Proof nets and semantics: coherence and acyclicity

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Introduction

The purpose of this short report is to investigate the relation between *coherence* (an algebraic relation between two points of a coherent space) and *acyclicity* (a geometrical property of graphs) for a specific class of graphs, first defined in 1987 by J.Y. Girard, called *proof structures*: we are going to show how both these properties allow to isolate among proof structures the ones coming from linear logic proofs (also called proof nets).

Proof nets were first defined in 1987 by J.Y. Girard, in [Gir87]; the definition was simplified in [Dan90] introducing the Danos-Regnier correctness criterion, which uses simple geometrical properties like acyclicity to characterize proof nets among proof structures. The denotational semantics of proof structures was also defined in 1987 by Girard (cf. [Gir87]), with the *experiment method*, which assigns to each proof structure a set of points of a coherent space as its interpretation. What we actually hold in this report is that for a proof structure *to have no cycles and to be interpreted by a clique (i.e. a set of points pairwise coherent) are the two sides of the same coin*: we then get a *semantical* way to discriminate between "good" proof structures (which correspond to proofs) and "bad" proof structures (which do not correspond to proofs).

There are also other ways of testing the "goodness" of a proof structure, based on the cut-elimination procedure, (see for example [Bech98]), that are *interactive* (we refer to [Gir01] for a general analysis of the notion of interaction in logic). The existence of both a semantical and an interactive criterion is not surprising at all: in fact it's well known from denotational semantics that the semantics of a proof takes in account all the possible interactions of the proof with an environment.

The proof that the interpretation of a proof net of MLL is a clique was first given by Girard in [Gir87], using the correctness criterion of trips; a proof of the extension of the theorem for MELL can be found in [Bar01]. Christian Retore, in [Ret97], proved, using the Danos-Regnier correctness criterion, that the interpretation of a proof structure of MLL which is not a proof net is not a clique.

In this report we prove an analogue of the theorem of Girard, and of its extension to MELL, using the Danos-Regnier criterion, and we give an alternative proof of the theorem of Retore for MLL, illustrating the differences between the two approaches; finally we explain why the extension for MELL is not straightforward from the result for MLL.

The report is divided in four parts:

Chapter 1 : In this chapter, after some basic definitions, we first define proof structures, then proof nets, both for MLL and MELL; we define altogether some important properties of proof nets, like the splitting property.

Chapter 2 : In this chapter we first present the coherent semantics of the formulas of linear logic, then we present the experiment method to define the interpretation of a proof structure in the coherent semantics, both for MLL and MELL.

Chapter 3 : In this chapter we prove the correspondence between coherence and acyclicity for the proof structures of MLL. The proof splits in two parts: we first prove that the interpretation of a proof net is a clique; then we prove that the interpretation of a "bad" proof structure is not a clique.

Chapter 4 : In the final chapter we prove that the interpretation of a proof net of MELL is a clique; then we explain why one cannot prove that the interpretation of a "bad" proof structure of MELL is clique by a straightforward extension of the proof for MLL.

Chapter 1

Proof nets

In this chapter, after introducing some basic definitions, we will define the notion of *proof structure* and of *proof net*, introducing the correctness criterion, both for MLL and MELL; we will also introduce some interesting geometrical property of the proof nets, namely the *splitting property*.

1.1 Syntactical preliminaries

We define the language of the fragments of linear logic we are interested in, i.e. propositional multiplicative linear logic without constant, and propositional multiplicative and exponential linear logic without constant.

Definition 1. *Let $\mathcal{V} = \{X,\ Y,\ Z, \ldots\}$ be a numerable set of propositional variables; the formulas of MLL are defined as follows:*

- *$X,\ Y,\ Z, \ldots$ and $X^\perp,\ Y^\perp,\ Z^\perp, \ldots$ are formulas of MLL*
- *if A and B are formulas of MLL, then $A \parr B$, $A \otimes B$ are formulas of MLL*

Linear negation is the connective defined from the following De Morgan laws:

$$(A \otimes B)^\perp = A^\perp \parr B^\perp$$

$$(A \parr B)^\perp = A^\perp \otimes B^\perp$$

Definition 2. *Let $\mathcal{V} = \{X,\ Y,\ Z, \ldots\}$ be a numerable set of propositional variables; the formulas of MELL are defined as follows:*

- *$X,\ Y,\ Z, \ldots$ and $X^\perp,\ Y^\perp,\ Z^\perp, \ldots$ are formulas of MELL*
- *if A and B are formulas of MELL, then $A \parr B$, $A \otimes B$ are formulas of MELL*
- *if A is a formula of MELL, then $?A$ and $!A$ are formulas of MELL*

Linear negation is the connective defined using the following De Morgan laws:

$(A \otimes B)^\perp = A^\perp \parr B^\perp$

$(A \parr B)^\perp = A^\perp \otimes B^\perp$

$(!A)^\perp = ?A^\perp$

$(?A)^\perp = !A^\perp$

We briefly recall the derivation rules of MELL in linear sequent calculus:

$$\frac{}{\vdash A,\ A^\perp}(Ax) \qquad \frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ A^\perp}{\vdash \Gamma,\ \Delta}(Cut)$$

$$\frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ B}{\vdash \Gamma,\ A \otimes B,\ \Delta}(\otimes) \qquad \frac{\vdash \Gamma,\ A,\ B}{\vdash\ \Gamma,\ A \parr B}(\parr)$$

$$\frac{\vdash ?\Gamma,\ A}{\vdash ?\Gamma,\ !A}(promotion) \qquad \frac{\vdash \Gamma,\ A}{\vdash \Gamma,\ ?A}(dereliction)$$

$$\frac{\vdash \Gamma}{\vdash \Gamma,\ ?A}(weakening) \qquad \frac{\vdash \Gamma,\ ?A,\ ?A}{\vdash \Gamma,\ ?A}(contraction)$$

In order to deal with proof structure we define another rule called Hyp

$$\frac{}{\vdash \Gamma}(Hyp)$$

Finally, we introduce another rule called *mix rule* to allow proof structures unconnected

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma,\ \Delta}(Mix)$$

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1.2 MLL proof nets

Definition 3 (Proof structures). *A proof structure of MLL is an oriented graph whose nodes are called links and whose edges are typed with MLL formulas;*

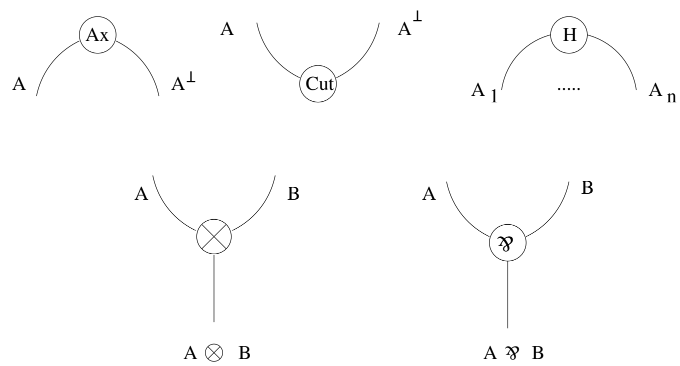


Fig.1 : the links of MLL

every link has a given number of incident edges, called the premises *of the link, and a given number of emergent edges, called the* conclusions *of the link. The links are the following:*

- *the* Axiom link *has no premise and two conclusions typed by dual formulas;*
- *the* Cut link *has two premises typed by dual formulas and no conclusion;*
- *The* Hypothesis link *(or H-link) has no premise and $n \geq 1$ conclusions, each of them typed by a formula;*
- *the* Tensor link *has two premises and one conclusion: if the left premise is typed by the formula A and the right premise is typed by the formula B, the conclusion is typed by the formula $A \otimes B$;*
- *the* Par link *has two premises and one conclusion: if the left premise is typed by the formula A and the right premise is typed by the formula B, the conclusion is typed by the formula $A \parr B$.*

Furthermore, a proof structure must satisfy the following conditions:

1. *every edge is the conclusion of a unique link;*
2. *every edge is the premise of at most one link;*

the edges which are not premise of a link are called the conclusions *of the proof structure, and the links they emerge from are called* terminal links

Definition 4 (Switching and correction graph). *Given a proof structure G we call* switching *for G a function S assigning to every instance of a Par link in G a value L (left) or R (right): if S is a switching for G, then the graph $S(G)$ obtained erasing for every instance of a Par link in G the left premise if $S(\parr) = R$ or the right premise if $S(\parr) = L$, is called* correction graph.

Definition 5 (Proof net). *Let R be a proof structure without Hypothesis link ; R is a* proof net *(resp.* AC proof net*) if and only if every correction graph S(R) is acyclic and connected (resp. acyclic).*

A proof net is a proof structure which correspond to a proof of linear sequent calculus; as we have noticed before, an AC proof net, instead, is a proof structure which corresponds to a proof of linear sequent calculus with the adjunction of the mix rule.

