Permutative Additives and Exponentials *

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Abstract. Permutative logic (PL) is a noncommutative variant of multiplicative linear logic (MLL) arising from recent investigations concerning the topology of linear proofs. Permutative sequents are structured as oriented surfaces with boundary whose topological complexity is able to encode some information about the exchange in sequential proofs. In this paper we provide a complete permutative sequent calculus by extending that one of PL with rules for additives and exponentials. This extended system, here called *permutative linear logic* (PLL), is shown to be a conservative extension of linear logic and able to enjoy cut-elimination. Moreover, some basic isomorphisms are pointed out.

1 Introduction

Linear logic (LL) presents a remarkable skill in emphasizing geometrical features of logical proofs. LL comes in fact with a double syntax: the usual one in terms of sequential rules, and a more geometrical one, constituted by a set of links which allow to turn sequential proofs into graphs called *proof-nets*. Proof-nets quotient on the class of linear demonstrations enabling to avoid pointless syntactical bureaucracies [9], [7].

Studies on logical noncommutativity take advantage from this more geometrical approach due to the fact that the use of the exchange in a sequential proof affects the genus of its corresponding net. *Cyclic logic* (namely, linear logic in which only cyclic exchanges are allowed) [16] constitutes a limit case in which cut-free proofs always induce planar proof-nets [1]. This kind of results have been progressively generalized by topological investigations on linear proofs due to Bellin, Fleury [6], Melliès [12] and Métayer [13]. In particular, Métayer has proposed a way to translate any proof-net Π into a (compact and orientable) surface with boundary $\mathscr{S}(\Pi)$, such that:

- the rank of $\mathscr{S}(\Pi)$ constitutes a lower bound for the complexity of the exchange inside the proof π sequentialisation of Π [13];
- $\mathscr{S}(\Pi)$ represents the minimal surface on which Π can be drawn without crossings [8].

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By stressing the fact that topology tells about exchange, the above-mentioned works have induced the following non-commutative variants of MLL: *planar logic* [12], the *calculus of surfaces* [8] and, later, *permutative logic* [5]. Such calculi stand out for dealing with sequents structured as orientable surfaces with boundary. Being more precise, each sequent turns out indexed by a natural number (counting the handles) and with formulas grouped into disjoint cycles forming in this way a permutation (denoting, cycle by cycle, each boundary-component). Such structures (permutations with attached a natural number) have been called *q-permutations* [5] and separately studied in [14].

Unlike planar logic and the calculus of surfaces, PL comes with two explicit structural rules of *divide* and *merge* (topologically corresponding to an amalgamated sum) and it enjoys the two fundamental proof-theoretical properties of cut-elimination and focussing. Moreover, thanks to permutative modalities and constants, PL provides a specific mechanism able to manage (topological) resources during proof-construction [5].

This paper should be considered as the continuation of the first one in which the multiplicative fragment of permutative logic has been introduced [5]. We propose in fact an enrichment of the PL calculus with rules for additives and exponentials, here called *permutative linear logic* (PLL). On the one hand, additives pose the problem of establishing when two PL sequents can be considered as having same context with respect to a fixed formula in each one of them. There are in fact two ways to introduce the &-rule in the context of PL: either by requiring the two premises to share their permutative structure or by enabling two premises having different structures to be "mixed". In accordance with the fact that the &-rule should be negative in sense of Andreoli's property of focussing, we decide to "officially" adopt the first solution. Nevertheless, we also propose an alternative version of the &-rule in which structural rules are compacted and optimized: this version would be useful towards both a semantics and a theory of proof-nets for PL (remark that, in proof-nets, structural transformations should not explicitly appear). On the other hand, exponentials are treated in the standard way, i.e. as central elements essentially aside from the inner structure of sequents [15]. The inference system obtained in this way is shown to be a conservative extension of LL and able to enjoy cut-elimination. By stressing this latter property, we recall the notion of logical isomorphism and we point out some basic permutative isomorphisms.

