An Algorithmic Characterisation of the Classification Theorem*

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Abstract. We consider the proof for the classification theorem of topological surfaces proposed by Massey in [1]. We arrange the proof at issue (which stresses the standard word-based treatment of surfaces) into a formal rewriting system $\mathcal{R}$. Moreover, we study the computational properties of two variants of $\mathcal{R}$: $\mathcal{R}_{\mathrm{or}}$, for dealing with words denoting orientable surfaces, and $\mathcal{R}_{\mathrm{nor}}$, for dealing with words denoting non-orientable surfaces. We show how such properties induce a proof, alternative to that one due to Massey, able to provide an algorithmic characterisation of the classification theorem.

1 Introduction

By following the standard terminology, we use the term *surfaces* for indicating compact and connected 2-manifolds. It is a well-known achievement in algebraic topology that, when surfaces are considered modulo homeomorphisms, their geometrical information can be encoded by a specific class of finite words. Such a word-based approach finds its most important theoretical application in the proof of the classification theorem. The classification theorem establishes that any surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a finite connected sum of projective planes; the sphere and the connected sum of tori are orientable, whereas the conncted sum of projective planes is non-orientable [1].

In the present work, we constantly refer to the proof of the classification proposed by Massey in [1]; the demonstration consists in:

- showing that any surface is homeomorphic to the connected sum of a finite number of tori and projective planes,
- establishing the basic homeomorphism between the connected sum of a torus with a projective plane and the connected sum of three projective planes.

As a first contribution, we arrange the word-based approach stressed by Massey into a formal rewriting system $\mathcal{R}$: the basic idea is that the process of normalisation in $\mathcal{R}$ represents the process of forming the quotient surface

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associate with a certain polygon through identification of paired edges. We investigate the computational properties of $\mathcal{R}$ and we "formalise" the Massey's proof within it.

As a second contribution, we introduce two variants of $\mathcal{R}$: $\mathcal{R}_{\text{or}}$, for dealing with words denoting orientable surfaces, and $\mathcal{R}_{\text{nor}}$, for dealing with words denoting non-orientable surfaces. We single out the computational properties of both $\mathcal{R}_{\text{or}}$ and $\mathcal{R}_{\text{nor}}$ and we deduce from them an alternative proof for the classification in which the basic homeomorphism used by Massey is not required. In particular, we show that:

- in $\mathcal{R}_{\text{or}}$, any word denoting an orientable surface can be transformed into the empty one by exclusively using the *torus* rule which geometrically corresponds to the formation of a torus (together with some other topologically neutral transformations);
- in $\mathcal{R}_{\text{nor}}$, any word denoting a non-orientable surface can be transformed into the empty one by exclusively using the *projective plane* rule which geometrically corresponds to the formation of a projective plane (together with some other topologically neutral transformations).

Such a proof constitutes an attempt to provide an algorithmic characterisation of the classification concerning the dynamical process that – through identification of paired edges – allows to pass from polygons to quotient surfaces.

2 The Systems $\mathcal{R}$, $\mathcal{R}_{\text{or}}$ and $\mathcal{R}_{\text{nor}}$

2.1 Basic Definitions

Consider an alphabet $\mathcal{A} \cup \bar{\mathcal{A}}$, where $\mathcal{A} = \{a, b, c, \ldots\}$ and $\bar{\mathcal{A}} = \{\bar{a}, \bar{b}, \bar{c}, \ldots\}$; the bar operation is an involution without fixed point: for any $x \in \mathcal{A} \cup \bar{\mathcal{A}}$, $\bar{\bar{x}} = x$ and $x \neq \bar{x}$. Finite words from $\mathcal{A} \cup \bar{\mathcal{A}}$ are indicated with small Greek letters; in particular, ϵ denotes the empty word. $|\alpha|$ is the multiset of letters occurring in α. The concatenation of two words is simply indicated by juxtaposing them: $\alpha \star \beta = \alpha\beta$. Words included between round brackets have to be considered modulo circular permutation of their letters: $(\alpha\beta) = (\beta\alpha)$; such words are called *cycles.*

