By arborization Lemma and Proposition 27, for every cone C of R there is exactly one positive link with depth 0 in C; we have to be sure that erasing the cone of C^n_R will not erase the unique positive link at depth 0 with respect to some other cone C. We proceed *ad absurdum*: let us suppose that an edge in the border of C^n_R is incident on a negative link m and emergent on a link p s.t. p is the unique positive link at depth 0 in C^m_R. Now, $p \in C^n_R$, and $p \in C^m_R$ so by arborescence and by hypothesis, $p \in C^n_R \subset C^m_R$, but then p has not depth 0 in C^m_R, contradicting the assumption. The rest of the proof follows by Remark 6, Proposition 27 and Proposition 29. $\square$

When we move to the full framework, including also not saturated J-proof nets, we notice that preservation of s-connectedness under reduction does not hold in general, as the following counterexample shows:



No matter what order we choose to reduce the cuts, we pass from an s-connected (not saturated) J-proof net to a not s-connected one.

We could imagine at least two ways to deal with this problem:

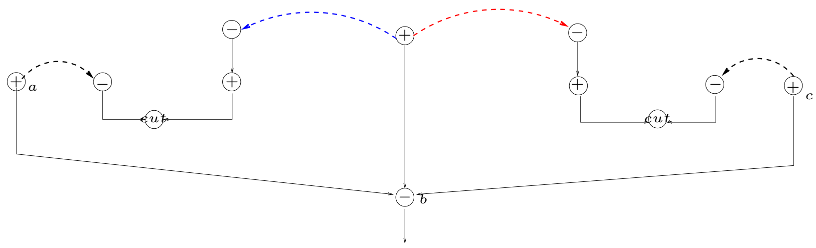
1. find another geometrical property to characterize the absence of Mix in J-proof nets;

2. modify the $+/-$ rewriting step in order to preserve s-connectedness.

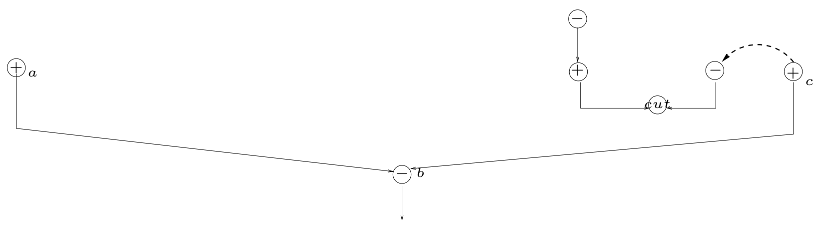
Concerning point 1., it seems quite hard to find a different characterization of Mix-free J-proof nets, not equivalent to s-connectedness.

Concerning point 2., one possible, radical solution is to modify the reduction step, by adding all the jumps needed to preserve s-connectedness.

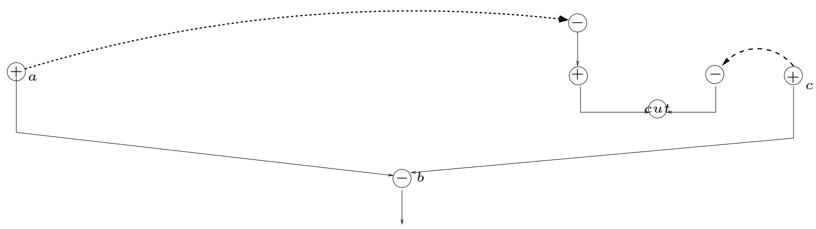
Let us try to apply such a method to the J-proof net in the counterexample above:



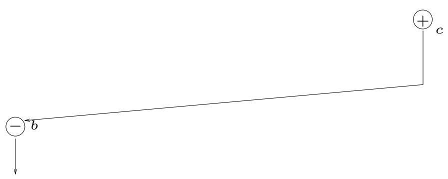
We reduce the left hand cut:



Now we try to add the jump needed to preserve s-connectedness:



If now we reduce the right hand cut, we get the following J-proof net:



We remark that (in this particular case) there are no other ways to add jumps in such a way to preserve s-connectdness. It is then easy to observe that if we inverse the two reductions, adding jumps to preserve s-connectedness in the same way, we get a different normal form: we have lost confluence. This opens a third, more interesting option, besides the two aforementioned ones: that, *in case of overlapping of cones, it is necessary to allow Mix in order to preserve confluence.* The analysis of the semantical and computational properties of J-proof nets, which constitutes the natural continuation of the present line of work, will serve as a starting point to study in details such a connection between Mix, overlapping of cones and reduction.