Definition 6 (Splitting link). *Let R be a proof structure; a Cut link or a Tensor link of R is said to be* splitting *for R, if the removal of the link yields two disjoint proof structures R_1 and R_2; a Par link of R is said to be* splitting *when the removal of the link yields two disjoint proof structures, R_1 and R_2, one having among the conclusions the premises of the Par link and the other having the conclusion of the Par link as the conclusion of an Hypothesis link .*

For the proof of the following lemma , we refer to the work of Vincent Danos [Dan90]

Lemma 1 (Splitting Par). *Let R be a proof net; if R has at least one Par link, it has a splitting par link.*

1.3 MELL proof net

Definition 7 (MELL proof structure). *A proof structure of MELL is an oriented graph whose nodes are called links and whose edges are typed with MELL formulas; every link has a given number of incident edges, called the* premises *of the link, and a given number of emergent edges, called the* conclusions *of the link. We add the following links to those of Definition 3:*

- *The !-link has one premise and one conclusion; if the premise is typed by the formula A, then the conclusion is typed by the formula $!A$*
- *The ?de-link has one premise and one conclusion; if the premise is typed by the formula A, then the conclusion is typed by the formula $?A$*
- *The ?w-link has no premise and one conclusion typed by the formula $?A$ for some formula A*
- *The ?co-link has $k \geq 2$ premises and one conclusion, all typed by the formula $?A$ for some formula A*
- *the pax link has one premise and one conclusion both typed by the formula $?A$ for some formula A.*

Besides those of Definition 3, a proof structure R of MELL must satisfy the following conditions:

1. *!-box condition:*

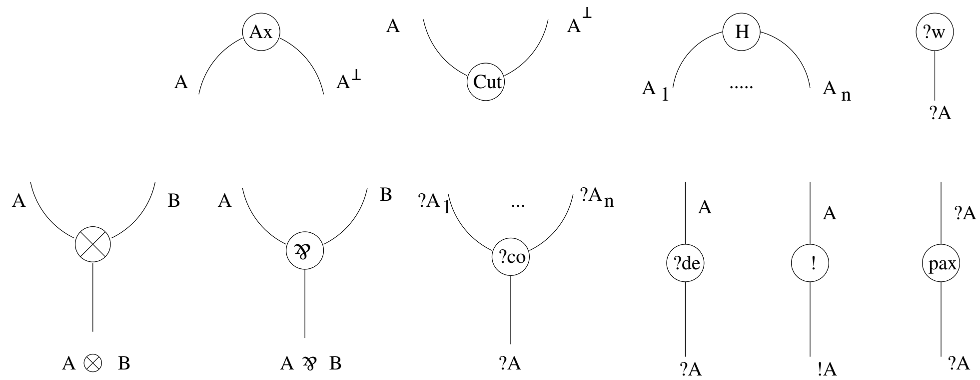


Fig.2 : the links of MELL

(a) *with each !-link n is associated a unique sub-graph B of R s.t. one among the conclusions of B is the conclusion of n and every other conclusion of B is the conclusion of a pax link; B is called an exponential box and n is called the* principal door *or* pal door *of B*

(b) *with each pax link n is associated an exponential box B of R s.t. one among the conclusions of B is the conclusion of n; the link n is called* auxiliary door *or* pax door *of B*

2. *nesting condition:*

   two boxes are either disjoint or included one in the other.

We shall say that a link or an edge of a given proof structure R has depth n in R, if it is contained in exactly n boxes of R; for a box B we shall say that B has depth n in R, if it is contained in exactly n boxes of R, all different from B. The depth of a proof structure is the maximal depth of its boxes.

Definition 8 (Graph with pairs). *The couple* $(G, App(G))$ *is called a* graph with pairs *when* G *is an oriented graph and* $App(G)$ *is a set of* n*-tuples* ($n \geq 2$) *of coincident edges, i.e. edges with the same target. Let* R *be a proof structure and let* $B_1, \ldots, B_k$ *be the boxes of* R *with depth 0. We are going to associate with* R *a set* $App(R)$ *and a graph with pairs* $R_{ap} = (G_R, App(R))$.

The graph G_R *is obtained from* R *in the following way:*
- substitute for each box B_i *with* p_i *conclusions* ($i \in \{1, \ldots, k\}$)*, an* H*-link with* p_i *conclusions*

The set $App(R)$ *contains the following (and only the following) m-tuples:*
- the couples of premises of every Par link of R *at depth 0*
- the p-tuples of premises of every ?co-link of R *at depth 0.*

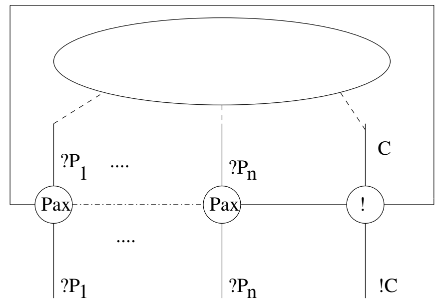


Fig.3 : a box of MELL

Definition 9 (Correctness graph). *Let R be a proof structure and let $B_1, \ldots, B_k$ be the boxes at depth 0 in R . Let $R_{ap} = (G_R, App(R))$ be the graph with pairs associated with R. A* switching *S for R is the choice of an edge for every n-tuple of $App(R)$. With each switching S is associated an unoriented graph $S(R)$, called* correctness graph *(which we will denote $S(R)$): for every n-tuple of $App(R)$, erase the edges of G_R which are not selected by S, and forget the orientation of the edges of the graph.*

Definition 10 (Proof net). *Let R be a proof structure which contains no instances of the H-link, and let $B_1, \ldots, B_k$ be the boxes at depth 0 in R. We say that R is an* AC proof net *when the following conditions are satisfied:*

1. *for every switching S of R, the correctness graph $S(R)$ is acyclic;*
2. *for every box $B_i \in \{B_1, \ldots, B_k\}$, the proof structure R_i contained in B_i is an AC proof net.*

Definition 11 (Splitting link). *Let R be a proof structure; a Cut link at depth 0 or a Tensor link at depth 0 of R is said to be* splitting *for R_{ap} , if the removal of the link ,with its conclusion, yields two disjoint proof structure R_1 and R_2; a ?co-link at depth 0 or a Par link at depth 0 n of R is said to be* splitting *for R_{ap} when the removal of the link yields two disjoint proof structures, R_1 and R_2, one having among the conclusion the premises of n and the other having the conclusion of n as the conclusion of an Hypothesis link .*

Lemma 2 (Splitting lemma). *Let R be a proof net; if R contains at least one Par link or ?co-link of depth 0, then one among these links is splitting for R_{ap}.*

References

Linear logic and proof nets were first defined in [Gir87]; for the presentation of MLL proof structures and the definition of MELL proof structures we refer to [Tor00].

Chapter 2

Experiments

In this chapter, after giving some generalities about coherent denotational semantics for MLL and MELL, we introduce the notion of semantics of a proof structure, using the *experiment method*, both for MLL and MELL; we also make a specific choice about the kind of coherent space we are going to use to interpret the atoms

2.1 Semantical preliminaries

Definition 12 (Coherent space). *A* coherent space $\mathcal{A}$ *is the data of a set* $|\mathcal{A}|$*, called the* web *of* $\mathcal{A}$*, and of a binary reflexive and symmetric relation denoted by* $\overset{\frown}{\smile}_{\mathcal{A}}$*, called the* coherence relation *on* $|\mathcal{A}|$

Notation 1. We use the following conventions:

- $x \frown_{\mathcal{A}} y$ stands for $x \overset{\frown}{\smile}_{\mathcal{A}} y$ and $x \neq y$
- $x \smile_{\mathcal{A}} y$ stands for $\neg(x \overset{\frown}{\smile}_{\mathcal{A}} y)$
- $x \asymp_{\mathcal{A}} y$ stands for $x \smile_{\mathcal{A}} y$ or $x \neq y$.

Definition 13 (Clique). *A* clique *of* $\mathcal{A}$ *is a multiset of elements of* $|\mathcal{A}|$ *pairwise coherent; we denote equally by* $\mathcal{A}$ *the set of the cliques of* $\mathcal{A}$*.*

An *interpretation* of MLL formulas is defined by induction on their complexity in this way: one associates some arbitrary coherent spaces with atomic formulas (we call such a choice *atom-interpretation* of MLL; obviously for every such choice one gets a different interpretation). Then the coherent spaces associated with compound formulas are defined as follows:

- $|\mathcal{A}^{\perp}| = |\mathcal{A}|$, and for every $x, y \in |\mathcal{A}|$, one has $x \overset{\frown}{\smile}_{\mathcal{A}^{\perp}} y$ iff $x \asymp_{\mathcal{A}} y$
- $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$, and for every $x, x' \in |\mathcal{A}|$ and $y, y' \in |\mathcal{B}|$, one has $(x, y) \overset{\frown}{\smile}_{\mathcal{A} \otimes \mathcal{B}} (x', y')$ iff $x \overset{\frown}{\smile}_{\mathcal{A}} x'$ and $y \overset{\frown}{\smile}_{\mathcal{B}} y'$

- $|\mathcal{A} \parr \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$, and for every $x, x' \in |\mathcal{A}|$ and $y, y' \in |\mathcal{B}|$, one has $(x, y) \stackrel{\frown}{\smile}_{\mathcal{A}\parr\mathcal{B}} (x', y')$ iff $x \frown_{\mathcal{A}} x'$ or $y \frown_{\mathcal{B}} y'$

To extend the semantics of MLL to MELL, we have to add the definitions of the meaning of the exponential connectives ! and ? to the previous:

- $|!\mathcal{A}| = \mathcal{A}_f$ whose elements are the finite elements of $\mathcal{A}$, and for every $x, y \in |!\mathcal{A}|$ one has $x \stackrel{\frown}{\smile}_{!\mathcal{A}} y$ iff $x \cup y \in \mathcal{A}$;
- $|?\mathcal{A}| = \mathcal{A}_f^{\perp}$ whose elements are the finite elements of $\mathcal{A}^{\perp}$, and for every $x, y \in |!\mathcal{A}^{\perp}|$ one has $x \stackrel{\frown}{\smile}_{?\mathcal{A}} y$ iff $x \cup y \notin \mathcal{A}^{\perp}$.