Definition 1 (TW, TC). *The set* $\mathtt{TW}$ *of topological words is the smallest set containing* ϵ *and such that: if* $\alpha\beta\gamma \in \mathtt{TW}$*,* $x \in \mathcal{A} \cup \bar{\mathcal{A}}$ *and* $x \notin |\alpha\beta\gamma|$*, then* $\alpha x \beta \bar{x} \gamma \in \mathtt{TW}$ *and* $\alpha x \beta x \gamma \in \mathtt{TW}$*. A cycle* (α) *is topological,* $(\alpha) \in \mathtt{TC}$*, if* $\alpha \in \mathtt{TW}$*.*

Example 1. $(\bar{a}\bar{b}cabc) \in \mathtt{TC}$; whereas $(\bar{a}\bar{b}cab), (\bar{a}\bar{b}cabcb) \notin \mathtt{TC}$.

Definition 2 (OTW and NTW, OTC and NTC). *The set* $\mathtt{OTW}$ *of orientable topological words, gathers all the* $\alpha \in \mathtt{TW}$ *such that: if* $x \in |\alpha|$*, then* $\bar{x} \in |\alpha|$*.* $\mathtt{NTW}$*, the set of non-orientable topological words, is the complement of* $\mathtt{OTW}$*, i.e.* $\mathtt{NTW} = \mathtt{TW} \backslash \mathtt{OTW}$*. A cycle* (α) *is orientable,* $(\alpha) \in \mathtt{OTC}$*, (resp. non-orientable,* $(\alpha) \in \mathtt{NTC}$*) if* $\alpha \in \mathtt{OTW}$ *(resp.* $\alpha \in \mathtt{NTW}$*).*

Example 2. $(\epsilon), (a\bar{a}b\bar{b}) \in \mathtt{OTC}$; whereas $(aab\bar{b}), (aabb) \in \mathtt{NTC}$.

Definition 3 ($\mathcal{R}$, $\mathcal{R}_{\mathbf{or}}$, $\mathcal{R}_{\mathbf{nor}}$). *Consider the following four transformations on topological cycles:*

- contraction: $(\alpha x\bar{x}) \rightarrow_{\mathtt{cont}} (\alpha)$
- permutation: $(\alpha x\beta\gamma\bar{x}) \rightarrow_{\mathtt{perm}} (\alpha x\gamma\beta\bar{x})$
- torus*:* $(x\alpha y\beta\bar{x}\gamma\bar{y}\delta) \rightarrow_{\mathtt{torus}} (\alpha\delta\gamma\beta)$
- projective plane: $(\alpha x\beta x) \rightarrow_{\mathtt{pjp}} (\alpha\bar{\beta})$.

The three rewriting systems $\mathcal{R}$, $\mathcal{R}_{\mathrm{or}}$ and $\mathcal{R}_{\mathrm{nor}}$ are defined according to the following table:

system	terms	transformations
$\mathcal{R}$	TC	cont, torus, pjp
$\mathcal{R}_{\mathrm{or}}$	OTC	cont, torus
$\mathcal{R}_{\mathrm{nor}}$	TC	cont, perm, pjp

Remark 1. The reader can easily check that $\mathcal{R}$, $\mathcal{R}_{\mathrm{or}}$ and $\mathcal{R}_{\mathrm{nor}}$ are indeed three rewriting systems: it is sufficient to remark that the set `TC` is closed under the four rules listed Definition 3 and that the set `OTC` is closed under the two $\mathcal{R}_{\mathrm{or}}$-transformations `cont` and `torus`.

Definition 4 (segment). *We say that β is a segment of (α), $\beta \sqsubseteq (\alpha)$, if there is a γ such that $(\alpha) = (\beta\gamma)$; we write $\beta \sqsubset (\alpha)$ for specifying that $\gamma \neq \epsilon$.*

Example 3. $\epsilon, a, b, ab, ba, aba, bab \sqsubset (abab)$ and $abab, baba \sqsubseteq (abab)$.