The extension we propose in these pages is legitimate by the need of having a complete counterpart of *non-commutative logic* [2] in which, instead of the usual approach rooted in serial and parallel combinators, logical non-commutativity is approached from the more geometrical point of view afforded by topology. In our opinion, this change of viewpoint may offer an interesting framework in which reconsider some of the typical problems related with non-commutativity, for instance, problems arising in studies on linguistics and concurrency.

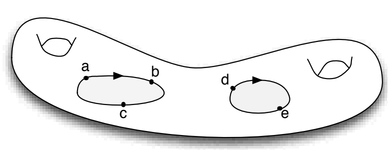
2 Multiplicative Permutative Logic

2.1 The Sequent Calculus

The well-known classification theorem for 2-dimensional surfaces says that any compact and connected orientable surface turns out to be homeomorphic to a sphere or to a connected sum of tori possibly with boundary [11]. If we consider an orientable surface $\mathscr{S}$ as the final result of identifying edges having same label in a set of polygons forming an its topological presentation, we have that each boundary-component will be formed by at least one edge. Let $\partial\mathscr{S}$ be the set of labels occurring on the boundary of $\mathscr{S}$: since fixed an orientation, we can notice that $\mathscr{S}$ induces a cyclic order on each one of the subsets of $\partial\mathscr{S}$ corresponding to boundary-components. In other words, we obtain nothing else but a permutation on $\partial\mathscr{S}$. The idea leading to the notion of q-permutation is that the basic information concerning any orientable connected surface $\mathscr{S}$ can always be encoded by a very easy mathematical structure consisting in a permutation σ (denoting cycle by cycle the boundary $\partial\mathscr{S}$) together with a natural number q counting the number of tori in the connected sum forming $\mathscr{S}$. Remark that the number of connected tori forming a surface $\mathscr{S}$ corresponds to the number of *handles* appearing on $\mathscr{S}$. In this way, q-permutations are able to characterize orientable connected surfaces modulo isomorphism, namely modulo homeomorphisms preserving the inner structure of the boundary together with an orientation.

Definition 1 (q-permutation, PL sequent). *A q-permutation is a triple (X, σ, p), where X is a finite set, σ is a permutation on X and $p \in \mathbb{N}$. A PL sequent is nothing else, but a q-permutation in which the support X is a set of linear formulas.*

Example 1. It is easy to check that the surface proposed below is well-characterized by the q-permutation $\{(a, b, c), (d, e)\}, 2$.



Notation. – Capital Greek letters Γ, Δ, Λ, ... denote series of elements, whereas (Γ) means that the series Γ is taken modulo cyclic exchange; Σ, Ξ, Ψ, ... denote sets of cycles. q-permutations are indicated with small Greek letters α, β, γ, ... Moreover, $|\alpha|$ and $\alpha[a]$ respectively denote the support of α and that $a \in |\alpha|$. In the sequel of this paper, it will be useful to adopt a simplified notation for q-permutations obtained by omitting the support: (X, σ, p), in which $\sigma = (\Gamma_1)(\Gamma_2)\dots(\Gamma_q)$, will become $\{(\Gamma_1), (\Gamma_2), \dots, (\Gamma_q)\}, p$. For any permutation σ, $\sigma^\bullet$ denotes the number of its cycles.

– PL sequents will be denoted in two ways. We write $\vdash_p (\Gamma_1), (\Gamma_2), \ldots, (\Gamma_q)$ for the PL sequent corresponding to the q-permutation $\{(\Gamma_1), (\Gamma_2), \ldots, (\Gamma_q)\}, p$. Otherwise, we directly write $\vdash \alpha$ for indicating the PL sequent corresponding to a certain q-permutation α.

Thanks to classification theorem, the topological complexity of an oriented surface $\mathscr{S}$ can be expressed by a couple of parameters (p, q), called the *genus* of the surface, such that $p \in \mathbb{N}$ is the number of handles of $\mathscr{S}$ and $q \in \mathbb{N}$ the number of pieces in which its boundary turns out to be decomposed. The rank can be straightforwardly obtained by the genus: $rk(\mathscr{S}) = 2p + q - 1$, if q is non-zero; $2p$, otherwise. Clearly we can associate genus and rank with q-permutations too.