2.2 Experiments for MLL proof structures

Notation 2. Let A be any MLL formula. By $|\mathcal{A}|$ we mean the web of the coherence space $\mathcal{A}$ associated with A by the coherent denotational semantics of MLL.

Definition 14 (Experiments). *Let R be a proof structure and e an application associating with every edge a of type A of R an element of $|\mathcal{A}|$; e is an* experiment *of R when the following conditions hold:*

- *if $a = a_1$ is the conclusion of an Axiom link with conclusions the edges a_1 and a_2 of type A and $A^{\perp}$ respectively, then $e(a_1) = e(a_2)$.*
- *if a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.*
- *if a is the conclusion of a Par (resp. Tensor) link with left premise a_1 and right premise a_2, then $e(a) = (x_1, x_2)$, where $e(a_1) = x_1$ and $e(a_2) = x_2$.*

If the conclusions of R are the edges $a_1, \ldots, a_l$ of type respectively $A_1, \ldots, A_l$ and e is an experiment of R such that $\forall i \in \{1, \ldots, l\}\ e(a_i) = x_i$ then we say that $(x_1, ..., x_l) \in |\mathcal{A}_1 \parr ... \parr \mathcal{A}_l|$ is the *conclusion* or the *result* of the experiment e of R, and we will denote it by $|e|$; the *interpretation* $[R]$ of a proof structure R is defined as the set of his results.

Notation 3. We adopt the following conventions:

1. Given a proof structure R with conclusions Γ and e_1, e_2 any two experiments of R, by $|e_1| \stackrel{\frown}{\smile} |e_2|$ (resp. by $|e_1| \frown |e_2|$, $|e_1| \smile |e_2|$, $|e_1| \asymp |e_2|$, $|e_1| = |e_2|$), we mean that $|e_1| \stackrel{\frown}{\smile}_{\parr\Gamma} |e_2|$ (resp. that $|e_1| \frown_{\parr\Gamma} |e_2|$, $|e_1| \smile_{\parr\Gamma} |e_2|$, $|e_1| \asymp_{\parr\Gamma} |e_2|$, $|e_1| =_{\parr\Gamma} |e_2|$).
2. To make things simpler, we will often refer to an edge by its type, when it can be done without ambiguity (for example we write "the link n with premises $A_1, \ldots, A_n$ and conclusion A" instead of "the link n with premises $a_1, \ldots, a_n$ of type $A_1, \ldots, A_n$ and conclusion the edge a of type A".)

3. Following the previous convention, given an edge a of type A of a proof structure R and two experiments e_1, e_2 of R, by $e_1(A) \smile\!\!\!\!\frown e_2(A)$ (resp. $e_1(A) \frown e_2(A)$, $e_1(A) \smile e_2(A)$, $e_1(A) \asymp e_2(A)$, $e_1(A) = e_2(A)$) we mean that $e_1(a) \smile\!\!\!\!\frown_{\mathcal{A}} e_2(a)$ (resp. $e_1(a) \frown_{\mathcal{A}} e_2(a)$, $e_1(a) \smile_{\mathcal{A}} e_2(a)$, $e_1(a) \asymp_{\mathcal{A}} e_2(a)$, $e_1(a) =_{\mathcal{A}} e_2(a)$).

2.3 Experiments for MELL proof structures

Definition 15 (Experiment). *Let e be an application which associates with every edge a of type A of a proof structure R of depth p a* multiset *$e(a)$ of elements of $|\mathcal{A}|$, in such a way that when a has depth 0 in R the multiset $e(a)$ contains exactly one element. The application e is an* experiment *of R when the following conditions hold:*
if $p = 0$, then:

- *if $a = a_1$ is the conclusion of an Axiom link with conclusions the edges a_1 and a_2 of type A and $A^{\perp}$ respectively, then $e(a_1) = e(a_2)$.*
- *if a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.*
- *if a is the conclusion of a Par (resp. Tensor) link with left premise a_1 and right premise a_2, then $e(a) = \{(x_1, x_2)\}$,where $e(a_1) = \{x_1\}$ and $e(a_2) = \{x_2\}$.*
- *if a is the conclusion of a ?de-link with premise a_1 , then $e(a) = \{\{x_1\}\}$, where $e(a_1) = \{x_1\}$.*
- *if a is the conclusion of a ?w-link, then $e(a) = \{\emptyset\}$.*
- *if a is the conclusion of a ?co-link of arity k ($k \geq 2$), with premises $a_1, \ldots, a_k$, then $e(a) = \{x_1 \cup \ldots \cup x_k\}$, where $e(a_i) = \{x_k\}$ (for every $i \in \{1, \ldots, k\}$).*

If $p > 0$ then e has to satisfy the same conditions as in case $p = 0$. Moreover, for every box B_n with depth 0 in R and whose pal door n has conclusion c of type $!C$ and whose pax doors have conclusions $a_1, \ldots, a_m$ ($m > 0$) of type , respectively, $?A_1, \ldots, ?A_m$, let $R_n = R_{B_n}$ be the biggest sub-proof structure of R contained in B_n (a sub-proof structure of R is a subgraph of R which is also a proof structure). Let c' be the premise of the !-link n and (for every $i \in \{1, \ldots, m\}$) let a'_i be the premise of the pax link of B_n having a_i as conclusion. Clearly c' and $a'_1, \ldots, a'_m$ are the conclusions of the proof structure R_n. In order for the application e to be an experiment of R, for every such box B_n there has to exist a unique multiset $\{e_1, \ldots, e_{k_n}\}$ ($k_n > 0$) of experiments of R_n satisfying the following conditions:

- *for every edge a of S_n, $e(a) = e_1(a) \cup \ldots \cup e_{k_n}(a)$,*
- *$e(c) = \{\{x_1, \ldots, x_{k_n}\}\}$, where $e_j(c') = \{x_j\}$ ($\forall j \in \{1, \ldots, k_n\}$),*

- $\forall i \in \{1, \ldots, m\}$ *one has* $e(a_i) = \{x_1^i \cup \ldots \cup x_{k_n}^i\}$, *where* $\forall j \in \{1, \ldots, k_n\}$ *we have* $e_j(a_i') = \{x_j^i\}$.

The result of an experiment is defined in the same way as in 2.2.

Remark 1. We notice that the previous definition implies that the following conditions are fulfilled (inductively, with respect to the depth)*:*

- the label $x_1 \cup \ldots \cup x_k$ of the conclusion a of type $?C$ of a $?co$- link with depth 0 satisfies $x_1 \cup \ldots \cup x_k \in |?\mathcal{C}| = \mathcal{C}^\perp$;
- the label $\{x_1, \ldots, x_{k_n}\}$ of the conclusion a of type $!C$ of a !-link with depth 0 satisfies $\{x_1, \ldots, x_{k_n}\} \in |!\mathcal{C}| = \mathcal{C}$
- the label $x_1^i \cup \ldots \cup x_{k_n}^i$ of the conclusion a_i of type $?A_i$ of a pax link with depth 0 satisfies $x_1^i \cup \ldots \cup x_{k_n}^i \in |?\mathcal{A}_i| = \mathcal{A}_i^\perp$.

References

Coherent semantics and experiments were first defined in [Gir87]; in this presentation we refer to [Tor01].

Chapter 3

Semantical characterization of acyclicity in MLL proof nets

In this chapter we will prove that the to be a proof net for a proof structure R of MLL corresponds to the fact that the results of all experiments of R form a clique; we first prove that the result of all the experiments on a proof net are a clique; then we prove that this holds only for the proof nets, and not for any proof structure.

3.1 The interpretation of a proof net is a clique

Remark 2. Let R be a non-empty proof structure, n one of his axiom links with conclusions $A, A^{\perp}$, and, let e_1, e_2 be any two experiments of R in any atom-interpretation of MLL. Then $e_1(A) \smile e_2(A)$ if and only if $e_1(A^{\perp}) \frown e_2(A^{\perp})$.

Remark 3. In a proof net R without Par link, every terminal Tensor link or Cut link n is splitting (otherwise there would be a cycle passing trough n).

Theorem 1. *Let R be a proof structure with conclusions Γ; if R is an AC proof net, then for all e_1, e_2 experiments of R in any atom-interpretation of MLL $|e_1| \frown\!\!\!\!\smile |e_2|$*

Proof. By induction on the number of the Par links and the number of the Cut links plus the number of the Tensor link of R, lexicographically ordered (# $\parr$,# cut +# $\otimes$) :

- (0,0): in this case all the links of R are axiom links: let's reason by contradiction and suppose that $|e_1| \smile |e_2|$, that is, there is a conclusion A of R s.t. $e_1(A) \smile e_2(A)$ and for every other conclusion $B \in \Gamma$ of R, $e_1(B) \asymp e_2(B)$; but A is a conclusion of an axiom link n, and by Remark 2 there is a conclusion $A^{\perp}$ of R s.t. $e_1(A^{\perp}) \frown e_2(A^{\perp})$: contradiction.