Definition 5 (block, blocked cycle). *A word β constitutes a* block *if $\beta \in \mathtt{OTW}$ and (β) cannot be transformed into (ϵ) by exclusively using* `cont` *transformations. A cycle $(\alpha) \in \mathtt{TC}$ is* blocked *if there is a $\beta \sqsubseteq (\alpha)$ such that β is a block.*

Example 4. $(ab\bar{a}\bar{b})$ and $(ab\bar{a}\bar{b}cc)$ are both blocked.

Definition 6 (blocking `pjp`). *A projective plane transformation $(\alpha) \rightarrow_{\mathtt{pjp}} (\alpha)'$ is said to be* blocking *if (α) is non-blocked and $(\alpha)'$ is blocked.*

Example 5. The following is a blocking transformation: $(abcacb) \rightarrow_{\mathtt{pjp}} (\bar{c}\bar{b}cb)$.

Definition 7 (monad). *A word $\mu \in \mathtt{TW}$ is a* monad *if, for any $\beta \sqsubset (\mu)$, $\beta \notin \mathtt{TW}$.*

Example 6. $ab\bar{a}\bar{b}$ is an orientable monad; $abc\bar{a}cb$ is a non-orientable monad.

2.2 Computational Properties

Notation. Consider a generic rewriting system $\mathfrak{R}$ standing for $\mathcal{R}$, $\mathcal{R}_{\mathrm{or}}$ or $\mathcal{R}_{\mathrm{nor}}$ and a certain $\mathfrak{R}$-rule $\mathfrak{r}$:

- $(\alpha) \rightarrow_{\mathfrak{R}} (\alpha)'$ means that $(\alpha)'$ is obtained by rewriting (α) through an $\mathfrak{R}$-transformation;

– $(\alpha) \rightarrow^*_{\mathfrak{R}} (\alpha)'$ means that there exists a chain (possibly empty) of $\mathfrak{R}$-transformations rewriting (α) into $(\alpha)'$;
– $(\alpha) \rightarrow^*_{\mathfrak{r}} (\alpha)'$ means that there exists a chain (possibly empty) made by successive repetitions of the transformation $\mathfrak{r}$ rewriting (α) into $(\alpha)'$.

Moreover, $\rightarrow_{\texttt{pjp}[x]}$ indicates the `pjp` transformation specifically involving the x-letters, for instance: $(\alpha x \beta x) \rightarrow_{\texttt{pjp}[x]} (\alpha \bar{\beta})$.

Theorem 1. *The system $\mathcal{R}$ strongly normalises.*

Proof. Simply by remarking that, if we attach to $(\alpha) \in \texttt{TC}$ a size equal to $card(|\alpha|)$, we have at disposal a measure which decreases for any $\mathcal{R}$-transformation admitted by α.

Theorem 2. *The system $\mathcal{R}$ enjoys the property of the uniqueness of the normal form and, in particular, (ϵ) is the normal form common to all the $(\alpha) \in \texttt{TC}$.*

Proof. The proof consists in showing that if $\alpha \neq \epsilon$, then (α) admits at least one of the three transformations forming $\mathcal{R}$. If $(\alpha) \in \texttt{NTC}$, by Definition 2, there is a letter x such that $\{x, x\} \subseteq |\alpha|$ and so the transformation $\rightarrow_{\texttt{pjp}[x]}$ is clearly admitted. For $(\alpha) \in \texttt{OTC}$, we proceed by induction. If $card(|\alpha|) = 2$, then $(\alpha) = (x\bar{x})$ and so we can apply a `cont` transformation. In case of $card(|\alpha|) > 2$, let $(\alpha) = (x\beta\bar{x}\gamma)$: we distinguish two cases. If $\beta \notin \texttt{TW}$, then (α) admits a `torus` transformation; otherwise, $\beta, \gamma \in \texttt{OTW}$ and so we can call up to the inductive hypothesis.