Example 2. If $\alpha = \{(a, b), (c), (d, e)\}, 2$, then its genus is given by the couple $(2, 3)$ and $rk(\alpha) = 6$.

The multiplicative permutative calculus is recalled in Table 1; moreover, the involutive duality is given by De Morgan rules:

$$
\begin{array}{llll}
(A \parr B)^\perp = B^\perp \otimes A^\perp & (\flat A)^\perp = \# A^\perp & \hbar^\perp = h & \perp^\perp = 1 \\
(A \otimes B)^\perp = B^\perp \parr A^\perp & (\# A)^\perp = \flat A^\perp & h^\perp = \hbar & 1^\perp = \perp .
\end{array}
$$

By the fact that basic commutations are not provable keeping the lowest topological complexity, PL turns out to be an inference system able to deal with logical noncommutativity. As suggested by some of the next propositions, basic commutations can be recovered throughout the two permutative modalities $\flat$ and $\#$. It is easy to check in fact that formulas marked with permutative modalities behave as central elements, namely they can be freely moved inside sequents without any cost in terms of topological complexity.

Notation. $A_1, \ldots, A_n \vdash B$ denotes the sequent $\vdash_0 (A_1^\perp, \ldots, A_n^\perp, B)$ and $A \dashv\vdash B$ denotes the two sequents $A \vdash B$ and $B \vdash A$.

Proposition 1. [5] *The following sequents are provable in PL:*

$$
\begin{array}{lll}
A \vdash \flat A & \flat A \vdash A \parr \hbar & \flat A \dashv\vdash \flat \# A \\
(A \parr B) \parr C \dashv\vdash A \parr (B \parr C) & \flat\flat A \dashv\vdash \flat A & A \parr \flat B \dashv\vdash \flat B \parr A \\
A \parr \perp \dashv\vdash A & \flat \perp \dashv\vdash \perp & \flat(A \parr \flat B) \dashv\vdash \flat A \parr \flat B \\
\perp \parr A \dashv\vdash A & \flat \hbar \dashv\vdash \hbar & \flat(A \parr B) \dashv\vdash \flat(B \parr A).
\end{array}
$$

We can easily prove that PL without specific constants and modalities and with indexes fixed in 0, exactly corresponds to Mellies' planar logic; if we require in addition the rank of sequents to be null, we just obtain cyclic logic [5].

2.2 Relaxation

We call *relaxation* the relation induced on q-permutations by the two structural rules divide and merge. In particular, we say that a q-permutation β relaxes another q-permutation α, $\alpha \succeq \beta$, if α can be rewritten into β throughout a series of stuctural rules. A more formal definition is provided below.

Table 1. The sequent calculus of permutative logic.

Identities

$$\text{ax.}\ \dfrac{}{\vdash_0 (A, A^\perp)} \qquad\qquad \mathit{cut}\ \dfrac{\vdash_d \Sigma, (\Gamma, A) \qquad \vdash_e \Theta, (\Delta, A^\perp)}{\vdash_{d+e} \Sigma, \Theta, (\Gamma, \Delta)}$$

Structural rules

$$\text{divide}\ \dfrac{\vdash_d \Sigma, (\Gamma, \Delta)}{\vdash_d \Sigma, (\Gamma), (\Delta)} \quad \text{merge}\ \dfrac{\vdash_d \Sigma, (\Gamma), (\Delta)}{\vdash_{d+1} \Sigma, (\Gamma, \Delta)}$$