6.3 Polarized proof nets and J-proof nets

Let us describe how to encode polarized proof nets of $MELLP$ into J-proof nets using an example. For a formal definition of polarized proof nets we refer to [Lau05]. Let us consider the polarized proof nets R_1, R_2 given resp. in Fig. 16, 17;

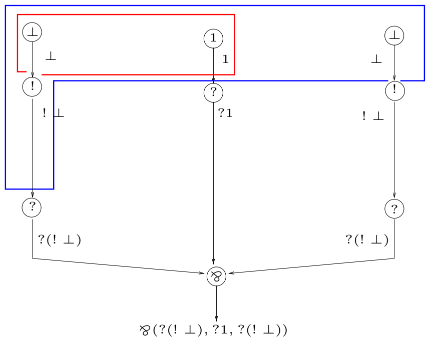


Figure 16: The polarized proof net R_1

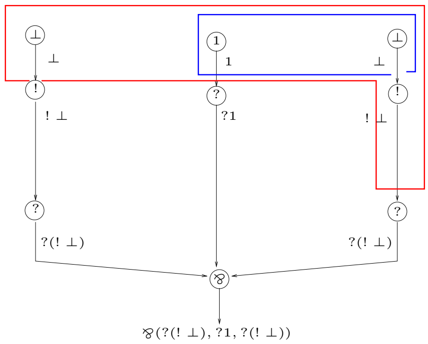


Figure 17: The polarized proof net R_2

We observe that the only difference between R_1, R_2 is in the inclusion relation between boxes: in R_1 the red exponential box is included in the blue one, while in R_2 it is the other way around.

Now we turn R_1 into a J-proof net R_1^J (pictured in Fig. 18) in two steps:

- first we transform each $MELLP$ link l of R_1 into an $MEHS$ link l' of R_1^J, by clustering together multiplicative and exponential links;
- then we add jumps to R_1^J, in such a way that if a link a belongs to the exponential box of a !-link b in R_1, then the image of a in R_1^J is in the cone of the corresponding premise of the image of b in R_1^J.

Now it is easy to see that the order associated with R_1^J is arborescent (because of the nesting condition on exponential boxes in R_1), and that R_1^J is s-connected (by the fact that R_1 is a polarized proof net), so by Proposition 29, R_1^J is saturated.

In the same way, we can associate with R_2 a saturated J-proof net R_2^J (pictured in Fig. 19).

We remark that the mapping of polarized proof nets into J-proof nets described above is injective (two different polarized proof nets are mapped into two different saturated J-proof nets), but it is not surjective (a J-proof net is the image of a polarized proof net only if is saturated). In Fig. 20 we provide an example of a J-proof net which is not the image of any polarized proof net (since its cones do not satisfy the nesting condition).

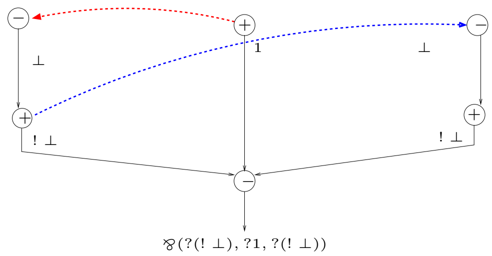


Figure 18: The saturated J-proof net associated with R_1

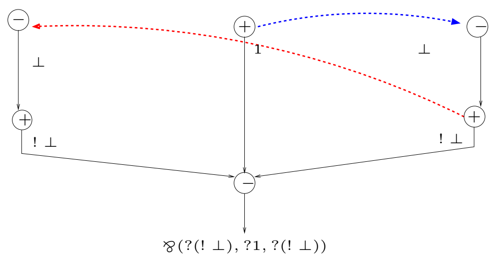


Figure 19: The saturated J-proof net associated with R_2

Acknowledgements

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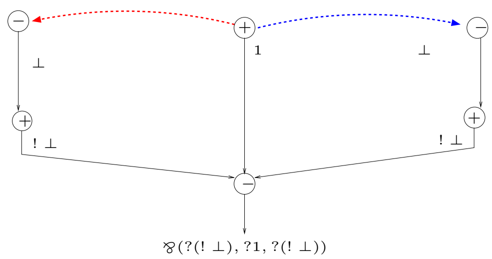

Figure 20: A J-proof net which is not the image of a polarized proof net

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