Corollary 1. *Theorems 1 and 2 hold true also when $\mathcal{R}$ is restricted to $\mathcal{R}_{\text{or}}$.*

Proof. Straightforwardly by Theorems 1 and 2 and by Remark 1.

Theorem 3. *For any blocked $(\alpha) \in \texttt{NTC}$ there is a non-blocked cycle $(\alpha)'$ such that $(\alpha) \rightarrow^*_{\texttt{perm}} (\alpha)'$.*

Proof. By Definition 7, we can suppose any orientable monad μ to be of the shape $\mu = y\mu_1\bar{y}\mu_2$ with $\mu_1, \mu_2 \neq \epsilon$. Moreover, remark that if $\mu, \nu \sqsubset (\alpha)$ are two monads, then $\mu = \nu$ or $|\mu| \cap |\nu| = \varnothing$.

Since $(\alpha) \in \texttt{NTC}$, we can refer to a letter x such that $\{x, x\} \subset |\alpha|$ and write $\alpha = \boldsymbol{x}\beta\boldsymbol{x}\gamma$ (the two letters x are emphasized in boldface style). Then consider an orientable monad $y\mu_1\bar{y}\mu_2$ occurring within the segment β, i.e. $\beta = \beta_1 y\mu_1\bar{y}\mu_2\beta_2$. We transform (α) as follows:

$$(\boldsymbol{x}\beta_1 y\mu_1\bar{y}\mu_2\beta_2\boldsymbol{x}\gamma) \rightarrow_{\texttt{perm}} (\boldsymbol{x}\beta_1\mu_2\beta_2\boldsymbol{x}y\mu_1\bar{y}\gamma).$$

We iterate this kind of permutation for all the orientable monads occurring in β and then we proceed in the analogous way for all the orientable monads $z\nu_1\bar{z}\nu_2$ occurring in $\gamma = \gamma_1 z\nu_1\bar{z}\nu_2\gamma_2$:

$$(\boldsymbol{x}\gamma_1 z\nu_1\bar{z}\nu_2\gamma_2\boldsymbol{x}\beta') \rightarrow_{\texttt{perm}} (\boldsymbol{x}\gamma_1\nu_2\gamma_2\boldsymbol{x}z\nu_1\bar{z}\beta').$$

By construction, the procedure ends with a non-blocked cycle $(\alpha)'$.

Example 7. We exemplify below the just illustrated de-blocking procedure:

$$(\boldsymbol{x}abc\bar{a}\bar{c}\bar{b}\boldsymbol{x}ded\bar{d}\bar{e}) \rightarrow_{\texttt{perm}} (\boldsymbol{x}\bar{c}\bar{b}\boldsymbol{x}abc\bar{a}ded\bar{d}\bar{e}) \rightarrow_{\texttt{perm}} (\boldsymbol{x}ded\bar{d}\bar{c}\bar{b}\boldsymbol{x}abc\bar{a}\bar{e}).$$

Theorem 4. *Let* (α) *be a non-blocked cycle:* (α) *admits a non-blocking* `pjp` *transformation.*

Proof. Suppose that (α) admits a blocking `pjp` transformation: by Definition 6, there are two words $\beta, \gamma \sqsubset (\alpha)$ such that $\bar{\beta}\gamma$ forms a block (and clearly $\beta, \gamma \neq \epsilon$, otherwise (α) would be blocked). The proof consists in showing that there is a letter $t \in |\beta| \cap |\gamma|$ such that $\rightarrow_{\texttt{pjp}[t]}$ is non-blocking. Let $(\alpha) = (\beta\delta\gamma\eta)$; we proceed by induction on $card(|\bar{\beta}\gamma|)$.

Base. Since $\bar{\beta}\gamma$ forms a block, we have that $card(|\bar{\beta}\gamma|) \geq 4$; consider the following three cases.