Logical rules

$$\parr\ \dfrac{\vdash_d \Sigma, (\Gamma, A, B)}{\vdash_d \Sigma, (\Gamma, A \parr B)} \qquad \otimes\ \dfrac{\vdash_d \Sigma, (\Gamma, A) \qquad \vdash_e \Theta, (\Delta, B)}{\vdash_{d+e} \Sigma, \Theta, (\Delta, \Gamma, A \otimes B)}$$

$$\flat\ \dfrac{\vdash_d \Sigma, (\Gamma), (A)}{\vdash_d \Sigma, (\Gamma, \flat A)} \qquad \#\ \dfrac{\vdash_d \Sigma, (\Gamma, A)}{\vdash_d \Sigma, (\Gamma), (\# A)}$$

$$\hbar\ \dfrac{\vdash_{d+1} \Sigma, (\Gamma)}{\vdash_d \Sigma, (\Gamma, \hbar)} \qquad h\ \dfrac{}{\vdash_1 (h)}$$

$$\perp\ \dfrac{\vdash_d \Sigma, (\Gamma)}{\vdash_d \Sigma, (\Gamma, \perp)} \qquad 1\ \dfrac{}{\vdash_0 (1)}$$

Definition 2. **[5]** *Relaxation is the smallest reflexive transitive relation* $\succcurlyeq$ *on q-permutations such that:*

- *divide:* $(X, \sigma, p) \succcurlyeq (X, \sigma', p)$, *where* σ' *is obtained from* σ *dividing one cycle* (Γ, Δ) *of* σ *into two:* (Γ) *and* (Δ)*;*
- *merge:* $(X, \sigma, p) \succcurlyeq (X, \sigma', p+1)$, *where* σ' *is obtained by* σ *merging two cycles* (Γ) *and* (Δ) *of* σ *into one:* (Γ, Δ)*;*
- degenerate merge*:* $(X, \sigma, p) \succcurlyeq (X, \sigma, p+1)$, *namely we can always merge an empty cycle to a cycle of* σ *increasing of one the number* p.

Remark 1. The last point of the previous definition (degenerate merge) is based on the idea that a PL sequent may be presented in various different ways: $\vdash_p \Sigma, (\Gamma)$ as well as $\vdash_p \Sigma, (\Gamma), ()$.

Remark 2. Any application of a divide or merge (possibly degenerate) rule on a certain q-permutation increases its rank and this is the reason for which relaxation induces a partial order on the set of q-permutations [5].

Theorem 1 (decision of relaxation). **[5],[14]** *For any pair of q-permutations* $\alpha = (X, \sigma, p)$ *and* $\beta = (X, \tau, q)$, *we have:*

$$\alpha \succeq \beta \ \Leftrightarrow\ q - p \geqslant \frac{n - (\sigma^{-1}\tau)^\bullet + \sigma^\bullet - \tau^\bullet}{2}.$$

3 Permutative Additives and Exponentials

3.1 The Sequent Calculus

Negative (resp. positive) connectives are those ones having a reversible (resp. not reversible) introduction rule. These notions arise inside the framework of Andreoli's studies on the property of focussing which allow to eliminate redundant non-determinism during proof-construction, by imposing a rigid alternation between clusters of negative and positive connectives [3]. The introduction of the basic version of the &-rule in which premises share their permutative structure (Definition 3), allow to classify the &-connective as a negative one, without altering the fundamental symmetry between negative and positive connectives we have in linear logic. It is easy to see that structural rules commute with negative ones; in the perspective of a focussed calculus, this aspect bears out our choice: it means that structural rules can be relegated between generalized positive and negative connectives, as a sort of shift rule changing the polarity.

Definition 3 (permutative additives). *Rules for additives are introduced as follows.*

$$\frac{\vdash_p \Sigma, (\Gamma, A)}{\vdash_p \Sigma, (\Gamma, A \oplus B)} \oplus_L \qquad \frac{\vdash_p \Sigma, (\Gamma, B)}{\vdash_p \Sigma, (\Gamma, A \oplus B)} \oplus_R$$

$$\frac{\vdash_p \Sigma, (\Gamma, A) \qquad \vdash_p \Sigma, (\Gamma, B)}{\vdash_p \Sigma, (\Gamma, A \& B)} \&$$

$$\frac{}{\vdash_p \Sigma, (\Gamma, \top)} \textit{true} \qquad \textit{(no rule for zero).}$$