- $(0, n+1)$: By the absence of Par links, every Cut link is splitting for R (otherwise there is a cycle in R passing trough n), and every terminal Tensor link is splitting for R (by the Remark 3); we will distinguish two cases, depending on the number of cut links in R:

  1. if $\#\text{Cut} \neq 0$: we take any cut link n, with premises A, $A^{\perp}$; n is splitting and by removing it, we get two disjoint proof structures R', R'' with conclusions respectively Γ', A and $A^{\perp}$, Γ'', with $\Gamma = \Gamma'$, Γ''; the experiment e_1 (resp. e_2) splits in two experiments e'_1, (resp. e'_2) of R' and e''_1 (resp.e''_2) of R''. R' and R'' are both AC proof nets and by induction hypothesis, $|e'_1| \overset{\frown}{\smile} |e'_2|$ and $|e''_1| \overset{\frown}{\smile} |e''_2|$. Let's look at what we get if we join R' and R'' via the link n: if $|e'_1| = |e'_2|$ and $|e''_1| = |e''_2|$ then $|e_1| = |e_2|$, by definition of experiment. Otherwise, let's suppose $|e'_1| \frown |e'_2|$ and for every $A' \in \Gamma'$, $e'_1(A') \asymp e'_2(A')$ (if there is an $A' \in \Gamma'$ s.t. $e'_1(A') \frown e'_2(A')$ we are already done); then it must be $e'_1(A) \frown e'_2(A)$, and then $e''_1(A^{\perp}) \smile e''(A^{\perp})$: so, there is an $A'' \in \Gamma''$ such that $e''_1(A'') \frown e''_2(A'')$, and then $|e_1| \frown |e_2|$.
  2. if $\#\text{Cut} = 0$: then any link in R is either an Axiom or a Tensor link. We know by Remark 3 that there is a terminal splitting Tensor link n with conclusion $A \otimes B$ and by removing it and its conclusion we get two proof structures R', with conclusions Γ', A, and R'' with conclusions B, Γ''; the experiment e_1 (resp. e_2) splits in two experiments e'_1, e''_1 (resp. e'_2, e''_2). R' and R'' are both proof nets and by induction hypothesis, $|e'_1| \overset{\frown}{\smile} |e'_2|$ and $|e''_1| \overset{\frown}{\smile} |e''_2|$. We join R' with R'' via the link n and we get back the proof net R with conclusions Γ', $A \otimes B$, Γ'' : if $|e'_1| = |e'_2|$ and $|e''_1| = |e''_2|$, then $|e_1| = |e_2|$, and we are done; if there is an $A' \in \Gamma'$ s.t. $e'_1(A') \frown e'_2(A')$ or if there is an $A'' \in \Gamma''$ s.t. $e''_1(A'') \frown e''_2(A'')$ then $|e_1| \frown |e_2|$, and we are done. Otherwise,let's suppose $e'_1(A) \frown e''_1(A)$ and $e''_1(B) \overset{\frown}{\smile} e''_2(B)$; then $e_1(A \otimes B) \frown e_2(A \otimes B)$, and $|e_1| \frown |e_2|$.

- $(n+1, k)$: let's suppose that for all conclusion $C \in \Gamma$ of R, $e_1(C) \asymp e_2(C)$. To conclude we have to prove that for all C $e_1(C) = e_2(C)$. We know by Lemma 1 that in R there exists a Splitting Par link p, with premises A and B and conclusion $A \parr B$; we remove it, in order to get two proof structures R_a, with conclusions Γ', A, B , and R_b with conclusions Γ'', in which the conclusion $A \parr B$ of p is the conclusion of an H link h (we assume $\Gamma = \Gamma'$, Γ''). It's easy to see that R_a is a proof net; in order to make R_b a proof net too, we change the link h in an Axiom link n adding to it a conclusion $A^{\perp} \otimes B^{\perp}$, which becomes a conclusion of R_b. We call e'_1 and e'_2 (resp. e''_1 and e''_2) the experiments obtained by restricting (and extending) e_1 and e_2 to R_a (R_b): R_a and R_b are both proof nets and by induction hypothesis, $|e'_1| \overset{\frown}{\smile} |e'_2|$ and $|e''_1| \overset{\frown}{\smile} |e''_2|$. Now we erase n and its conclusion $A^{\perp} \otimes B^{\perp}$, we join R_a with R_b via p and we look at the coherence of e_1 with e_2 in $A \parr B$. Remember that for every $C \in \Gamma$ we have

$e_1(C) \asymp e_2(C)$ and then for every $C \in \Gamma'$, $e'_1(C) \asymp e'_2(C)$ and for every $C \in \Gamma''$, $e''_1(C) \asymp e''_2(C)$. We have then three cases:

1. $e_1(A \parr B) \frown e_2(A \parr B)$: then $e''_1(A^\perp \otimes B^\perp) \smile e''_2(A^\perp \otimes B^\perp)$ in R_b and that's not possible by induction hypothesis;
2. $e_1(A \parr B) \smile e_2(A \parr B)$: then $e'_1(A) \asymp e'_2(A)$ and $e'_1(B) \asymp e'_2(B)$; furthermore, $e'_1(A) \smile e'_2(A)$ or $e'_1(B) \smile e'_2(B)$, and that's not possible by induction hypothesis;
3. $e_1(A \parr B) = e_2(A \parr B)$: it's the only possible case. If this holds, by induction hypothesis and the hypothesis that for all conclusion $C \in \Gamma$ of R, $e_1(C) \asymp e_2(C)$, we get for all $A' \in \Gamma'$, $e'_1(A') = e'_2(A')$ and for all $A'' \in \Gamma''$, $e''_1(A'') = e''_2(A'')$. Then for all $C \in \Gamma$, $e_1(C) = e_2(C)$.

□

3.2 The interpretation of a cyclic structure is not a clique

Remark 4. In this section, we assume that the coherent spaces we are working with, are s.t. in every coherent space S, there are at least two points of the web coherent and at least two points of the web incoherent; we call the two coherent spaces minimal (in the sense of the number of points of the web) with respect to this condition CoH and $InCoH$, defined as follows:

1. CoH: its web is composed by three points α, β, γ, with the following coherence relation: $\alpha \frown_{CoH} \beta$, $\alpha \frown_{CoH} \gamma$, $\beta \smile_{CoH} \gamma$.

2. $InCoH$: the dual of CoH.

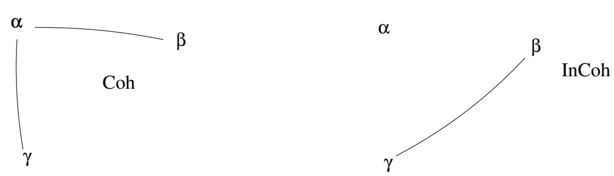


Fig.4 : the coherent spaces Coh and InCoh

Definition 16. *A* 3-interpretation *of MLL is an atom-interpretation of MLL which assigns to every atomic formulas one of the coherent spaces CoH and $InCoH$.*

Proposition 1. *Given a 3-interpretation of MLL, for the interpretation $\mathcal{A}$ of any formula A of MLL the following property holds:*

- *in $|\mathcal{A}|$ there are at least two points strictly coherent, and at least two points strictly incoherent.*

Proof. By induction on the complexity of $\mathcal{A}$:

1. if $\mathcal{A} = CoH$ or $\mathcal{A} = InCoH$ then we are done;

2. If $\mathcal{A} = \mathcal{C} \otimes \mathcal{D}$, then for coherence we take two points (x, y) and (x', y'), where x, x' are two points of $|\mathcal{C}|$ coherent (they exists by induction hypothesis on $\mathcal{C}$) and y, y' are two points of $|\mathcal{D}|$ coherent (they exists by induction hypothesis on $\mathcal{D}$), so (x, y) and (x', y') are coherent in $\mathcal{C} \otimes \mathcal{D}$; for incoherence we take two points (x, y) and (x', y'), where x, x' are any two points of $|\mathcal{C}|$ and y, y' are two points of $|\mathcal{D}|$ strictly incoherent (they exists by induction hypothesis on $\mathcal{D}$), so (x, y) and (x', y') are incoherent in $\mathcal{C} \otimes \mathcal{D}$;

3. If $\mathcal{A} = \mathcal{C} \parr \mathcal{D}$, then for coherence we take two points (x, y) and (x', y'), where x, x' are two points of $|\mathcal{C}|$ strictly coherent (they exists by induction hypothesis on $\mathcal{C}$) and y, y' are any two points of $|\mathcal{D}|$, so (x, y) and (x', y') are coherent in $\mathcal{C} \parr \mathcal{D}$; for incoherence we take two points (x, y) and (x', y'), where x, x' are two points of $|\mathcal{C}|$ incoherent (they exists by induction hypothesis on $\mathcal{C}$) and y, y' are two points of $|\mathcal{D}|$ incoherent (they exists by induction hypothesis on $\mathcal{D}$), so (x, y) and (x', y') are incoherent in $\mathcal{C} \parr \mathcal{D}$;