- $\beta = tq$ and $\gamma = qt$. We perform the following transformation:

$$(tq\delta qt\eta) \rightarrow_{\texttt{pjp}} (t\bar{\delta}t\eta);$$

  $\bar{\delta}$ and η do not contain blocks (otherwise (α) would be blocked), so $\rightarrow_{\texttt{pjp}[q]}$ is non-blocking.
- $\beta = t$ and $\gamma = qt\bar{q}$ (or vice versa). We perform the following transformation:

$$(t\delta qt\bar{q}\eta) \rightarrow_{\texttt{pjp}} (\bar{q}\bar{\delta}\bar{q}\eta)$$

  and so $\rightarrow_{\texttt{pjp}[t]}$ is non-blocking as well.
- The case $\beta = \gamma = tq$ has to be rejected, because the word $\bar{q}\bar{t}tq$ is not a block.

Step. The non-blocked *status* of (α) ensures the presence of a letter $t \in |\beta| \cap |\gamma|$; let $(\alpha) = (\beta_1 t\beta_2\delta\gamma_1 t\gamma_2\eta)$, transform (α) as follows

$$(\beta_1 t\beta_2\delta\gamma_1 t\gamma_2\eta) \rightarrow_{\texttt{pjp}} (\beta_1\bar{\gamma}_1\bar{\delta}\bar{\beta}_2\gamma_2\eta)$$

and suppose $\rightarrow_{\texttt{pjp}[t]}$ to be blocking. Since $\bar{\beta}_2\bar{t}\bar{\beta}_1\gamma_1 t\gamma_2 \in \texttt{OTW}$, we have that $z \in |\beta_1\bar{\gamma}_1| \Rightarrow \bar{z} \notin |\bar{\beta}_2\gamma_2|$, i.e. the two words $\beta_1\bar{\gamma}_1$ and $\bar{\beta}_2\gamma_2$ cannot share a block. Moreover remark that, as (α) is non-blocked, the six words $\beta_1, \bar{\gamma}_1, \bar{\delta}, \bar{\beta}_2, \gamma_2$ and η cannot contain blocks. At this point, it is easy to show that a block occurs in $\beta_1\bar{\gamma}_1$ or in $\bar{\beta}_2\gamma_2$. We consider the following five cases:

- if $\beta_1\bar{\gamma}_1, \bar{\beta}_2\gamma_2 \neq \epsilon$, then clearly $\beta_1\bar{\gamma}_1$ or $\bar{\beta}_2\gamma_2$ contains a block;
- if $\beta_1\bar{\gamma}_1 = \epsilon$, then $\bar{\beta}_2\bar{t}t\gamma_2$ is a block and so is $\bar{\beta}_2\gamma_2$;
- if $\bar{\beta}_2\gamma_2 = \epsilon$, then $\bar{t}\bar{\beta}_1\gamma_1 t$ is a block and so are $\bar{\beta}_1\gamma_1$ and $\beta_1\bar{\gamma}_1$;
- if $\beta_1 = \epsilon$ and $\gamma_1 \neq \epsilon$ (or vice versa), then $\bar{\beta}_2\gamma_2$ contains a block (recall that β_1 and γ_1 cannot contain blocks);
- if $\beta_2 = \epsilon$ and $\gamma_2 \neq \epsilon$ (or vice versa), then $\beta_1\bar{\gamma}_1$ contains a block (recall that β_2 and γ_2 cannot contain blocks).

For concluding, in every case there are two words $\beta'_i, \gamma'_i \sqsubset (\alpha)$, with $i \in \{1, 2\}$, such that: $|\beta'_i| \subseteq |\beta_i|$ and $|\gamma'_i| \subseteq |\gamma_i|$, $\bar{\beta'}_i\gamma'_i$ forms a block and, since $card(|\bar{\beta}_i\gamma_i|) < card(|\bar{\beta}\gamma|)$, we have $card(|\bar{\beta'}_i\gamma'_i|) < card(|\bar{\beta}\gamma|)$.