Definition 4 (permutative exponentials). *Rules for permutative exponentials are introduced as follows.*

$$\frac{\vdash_p \Sigma, (\Gamma, A)}{\vdash_p \Sigma, (\Gamma, ?A)} \text{dereliction} \qquad \frac{\vdash_p \Sigma, (\Gamma)}{\vdash_p \Sigma, (\Gamma, ?A)} \text{weakening}$$

$$\frac{\vdash_p \Sigma, (\Gamma, ?A), (?A, \Delta)}{\vdash_p \Sigma, (\Gamma, ?A), (\Delta)} \text{contraction} \qquad \frac{\vdash_0 ?\Sigma, (?\Gamma, A)}{\vdash_0 ?\Sigma, (?\Gamma, !A)} \text{promotion}$$

Remark 3. By making the following two rules of *center* derivable, contraction rule induces the following two rules of *center*.

$$\frac{\vdash_p \Sigma, (\Gamma, ?A, \Delta)}{\vdash_p \Sigma, (\Gamma, \Delta, ?A)} \text{center(1)} \quad \cong \quad \cfrac{\cfrac{\cfrac{\vdash_p \Sigma, (\Gamma, ?A, \Delta)}{\vdash_p \Sigma, (\Gamma, ?A, \Delta, ?A)} \text{weak.}}{\vdash_p \Sigma, (\Gamma, \Delta, ?A), (?A)} \text{divide}}{\vdash_p \Sigma, (\Gamma, \Delta, ?A)} \text{contr.}$$

$$\frac{\vdash_p \Sigma, (\Gamma, ?A), (\Delta)}{\vdash_p \Sigma, (\Gamma), (?A, \Delta)} \text{center(2)} \quad \cong \quad \cfrac{\cfrac{\vdash_p \Sigma, (\Gamma, ?A), (\Delta)}{\vdash_p \Sigma, (\Gamma, ?A), (?A, \Delta)} \text{weak.}}{\vdash_p \Sigma, (\Gamma), (?A, \Delta)} \text{contr.}$$

As for the two permutative modalities $\flat$ and $\#$, formulas marked with exponentials behave as central elements, in other words they turn out to be essentially aside from the inner permutative structure of sequents. This is consistent with the standard treatment of exponentials in non-commutative systems, for instance in non-commutative logic [15]. In spite of their centrality, permutative modalities and exponentials remain two distinct logical objects. In fact, unlike permutative modalities, permutative exponentials allow to recover the basic properties concerning exponentials we have linear logic. In particular, as we will show in the next theorem, we can provide a proof for $\flat A \vdash ?A$, but the converse does not hold.

Theorem 2. *The following propositions are provable in PLL.*

- *Commutations:* $A\&B \dashv\vdash B\&A$*;* $!A \otimes B \dashv\vdash B \otimes !A$*;* $!(A\otimes B) \dashv\vdash !(B \otimes A)$*.*
- *Associativity:* $A\&(B\&C) \dashv\vdash (A\&B)\&C$*.*
- *Distributivity:* $A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$*.*
- *Constants:* $!\top \dashv\vdash 1$*;* $A\&\top \dashv\vdash A$*;* $A \parr \top \dashv\vdash \top$*.*
- *Exponentials:* $!!A \dashv\vdash !A$*;* $!(A\&B) \dashv\vdash (!A)\otimes(!B)$*;* $?\flat A \vdash \flat ?A$*;* $\flat A \vdash ?A$*.*

Proof. We respectively report the proofs of the sequents $A\&B \vdash B\&A$, $!A \vdash A \otimes A$, $?\flat A \vdash \flat ?A$, $!A \otimes B \vdash B \otimes !A$ and $!A \vdash \flat A$.