□

Lemma 3. *Let R be an AC proof net without Cut and Par link and connected. Then for every A, B distinct conclusions of R there exist two experiments e_1, e_2 of R in any 3-interpretation of MLL s.t $e_1(A) \smile e_2(A)$, $e_1(B) \frown e_2(B)$ and for all other conclusion $E \neq A, B$ of R, $e_1(E) \asymp e_2(E)$.*

Proof. The proof is by induction on the number n of Tensor links in R:

- if $n = 0$, then the lemma follows by Remark 2;
- Otherwise, there is a terminal Tensor link with conclusion $C \otimes D$ which is splitting for R, and it joins a proof net R_1 with conclusions Γ, C with a proof net R_2 with conclusions D, Δ . We have three cases:
    1. $A, B \in \Gamma$ ($A, B \in \Delta$): then by induction hypothesis, there exist e'_1, e'_2 experiments of R_1 s.t. $e'_1(A) \smile e'_2(A)$ and $e'_1(B) \frown e'_2(B)$, while $e'_1(C) \asymp e'_2(C)$; now if we take e''_1, e''_2 two equal experiments of R_2, from e'_1, e''_1 and e'_2, e''_2 we get two experiments e_1, e_2 for R such that $e_1(C \otimes D) \asymp e_2(C \otimes D)$ and we are done (same reasoning for $A, B \in \Delta$);

2. $A \in \Gamma$, $B = C \otimes D$ ($A \in \Delta$, $B = C \otimes D$): then by induction hypothesis for R_1 there exist two experiments e'_1, e'_2 s.t. $e'_1(A) \smile e'_2(A)$, $e'_1(C) \frown e'_2(C)$ and for every $E \in \Gamma$, $E \neq A$, C $e_1(E) \asymp e_2(E)$; now if we take e''_1, e''_2 two equal experiments of R_2, from e'_1, e''_1 and e'_2, e''_2 we get two experiments e_1, e_2 for R such that $e_1(C \otimes D) \frown e_2(C \otimes D)$ and we are done (same reasoning for $A \in \Delta$, $B = C \otimes D$);
3. $B \in \Gamma$, $A = C \otimes D$ ($B \in \Delta$, $A = C \otimes D$): then by induction hypothesis for R_1 there exist two experiments e'_1, e'_2 s.t. $e'_1(B) \frown e'_2(B)$, $e'_1(C) \smile e'_2(C)$, and for every $E \in \Gamma$, $E \neq B$, C $e_1(E) \asymp e_2(E)$; now if we take e''_1, e''_2 two equal experiments of R_2, from e'_1, e''_1 and e'_2, e''_2 we get two experiments e_1, e_2 for R such that $e_1(C \otimes D) \smile e_2(C \otimes D)$ and we are done (same reasoning for $B \in \Delta$, $A = C \otimes D$);
4. $A \in \Gamma$, $B \in \Delta$ ($B \in \Gamma$, $A \in \Delta$): then by induction hypothesis for R_1 there exist two experiments e'_1, e'_2 s.t. $e'_1(A) \smile e'_2(A)$, $e'_1(C) \frown e'_2(C)$ and for every $E \in \Gamma$, $E \neq A$, C $e_1(E) \asymp e_2(E)$, and by induction hypothesis for R_2 there exist two experiments e''_1, e''_2 s.t. $e''_1(D) \smile e''_2(D)$, $e''_1(B) \frown e''_2(B)$ and for every $E \in \Gamma$, $E \neq B$, D $e_1(E) \asymp e_2(E)$; from e'_1, e''_1 and e'_2, e''_2 we get two experiments e_1, e_2 for R s.t. $e_1(C \otimes D) \smile e_2(C \otimes D)$ and we conclude (same reasoning for $B \in \Gamma$, $A \in \Delta$).

□

Remark 5. In the proof of Lemma 3, our choice of CoH and $InCoH$ as the coherent spaces interpreting the atomic formulas, is exactly what make the proof work; if we had chose coherent spaces smaller, (for example of two points), the theorem would be trivially false.

To see it, is sufficient to take a proof net consisting only of an Axiom link whose edges $A, A^{\perp}$ are of atomic type; if one interprets A, $A^{\perp}$ with coherent spaces of two points, which are incoherent in $\mathcal{A}$, coherent in $\mathcal{A}^{\perp}$, the Lemma doesn't hold; to make it work, we have to consider coherent spaces with at least three points, two points incoherent and two points coherent, i.e. CoH and $InCoh$.

Definition 17. *Let R be a proof structure, and n be a link of R. If n is different from an Axiom link or an H-link, and if the unique conclusion of n is a premise of the link n' of R, we say that n' is the* successor of n.

Definition 18. *Let R be a proof structure and a, a' be two edges of R. We say that there is a* direct path from a to a' *when there is a path from a to a' following the orientation of R (if it exists, this path is unique by definition of proof structure).*

Remark 6. Given a proof structure R with conclusions Γ and without Cut links, and given an edge a of R, there is a unique $C \in \Gamma$ s.t there exist a direct path from a to C, and we call it *the conclusion of R reachable from a.*

Definition 19. *Let R be a cut-free proof structure with conclusions Γ, a be an edge of R, and $C \in \Gamma$ the conclusion of R reachable from a; the* cluster of links associated with a *is the set of links of R belonging to the direct path from a to C.*

Definition 20. *Let R be a cut-free proof structure which contains a non-terminal Par link n with conclusion $A \parr B$; we define the proof structure $R^{\parr}$ from R as follows:*

1. *select one of the premises of n: call it a_{sel};*
2. *erase the link n with its conclusion, and join a_{sel} with the successor n' of n, if it exists (otherwise a_{sel} is a conclusion of $R^{\parr}$);*
3. *replace every edge d of type D of the direct path from $A \parr B$ to the conclusion of R reachable from a_{sel} by the same edge but now with type $D[A/A \parr B]$ (resp. $D[B/A \parr B]$)*
4. *the premise not chosen at point 1 is a new conclusion of $R^{\parr}$.*

We call such an operation $\parr$-opening of n in R *(the inverse operation,which allows to get back from $R^{\parr}$ to R, is called* $\parr$-closing of n in R*).*

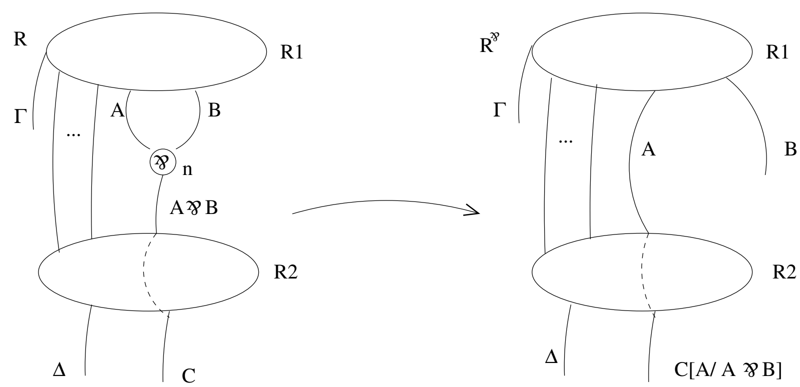


Fig.3: ⅋ –opening of n in R

Remark 7. Notice that:

1. If R is an AC proof net (resp. a proof net) then $R^{\parr}$ is an AC proof net (resp. a proof net)
2. If R is not an AC proof net (resp. a proof net) then it is always possible to choose the edge a_{sel} of the previous construction in such a way that $R^{\parr}$ is not an AC proof net (resp. a proof net) too.

Theorem 2. *Let R be a proof structure with conclusions Γ and without Cut links; for all e_1, e_2 experiments of R in any 3-interpretation of MLL, if $|e_1| \overset{\frown}{\smile} |e_2|$, then R is an AC proof net.*

Proof. The proof is by contraposition, i.e., we prove that if R is not an AC proof net, then there exist two experiments $e_1,\ e_2$ of R s.t. $|e_1| \smile |e_2|$. We reason by induction on the number of the Par links and number of the Tensor link in R, lexicographically ordered ($\# \parr, \# \otimes$):

- (0, 0) If there are only Axiom links, R is correct;
- (0, $n+1$) We have two cases:

  1. there exists a terminal Tensor link n with conclusion $A \otimes B$ such that its removal yields a proof structure R_1 with conclusions Γ, A , B, which is still incorrect; then by induction hypothesis, there exist two experiments e_1, e_2 of R_1 such that $|e_1| \smile |e_2|$, that is there exists a conclusion C s.t. $e_1(C) \smile e_2(C)$, and for every conclusion D of R_1, $e_1(D) \asymp e_2(D)$. If $e_1(A) \asymp e_2(A)$ and $e_1(B) \asymp e_2(B)$ then joining A with B via n yields $e_1(A \otimes B) \asymp e_2(A \otimes B)$; if $e_1(A) \smile e_2(A)$ or $e_1(B) \smile e_2(B)$, then joining A with B via n yields $e_1(A \otimes B) \smile e_2(A \otimes B)$;
  2. otherwise, we choose one terminal Tensor link n with conclusion $A \otimes B$ such that by removing it we get a proof structure R_1 of conclusions Γ, A, B, which is acyclic; the conclusions A and B have to be in the same connected component R_1' of R_1 (otherwise there would not be a cycle passing trough n); but then R_1' is an AC proof net, connected and without Cut and Par links, and by Lemma 3 there exist two experiments e_1, e_2 s.t. $e_1(A) \smile e_2(A)$, $e_1(B) \frown e_2(B)$ and for all other conclusions D of R_1', $e_1(D) \asymp e_2(D)$; then adding the link n the same two experiments give $e_1(A \otimes B) \smile e_2(A \otimes B)$;

- ($n+1$, k): We select a Par link n of type $A \parr B$ such that n is a terminal link, or has below it only Tensor links (obviously such a link necessarily exists).