Corollary 2. *If* $(\alpha) \in \mathtt{NTC}$, *then* $(\alpha) \rightarrow^*_{\mathcal{R}_{\mathrm{nor}}} (\epsilon)$.

Proof. At first remark that, by Definition 5, if $(\alpha) \in \mathtt{OTC}$ does not form a block, then $(\alpha) \rightarrow^*_{\mathtt{cont}} (\epsilon)$. For any $(\alpha) \in \mathtt{NTC}$, we can produce a chain $(\alpha) \rightarrow^*_{\mathcal{R}_{\mathrm{nor}}} (\epsilon)$ by combining Theorems 3 and 4 as indicated in Figure 1.

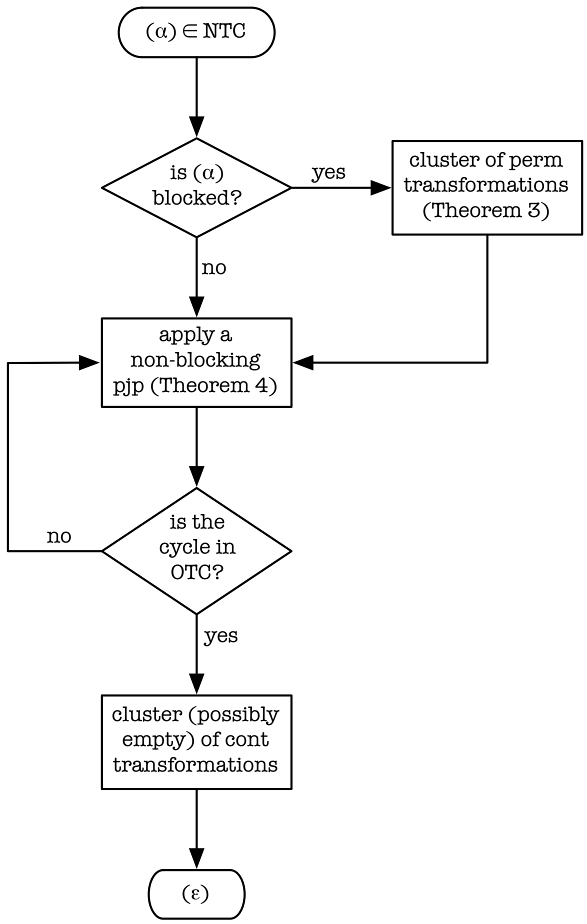


Fig. 1. The normalisation algorithm for non-orientable cycles.

Remark 2. Unlike $\mathcal{R}$ and $\mathcal{R}_{\mathrm{or}}$, the system $\mathcal{R}_{\mathrm{nor}}$ does not strongly normalise; this is due to the fact that $\mathcal{R}_{\mathrm{nor}}$ includes the `perm` transformation whose effect simply consists in permuting some letters inside a cycle without decreasing the cardinality. Moreover, in $\mathcal{R}_{\text{nor}}$, we also lose the uniqueness of the normal form; consider the following example:

- $(aabc\bar{b}\bar{c}) \rightarrow_{\texttt{pjp}} (bc\bar{b}\bar{c})$,
- $(aabc\bar{b}\bar{c}) \rightarrow_{\texttt{perm}} (abc\bar{b}a\bar{c}) \rightarrow_{\texttt{pjp}} (b\bar{c}\bar{b}\bar{c}) \rightarrow_{\texttt{pjp}} (bb) \rightarrow_{\texttt{pjp}} (\epsilon)$.

The table below summarises the results obtained in this section:

system	**normalisation**	**normal form**	**reduction chain**
$\mathcal{R}$	strong	unique	$\rightarrow^*_{\texttt{pjp}}\rightarrow^*_{\texttt{torus}}\rightarrow^*_{\texttt{cont}}$
$\mathcal{R}_{\text{or}}$	strong	unique	$\rightarrow^*_{\texttt{torus}}\rightarrow^*_{\texttt{cont}}$
$\mathcal{R}_{\text{nor}}$	weak	not unique	$\rightarrow^*_{\texttt{perm}}\rightarrow^*_{\texttt{pjp}}\rightarrow^*_{\texttt{cont}}$

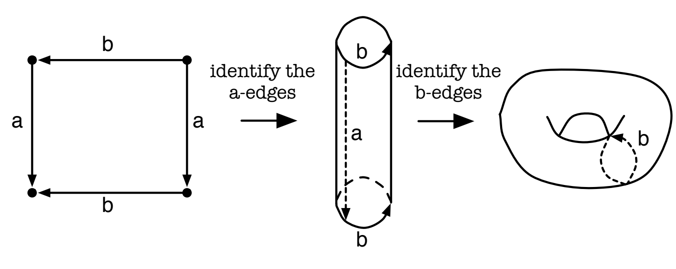
3 Geometrical Interpretation

According to the instructions reported below, we can associate to any $(\alpha) \in \texttt{TC}$ a polygon having the edges labelled and oriented.

1. Consider a polygon having $card(|\alpha|)$ edges.
2. Let $\alpha^\star$ be the sequence obtained from α by forgetting everywhere the bar; start from an arbitrary edge and label all the edges by following the clockwise order: if $\alpha^\star = a_1 a_2 \ldots a_n$, label the first one with a_1, the second one with a_2 and so on.
3. For all the $1 \leq i \leq n$: if a_i occurs in α without the bar, orient the a_i-edge according to the clockwise direction; otherwise, follow to the anticlockwise direction.

According to the standard terminology, we refer to compact and connected 2-manifolds by simply calling them *surfaces*; in the sequel of this paper, surfaces are always considered modulo homeomorphisms. We write $\mathscr{U}_{(\alpha)}$ for denoting the *quotient surface* associated with the cycle\polygon (α); this is the surface obtained, as usual, through identification of paired edges according to their orientation [1].

Example 8. We associate with the topological cycle $(ab\bar{a}\bar{b})$ the square reported below; as the reader can see, the resulting quotient surface $\mathscr{U}_{(ab\bar{a}\bar{b})}$ is a torus.



We recall two basic achievements in algebraic topology [1]:

- for any $(\alpha) \in \texttt{TC}$, the surface $\mathscr{U}_{(\alpha)}$ is unique,
- the set $\{\mathscr{U}_{(\alpha)} : (\alpha) \in \texttt{TC}\}$ ranges over the whole class of surfaces.

The spere, the torus and the projective plane are respectively indicated with $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{P}$; $\mathscr{U}\#\mathscr{V}$ denotes the connected sum of the two surfaces $\mathscr{U}$ and $\mathscr{V}$; $\mathscr{U}^n$, with $n \in \mathbb{N}^+$, stands for the connected sum of n copies of $\mathscr{U}$. The set of topological surfaces together with the connected sum operation form a monoid in which $\mathscr{S}$ plays the role of the neutral element: for any $\mathscr{U}$, $\mathscr{U}\#\mathscr{S} = \mathscr{U}$ [1].

The next theorem provides a geometrical interpretation of the rules listed in Definition 3: the basic idea is that the process of normalisation (in $\mathcal{R}$, $\mathcal{R}_{\mathrm{or}}$ and $\mathcal{R}_{\mathrm{nor}}$) represents, step by step, the process of forming the quotient surface associate with a certain cycle\polygon, through identification of paired edges.

Theorem 5.
- *If $(\alpha) \rightarrow_{\texttt{cont}} (\alpha)'$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha)'}$;*
- *if $(\alpha) \rightarrow_{\texttt{perm}} (\alpha)'$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha)'}$;*
- *if $(\alpha) \rightarrow_{\texttt{torus}} (\alpha)'$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha)'}\#\mathscr{T}$;*
- *if $(\alpha) \rightarrow_{\texttt{pjp}} (\alpha)'$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha)'}\#\mathscr{P}$.*

Proof. The reader can find all the details in [2, 3].