  – If n is a terminal link, then we erase n with its conclusion, and we get a new proof structure R' with conclusions Γ, A, B; R' is incorrect too (erasing n doesn't eliminate any cycle) then we can apply the induction hypothesis; so there are two experiments e_1, e_2 of R' s.t. $|e_1| \smile |e_2|$, that is there exists a conclusion C of R' s.t. $e_1(C) \smile e_2(C)$, and for every conclusions D of R', $e_1(D) \asymp e_2(D)$. If $e_1(A) \smile e_2(A)$ or $e_1(B) \smile e_2(B)$ then if we join A with B via n the same two experiments give $e_1(A \parr B) \smile e_2(A \parr B)$ in R, and we are done; if $e_1(A) \asymp e_2(A)$ and $e_1(B) \asymp e_2(B)$ then there exist one $C \in \Gamma$ s.t. $e_1(C) \smile e_2(C)$ in R' and so in R too while the same two experiments give $e_1(A \parr B) \asymp e_2(A \parr B)$ in R, and we are done.

– Otherwise, we $\parr$-open n, selecting the premise of n in such a way that $R^{\parr}$ is still incorrect (following Remark 7). Suppose for example that the selected premise is A; we get a proof structure $R^{\parr}$ of conclusions Γ, B, which is incorrect too. We are in condition to apply the induction hypothesis, so there exist two experiments e_1, e_2 of $R^{\parr}$ s.t $|e_1| \smile |e_2|$, that is there exists a conclusion C of $R^{\parr}$ s.t. $e_1(C) \smile e_2(C)$, and for every conclusions D of $R^{\parr}$, $e_1(D) \asymp e_2(D)$. Now to conclude we have to prove that, when we $\parr$-close n, e_1 and e_2 are the experiments of R we are searching for; first notice that $\parr$-closing n has the only effect of (possibly) changing the coherence relation on the edges of the direct path from $A \parr B$ to the conclusion of R reachable from $A \parr B$: everywhere else nothing changes in R w.r.t. $R^{\parr}$. Then we have three cases, depending on the coherence relation of e_1 and e_2 in A:

  1. $e_1(A) \smile e_2(A)$: then in the conclusion C reachable from A in $R^{\parr}$, $e_1(C) \smile e_2(C)$, because in the cluster of links associated with A there are only Tensor links; then $\parr$-closing n we get $e_1(A \parr B) \smile e_2(A \parr B)$ and for the same reason $e_1(C) \smile e_2(C)$ in R, and we are done;
  2. $e_1(A) \frown e_2(A)$: then in the conclusion C reachable from A in $R^{\parr}$, $e_1(C) \smile e_2(C)$, because $e_1(A) \neq e_2(A)$; so there is necessarily a premise d of a Tensor link in the direct path from A to C s.t. $e_1(d) \smile e_2(d)$; so if we $\parr$-close n we get $e_1(A \parr B) \frown e_2(A \parr B)$ and for the same reason (i.e. thanks to the coherence relation on d), $e_1(C) \smile e_2(C)$ in R, and we are done;
  3. $e_1(A) = e_2(A)$: then in the conclusion C reachable from A in $R^{\parr}$, $e_1(C) \asymp e_2(C)$; here we have to look at the coherence relation of e_1 and e_2 in B, and there are two subcases:
     (a) $e_1(B) = e_2(B)$: then if we $\parr$-close n we get $e_1(A \parr B) = e_2(A \parr B)$, and this doesn't modify the coherence relation of e_1 and e_2 in C nor in any other conclusion, which means that there exists a conclusion D of $R^{\parr}$ and of R s.t. $e_1(D) \smile e_2(D)$.;
     (b) $e_1(B) \smile e_2(B)$: then if we $\parr$-close n we get $e_1(A \parr B) \smile e_2(A \parr B)$, and this forces $e_1(C) \smile e_2(C)$ because in the cluster of link associated with $A \parr B$ there are only Tensor links.

□

3.3 Comparison with Retore's proof

In [Ret97] Christian Retore gives a proof of Theorem 2 that is very similar to this, but which differs for some features. Essentially, Retore proves the theorem

Fig.6 : the two dual coherent spaces N and Z

in such a way that one can extract an algorithm out of it, which decides whether a proof structure is a proof net or not. In order to do that, he does the following things:

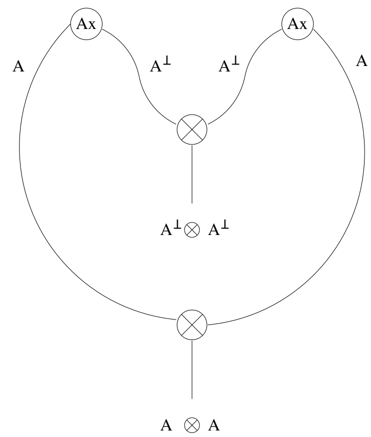
- he chooses a different kind of coherent spaces, that he calls N and Z, to interpret atomic formulas; the peculiarity of this two dual coherent spaces is that every point of the web is coherent with exactly one point and incoherent with exactly another point.

- given this interpretation (that he calls NZ-interpretation), he states an analogue of Lemma 3: given any experiment e_1 of a proof net R, possibly containing ⅋-links, there exists another experiment e_2 of R such that for every A, B conclusions of R, s.t. there is a path from A to B in a correction graph of R, $e_1(A) \frown e_2(A)$, $e_1(B) \smile e_2(B)$ and for all other conclusions C of R, $e_1(C) \asymp e_2(C)$.

  What we want to stress here is that the lemma, formulated this way, doesn't hold in our setting due to the different choice of atom-interpretation we made: in fact in CoH and $InCoH$ there is a point α which is respectively coherent with all the others in CoH, and incoherent with all the others in $InCoH$; so if we take a proof net consisting in only an axiom link with conclusions A, interpreted by $InCoH$ and $A^{\perp}$, interpreted by CoH, if $e_1(A) = \alpha$ then the lemma doesn't hold.

- then he uses this lemma to prove the theorem using the same kind of formulation; the proof is by contraposition, as in our case, so what he actually proves is that, if R is a proof structure which is not a proof net, given any experiment e_1 of R in any NZ-interpretation, there exists another experiment e_2 of R s.t. $|e_1| \smile |e_2|$.

  The theorem too, formulated in this way, don't holds in our setting, due to a simple counter example: let's take a cyclic proof structure like this one:



Given this structure, if we interpret $A^{\perp}$ with $InCoH$, and we fix an experiment e_1 such that $e_1(A^{\perp}) = \alpha$ for both the instances of $A^{\perp}$, then there isn't any other experiment e_2 s.t. $e_1(A^{\perp} \otimes A^{\perp}) \smile e_2(A^{\perp} \otimes A^{\perp})$ *and* $e_1(A \otimes A) \smile e_2(A \otimes A)$. This proves that the setting of Retore is the best possible (in terms of number and coherence of the points of the coherent spaces interpreting the atomic formulas) in order to prove the theorem in the way he proves it.

Proving the theorem using Retore's method, provides a simple algorithm to verify the correctness of a proof structure R: it simply consists of :

1. Choose an arbitrary NZ-interpretation
2. Choose an experiment e_1 of R
3. Check that any other experiment e_2 of R satisfies $|e_1| \frown |e_2|$ (the number of possible e_2 is finite, because N and Z are finite).

This algorithm, although simple, is not very efficient; if the proof structure has N Axiom links, one must verify 4^N possible experiments, making its complexity altogether exponential.

References

A different, stronger version of Theorem 1 was first proved in [Gir87]; for the proof of Theorem 2 we have been inspired from [Ret97].

Chapter 4

About the extension to MELL

In this chapter first we will state the extension of theorem 1 to MELL; then we show that the extension of Theorem 2 is not straightforward.

4.1 The interpretation of a proof net of MELL is a clique

Remark 8. In a proof net R without Par link and $?co$-link at depth 0, every terminal Tensor link or Cut link at depth 0 n is splitting (otherwise there would be a cycle in some correction graph passing trough n).

Proposition 2. *Let R be proof structure of depth 0, and $e_1,\ e_2$ any two experiments of R in any atom-interpretation of MELL. If for all the Axiom links n of R with conclusions $a,\ a'$, $e_1(a) = e_2(a) = e_1(a') = e_2(a')$, then $|e_1| = |e_2|$.*

Proof. Trivial, due to the fact that, at depth 0, once a value for the conclusions of the axiom links is given, building an experiment is a deterministic operation. □

Remark 9. The $?de$-link preserves the coherence relation between the values assigned by any two experiments on its premise: if R is a proof structure, e', e'' any two experiments of R, a any edge of R premise of a $?de$-link with conclusion c, if $e'(a) \frown e''(a)$ (resp. $e'(a) \smile e''(a),\ e'(a) = e''(a)$), then $e'(c) \frown e''(c)$ (resp. $e'(c) \smile e''(c),\ e'(c) = e''(c)$).