Remark 3. Theorem 5 induces a very easy algorithm for computing the quotient surface $\mathscr{U}_{(\alpha)}$ associated with any cycle\polygon $(\alpha) \in \texttt{TC}$. At first remark that $\mathscr{U}_{(\epsilon)} = \mathscr{S}$: $(x, \bar{x})$ constitutes the canonical cycle denoting the sphere, $\mathscr{U}_{(x,\bar{x})} = \mathscr{S}$ [1]; therefore, by Theorem 5, we have $\mathscr{U}_{(x,\bar{x})} = \mathscr{U}_{(\epsilon)} = \mathscr{S}$. Then, normalise (α) through a chain $(\alpha) \rightarrow^*_{\mathcal{R}} (\epsilon)$ and suppose that in such a chain p `torus` applications and q `pjp` applications occur. By Theorem 5 we have: $\mathscr{U}_{(\alpha)} = \mathscr{S}\#\mathscr{T}^p\#\mathscr{P}^q = \mathscr{T}^p\#\mathscr{P}^q$.

3.1 The Classification Theorem

Lemma 1. *$\mathscr{T}\#\mathscr{P} = \mathscr{P}^3$.*

Proof. We normalise the cycle $(abcacb)$ by following the two chains below:

- $(abcacb) \rightarrow_{\texttt{pjp}} (\bar{c}\bar{b}cb) \rightarrow_{\texttt{torus}} (\epsilon)$;
- $(abcacb) \rightarrow_{\texttt{pjp}} (ab\bar{a}b) \rightarrow_{\texttt{pjp}} (aa) \rightarrow_{\texttt{pjp}} (\epsilon)$.

By Remark 3, we have: $\mathscr{U}_{(abcacb)} = \mathscr{T}\#\mathscr{P}$ and $\mathscr{U}_{(abcacb)} = \mathscr{P}^3$; since the quotient surface associated with $(abcacb)$ is unique: $\mathscr{T}\#\mathscr{P} = \mathscr{P}^3$.

Theorem 6 (classification). *For any surface $\mathscr{U}$:*

- *$\mathscr{U} = \mathscr{S}$ or*
- *$\mathscr{U} = \mathscr{T}^p$, with $p \in \mathbb{N}^+$, or*
- *$\mathscr{U} = \mathscr{P}^q$, with $q \in \mathbb{N}^+$.*

Proof (Massey). Let $p, q \in \mathbb{N}^+$; the proof can be easily obtained by combining the following three propositions:

1. if $(\alpha) \in$ `OTC`, then $\mathscr{U} = \mathscr{S}$ or $\mathscr{U} = \mathscr{T}^p$ (Remark 3 and Corollary 1);
2. if $(\alpha) \in$ `NTC`, then $\mathscr{U} = \mathscr{P}^q$ or $\mathscr{U} = \mathscr{T}^p \# \mathscr{P}^q$ (Remark 3 and Theorem 1);
3. $\mathscr{T}^p \# \mathscr{P}^q = \mathscr{P}^{2p+q}$ (Lemma 1).

Proof (algorithmic characterisation). The proof comes straightforwardly by the following two propositions:

1. if $(\alpha) \in$ `OTC`, then $\mathscr{U} = \mathscr{S}$ or $\mathscr{U} = \mathscr{T}^p$ (Remark 3 and Corollary 1);
2. if $(\alpha) \in$ `NTC`, then $\mathscr{U} = \mathscr{P}^q$ (Remark 3 and Corollary 2).

Remark 4. With respect to Massey's proof, this latter alternative demonstration provides some new information concerning the dynamical side of the classification, i.e. the side concerning the process that – through identification of paired edges – allows to pass from polygons to their corresponding quotient surfaces. In detail:

- if $(\alpha) \in$ `OTC`, then (α) can be transformed into (ϵ) by exclusively achieving tori;
- if $(\alpha) \in$ `NTC`, then (α) can be transformed into (ϵ) by exclusively achieving projective planes.

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