Theorem 3. *Let R be a proof structure of MELL with conclusions Γ; if R is an AC proof net then for all e_1, e_2 experiments of R in any atom-interpretation of MELL, $|e_1| \stackrel{\frown}{\smile} |e_2|$.*

Proof. By induction on the depth p of R, the number of the Par links plus the number of the $?co$-links at depth 0 and the number of the Cut links plus the number of the Tensor links at depth 0 of R, lexicographically ordered (p, $\# \parr + \#?co$, $\#\mathrm{cut} + \#\otimes$):

- (0,0,0) : let's suppose that for all Axiom links n of conclusions a, a', $e_1(a) = e_2(a)$ and $e_1(a') = e_2(a')$; then by Proposition 2 and Remark 9 we know that $|e_1| = |e_2|$; otherwise for some n, $e_1(a) \frown e_2(a)$ or $e_1(a) \smile e_2(a)$. In the first case, by Remark 9, there is a $C \in \Gamma$ s.t. $e_1(C) \frown e_2(C)$, and we are done; in the second case, by Remark 3 $e_1(a') \frown e_2(a')$ and we conclude as in the previous case; in any case the presence of any $?we$-link does not influence the coherence of $|e_1|$ and $|e_2|$.

- (p,0, $n+1$) : by the absence of any Par link or co-link at depth 0 in R we know by Remark 8 that every terminal Tensor link and every Cut link is splitting for R; we will distinguish two cases, depending on the number of Cut links at depth 0 in R:

  1. if $\#\mathrm{Cut} \neq 0$: we take any cut link n, with premises A, $A^{\perp}$; n is splitting and by removing it, we get two disjoint proof structures R', R'' with conclusions respectively Γ', A and $A^{\perp}$, Γ'', with $\Gamma = \Gamma'$, Γ''; the experiment e_1 (resp. e_2) splits in two experiments e_1', (resp. e_2') of R' and e_1'' (resp. e_2'') of R''. R' and R'' are both AC proof nets and by induction hypothesis, $|e_1'| \stackrel{\frown}{\smile} |e_2'|$ and $|e_1''| \stackrel{\frown}{\smile} |e_2''|$. Let's look at what we get if we join R' and R'' via the link n: if $|e_1'| = |e_2'|$ and $|e_1''| = |e_2''|$ then $|e_1| = |e_2|$, by definition of experiment. Otherwise, let's suppose $|e_1'| \frown |e_2'|$ and for every $A' \in \Gamma'$, $e_1'(A') \asymp e_2'(A')$ (if there is an $A' \in \Gamma'$ s.t. $e_1'(A') \frown e_2'(A')$ we are already done); then it must be $e_1'(A) \frown e_2'(A)$,and via the Cut link n, $e_1''(A^{\perp}) \smile e''(A^{\perp})$: so, there is an $A'' \in \Gamma''$ such that $e_1''(A'') \frown e_2''(A'')$, and then $|e_1| \frown |e_2|$.
  2. if $\#\mathrm{Cut} = 0$: if there is a terminal Tensor link n with conclusion $A \otimes B$, then it is splitting by Remark 8 and removing it and its conclusion we get two proof structures R', with conclusions Γ', A, and R'' with conclusions B, Γ''; the experiment e_1 (resp. e_2) splits in two experiments e_1', (resp. e_2') of R' and e_1'' (resp. e_2'') of R''. R' and R'' are both proof nets and by induction hypothesis, $|e_1'| \stackrel{\frown}{\smile} |e_2'|$ and $|e_1''| \stackrel{\frown}{\smile} |e_2''|$. We join R' with R'' via the link n and we get back the proof net R with conclusions Γ', $A \otimes B$, Γ'': if $|e_1'| = |e_2'|$ and $|e_1''| = |e_2''|$, then $|e_1| = |e_2|$, and we are done; if there is an $A' \in \Gamma'$ s.t. $e_1'(A') \frown e_2'(A')$ or if there is an $A'' \in \Gamma''$ s.t. $e_1''(A'') \frown e_2''(A'')$ then $|e_1| \frown |e_2|$, and we are done. Otherwise, let's suppose $e_1'(A) \frown e_1''(A)$ and $e_1''(B) \stackrel{\frown}{\smile} e_2''(B)$; then $e_1(A \otimes B) \frown e_2(A \otimes B)$, and $|e_1| \frown |e_2|$. If there is not any Terminal tensor link, then there is an edge a conclusion of a Tensor link at depth 0 of R s.t. every link below a is a $?de$-links; then we consider the proof net R^* obtained erasing all

the $?de$-links below a in R; for R^* $|e_1| \frown |e_2|$ and then by Remark 9, this holds for R too.

- $(p, m+1, 0)$: Let's suppose that for all conclusion c of type $C \in \Gamma$ of R, $e_1(c) \asymp e_2(c)$. To conclude we have to prove that for all c $e_1(c) = e_2(c)$. We know by Lemma 2 that in R there exists a splitting Par link p, or there exists a splitting $?co$-link n; so we have two cases:

  1. if there exists a splitting Par link p , with premises a of type A and b of type B and conclusion d of type $A \parr B$ we remove it, in order to get two proof structures R_a, with conclusions Γ', a of type A, b of type B, and R_b with conclusions Γ'', in which the conclusion d of type $A \parr B$ of p is the conclusion of an H link h (we assume $\Gamma = \Gamma',\ \Gamma''$). It's easy to see that R_a is a proof net; in order to make R_b a proof net too, we change the link h in an Axiom link n adding to it a conclusion f of type $A^\perp \otimes B^\perp$, which becomes a conclusion of R_b. We call e'_1 and e'_2 (resp. e''_1 and e''_2) the experiments obtained by restricting (and extending) e_1 and e_2 to R_a (R_b): R_a and R_b are both proof nets and by induction hypothesis, $|e'_1| \frown |e'_2|$ and $|e''_1| \frown |e''_2|$. Now we erase n and its conclusion f of type $A^\perp \otimes B^\perp$, we join R_a with R_b via p and we look at the coherence of e_1 with e_2 in d of type $A \parr B$. Remember that for every $c \in \Gamma$ we have $e_1(c) \asymp e_2(c)$ and then for every $c' \in \Gamma'$, $e'_1(c) \asymp e'_2(c)$ and for every $c'' \in \Gamma''$, $e''_1(c'') \asymp e''_2(c'')$. We have then three cases:

     (a) $e_1(d) \frown e_2(d)$: then $e''_1(f) \smile e''_2(f)$ in R_b and that's not possible by induction hypothesis;

     (b) $e_1(d) \smile e_2(d)$: then $e'_1(a) \asymp e'_2(a)$ and $e'_1(b) \asymp e'_2(b)$; furthermore, $e'_1(a) \smile e'_2(a)$ or $e'_1(b) \smile e'_2(b)$, and that's not possible by induction hypothesis, applied to R_a;

     (c) $e_1(d) = e_2(d)$: it's the only possible case. If this holds, by the hypothesis that for all conclusion c of type $C \in \Gamma$ of R, $e_1(c) \asymp e_2(c)$, applying the induction hypothesis to R', we get for all a' of type $A' \in \Gamma'$, $e'_1(a') = e'_2(a')$ and applying the induction hypothesis to R'' for all a'' of type $A'' \in \Gamma''$, $e''_1(a'') = e''_2(a'')$. Then for all c of type $C \in \Gamma$, $e_1(c) = e_2(c)$.

  2. if there exists a splitting $?co$-link n with premises $?A_1, \ldots, ?A_n$ and conclusion $?A$ we remove it, in order to get two proof structures R_a, with conclusions Γ' and $a_1, \ldots, a_n$ of type $?A_1, \ldots, ?A_n$, and R_b with conclusions Γ'' ,in which the conclusion a of type $?A$ of n is the conclusion of an H link h. It's easy to see that R_a is a proof net; in order to make R_b a proof net too, we change the link h in an Axiom link k adding to it a conclusion b of type $!A^\perp$, which becomes a conclusion of R_b. We call e'_1 and e'_2 (resp. e''_1 and e''_2) the experiments obtained by restricting (and extending) e_1 and e_2 to R_a (R_b): R_a and R_b are both proof nets and by induction hypothesis, $|e'_1| \frown |e'_2|$

and $|e''_1| \frown\!\!\!\!\smile |e''_2|$. Now we erase k and its conclusion b of type $!A^\perp$, we join R_a with R_b via n and we look at the coherence of e_1 with e_2 in a of type $?A$. Remember that for every $c \in \Gamma$, we have $e_1(c) \asymp e_2(c)$ and then for every $c' \in \Gamma'$, $e'_1(c') \asymp e'_2(c')$ and for every $c'' \in \Gamma''$, $e''_1(c'') \asymp e''_2(c'')$. We have then three cases:

(a) $e_1(a) \frown e_2(a)$: then $e''_1(b) \smile e''_2(b)$ in R_b and that's not possible by induction hypothesis;

(b) $e_1(a) \smile e_2(a)$: then there exist an $l \in \{1, \dots, n\}$ s.t. $e'_1(a_l) \smile e'_2(a_l)$ while for all $k \in \{1, \dots, n\}$ $e'_1(a_k) \asymp e'_2(a_k)$ in R_a , and that's not possible by induction hypothesis.

(c) $e_1(a) = e_2(a)$: it's the only possible case. If this holds, by the hypothesis that for all conclusion c of type $C \in \Gamma$ of R, $e_1(c) \asymp e_2(c)$, applying the induction hypothesis to R' we get for all a' of type $A' \in \Gamma'$, $e'_1(a') = e'_2(a')$ and applying the induction hypothesis to R'' for all a'' of type $A'' \in \Gamma''$, $e''_1(a'') = e''_2(a'')$. Then for all c of type $C \in \Gamma$, $e_1(c) = e_2(c)$.

- $(p, 0, 0)$: In this case suppose that there is some box B of R of depth 0 s.t. its conclusions $p_1, \dots, p_n$ of type respectively $?P_1, \dots, ?P_n$ and c of type $!C$ are conclusions of R (by simplicity's sake, we say that they are all the conclusions of R; once proven in this case the case in which there are also other conclusions is trivial); we call R' the maximal proof net contained in B, with conclusions the premises $p'_1, \dots, p'_n$ of the auxiliary doors with conclusions $p_1, \dots, p_n$ of B, and the premise c' of the principal door with conclusion c of B; we know by induction hypothesis that for every e', e'' experiments of R', $|e'| \frown\!\!\!\!\smile |e''|$. Let e_1, e_2 be any two experiments of R with result γ_1, (resp.γ_2) $\in$ $|!\mathcal{C} \parr ?\mathcal{P}_1 \parr \dots \parr ?\mathcal{P}_n|$; we want to prove that, if e_1, e_2, $|e_1| \asymp |e_2|$, then $|e_1| = |e_2|$. If $|e_1| \asymp |e_2|$ this means that $\forall i \in \{1, \dots, n\}$, $e_1(p_i) \asymp e_2(p_i)$ and $e_1(c) \asymp e_2(c)$. Let's suppose that $e_1(c) = \emptyset$: then $e_2(c) = \emptyset$; otherwise, by the fact that $e_1(c) \smile e_2(c)$, $e_1(c) \cup e_2(c)$ would not be a clique; but that's not possible, for $e_2(c)$ is a clique. Then if one among $e_1(c)$, $e_2(c)$ is empty then they are both empty, but then $|e_1| = |e_2|$. Now suppose that $e_1(c)$, $e_2(c)$ are both non-empty: we know then that $\forall i$, $e_1(p_i) = \{e^1_1(p'_i) \cup \dots \cup e^{k_1}_1(p'_i)\}$ and $e_2(p_i) = \{e^1_2(p'_i) \cup \dots \cup e^{k_2}_2(p'_i)\}$, where for all $l \in \{1, \dots, k_1\}$, and for all $m \in \{1, \dots, k_2\}$ e^l_1 and e^m_2 are experiments of R'; if $e_1(p_i) \asymp e_2(p_i)$ this means that $e_1(p_i) \cup e_2(p_i)$ is a clique of $\mathcal{P}^\perp_i$, and by the fact that every subset of a clique is a clique too, $\forall i \in \{1, \dots, n\}$, $\forall l \in \{1, \dots, k_1\}$, $\forall m \in \{1, \dots, k_2\}$ $e^l_1(p'_i) \asymp e^m_2(p'_i)$. Now we have two cases, depending on the coherence relation of e_1 and e_2 in c:

  1. $e_1(c) \smile e_2(c)$: in this case $e_1(c) = \{x_1, \dots, x_{k_1}\}$, where for $t \in \{1, \dots, k_1\}$, $e^t_1(c') = \{x_t\}$, and $e_2(c) = \{y_1, \dots, y_{k_2}\}$, where for $s \in \{1, \dots, k_2\}$ $e^s_2 = \{y_s\}$, and $e_1(c) \cup e_2(c)$ is not a clique of $\mathcal{C}$; then there are some e^t_1, e^s_2 s.t. $e^t_1(c') \smile e^s_2(c')$ while for all $l \in \{1, \dots, k_1\}$, for all $m \in \{1, \dots, k_2\}$ and for all $i \in \{1, \dots, n\}$, $e^l_1(p_i) \asymp e^m_2(p_i)$;

this imply $|e_1^t| \smile |e_2^s|$ in R' and that's not possible by induction hypothesis: contradiction.

2. $e_1(c) = e_2(c)$: this is the only possible case. If this holds, $e_1(c) = \{x_1, \ldots, x_{k_1}\}$, $e_2(c) = \{y_1, \ldots, y_{k_2}\}$, and $k_1 = k_2 = k$; we reorder $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ in such a way that $x_1 = y_1, \ldots, x_k = y_k$; if $e_1^1(c') = x_1, \ldots, e_1^k(c') = x_k$ and $e_2^1(c') = y_1, \ldots, e_2^k(c') = y_k$ then $\forall i \in \{1, \ldots, k\}$, $e_1^i(c') = e_2^i(c')$. We already know that for all the $p'_1, \ldots, p'_n$ premises of the auxiliary doors of R, for all $j \in \{1, \ldots, n\}$, $e_1^i(p'_j) \asymp e_2^i(p'_j)$, and by induction hypothesis that $|e_1^i| \stackrel{\frown}{-} |e_2^i|$; by this, and the fact that, $\forall i$, $e_1^i(c') = e_2^i(c')$, we can say that, $\forall j$, $e_1^i(p'_j) = e_2^i(p'_j)$; this implies that for all $p_1, \ldots, p_n$ conclusions of the auxiliary doors of R, $\forall j \in \{1, \ldots, n\}$, $e_1(p_j) = e_2(p_j)$, and then $|e_1| = |e_2|$.

If there is not a box at depth 0 whose conclusions are all conclusions of the proof net, then there must be some cluster of $?de$-links below some of the conclusions of the box; in this case we erase the clusters, getting a proof net R^* s.t. $|e_1| = |e_2|$; by Remark 9 this holds for R too. In case there are several boxes of depth 0, we argue in the same way for each of them.

□

4.2 Is it the case that the interpretation of a cyclic structure of MELL is not a clique?

We just stated and proved the extension of Theorem 1 to MELL, and so it would be reasonable to imagine that Theorem 2 has a straightforward extension to MELL; unfortunately, that's not the case, and now we will briefly show why.

The proof of Theorem 2 is a proof by contraposition (i.e. given a cyclic structure R we search for two experiments on R whose results are incoherent) and by induction (on the depth of R); to extend it to MELL, in the induction step, we have to deal with the case of the presence of some $?co$-link at depth 0.

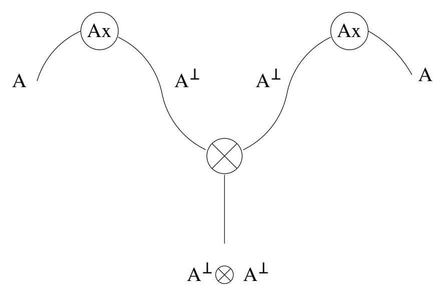
The most natural way to deal with this case, is to argue as in the case of the Par link, designing a procedure of $?co$-opening of a $?co$-link, in any way similar to that of ⅋-opening, state the induction hypothesis for the opened structure R', and then prove that $?co$-closing R' preserves the induction hypothesis.

Here we have the first problem: the induction hypothesis for R' tell us that there are two experiments of R' whose results are incoherent; but when we $?co$-close R', *not necessarily all experiments of R' become experiments of R*; in fact in the case of the $?co$-link, if $\{x_1\}, \ldots, \{x_n\}$ are the labels assigned by an experiment e to the premises $?A_1, \ldots, ?A_n$ of a $?co$-link of R, e is an experiment of R iff the label $\{x_1 \cup \ldots \cup x_n\}$ assigned by e to the conclusion $?A$ of the $?co$-link is a clique of $\mathcal{A}^{\perp}$; and that's not the case for all experiments of R'; so the two incoherent experiment of R' we get from the induction hypothesis may not be experiments of R too.

The problem can be partially solved if, instead of any two experiments, we search for two experiments *simple* (see [Tor01]): an experiment e is simple when it is obsessional (see [Tor01]) and $\forall a,\ a'$ edges of the same atomic type of R such that $a \neq a'$, the unique element of $e(a)$ is equal to the unique element of $e(a')$.

In this case, there is a lemma, proven in [Tor01], which says that $\forall a, b$ edges of type $?A$ of R, if e is a simple experiment, then $e(a) \asymp e(b)$; using this lemma in our proof, we can say that every simple experiment of R' is an experiment of R too, and the case of the $?co$-link can be handled straightforwardly.

But, if we want modify our proof in order to search only for simple experiments, we have to modify consequently all the lemmas we use in the proof, in particular Lemma 3, which has to be restricted only to the case of simple experiments; unfortunately (again!), this cannot be done, because the restriction of Lemma 3 to simple experiments is trivially false: as a very simple counterexample one can consider the following proof net, interpreting A with $InCoH$:



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We then showed that it's not possible extend our theorem in a direct way; this don't close the door to *other* ways of extending it, (very probably it can be extended), but the extension is not an immediate consequence of the result for the multiplicative fragment.

References

An alternative proof of Theorem 3 is given in [Bar01]; a complete extension of Theorem 2 to MELL for pure proof nets is given in [DuqVdW].

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