$$\dfrac{\otimes_R\,\dfrac{\text{ax.}\,\dfrac{}{\vdash_0 (B^\perp, B)}}{\vdash_0 (B^\perp \oplus A^\perp, B)} \qquad \dfrac{\dfrac{}{\vdash_0 (A^\perp, A)}\,\text{ax.}}{\vdash_0 (B^\perp \oplus A^\perp, A)}\,\otimes_L}{\vdash_0 (B^\perp \oplus A^\perp, B\&A)}\,\&$$

$$\dfrac{\dfrac{\text{der.}\,\dfrac{\text{ax.}\,\dfrac{}{\vdash_0 (A^\perp, A)}}{\vdash_0 (?A^\perp, A)} \qquad \dfrac{\dfrac{\dfrac{}{\vdash_0 (A, A^\perp)}\,\text{ax.}}{\vdash_0 (A, ?A^\perp)}\,\text{der.}}{\vdash_0 (A), (?A^\perp)}\,\text{div.}}{\vdash_0 (?A^\perp, A \otimes A), (?A^\perp)}\,\otimes}{\vdash_0 (?A^\perp, A \otimes A)}\,\text{contr.}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (A^\perp, A)}\,\text{ax.}}{\vdash_0 (A^\perp, ?A)}\,\text{der.}}{\vdash_0 (\#A^\perp), (?A)}\,\#}{\vdash_0 (!\#A^\perp), (?A)}\,!}{\vdash_0 (!\#A^\perp, \flat ?A)}\,\flat$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{ax.}\,\dfrac{}{\vdash_0 (B^\perp, B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (A, A^\perp)}\,\text{ax.}}{\vdash_0 (A, ?A^\perp)}\,\text{der.}}{\vdash_0 (!A, ?A^\perp)}\,!}{\vdash_0 (!A), (?A^\perp)}\,\text{div.}}{\vdash_0 (B^\perp, B \otimes !A), (?A^\perp)}\,\otimes}{\vdash_0 (B^\perp, ?A^\perp, B \otimes !A), (?A^\perp)}\,\text{weak.}}{\vdash_0 (B^\perp, ?A^\perp, B \otimes !A)}\,\text{contr.}}{\vdash_0 (B^\perp \parr ?A^\perp, B \otimes !A)}\,\parr
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (A^\perp, A)}\,\text{ax.}}{\vdash_0 (A^\perp, ?A)}\,\text{der.}}{\vdash_0 (\#A^\perp), (?A)}\,\#}{\vdash_0 (\#A^\perp, ?A)}\,\text{center}(2)$$

3.2 Embedding Linear Logic

Definition 5. *We define the function "$p\ell$" from LL to PL formulas in the following way. If p is an atom or a constant, then $p^{p\ell} = p$; moreover:*

$$
\begin{array}{ll}
(A^{\perp})^{p\ell} = (A^{p\ell})^{\perp} & \\
(A \parr B)^{p\ell} = A^{p\ell} \parr \flat B^{p\ell} & (A \otimes B)^{p\ell} = \# A^{p\ell} \otimes B^{p\ell} \\
(A \& B)^{p\ell} = A^{p\ell} \& B^{p\ell} & (A \oplus B)^{p\ell} = A^{p\ell} \oplus B^{p\ell} \\
(?A)^{p\ell} = ?A^{p\ell} & (!A)^{p\ell} = !A^{p\ell}
\end{array}
$$

This function can be extended to sequents by mapping any set of formulas Σ into the identical permutation, namely: if $\Sigma = A_1, A_2, \ldots, A_n$, then $\Sigma^{p\ell} = (A_1^{p\ell}), (A_2^{p\ell}), \ldots, (A_n^{p\ell})$.

Theorem 3. *A sequent $\vdash \Sigma$ is provable in LL if, and only if, $\vdash_0 \Sigma^{p\ell}$ is provable in PLL.*

Proof. ($\Rightarrow$) We proceed by induction on the length of the LL proof $\pi \vdash \Sigma$. The base is easily verified as follows:

$$
\frac{}{\vdash A, A^{\perp}}\,\text{ax.} \quad \overset{p\ell}{\longmapsto} \quad \frac{\dfrac{}{\vdash_0 (A, A^{\perp})}\,\text{ax.}}{\vdash_0 (A), (A^{\perp})}\,\text{divide}
$$

Then we consider some induction steps; the missing cases are immediate.

$$
\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad \overset{p\ell}{\longmapsto} \quad \frac{\dfrac{\vdash \Gamma^{p\ell}, (A^{p\ell}), (B^{p\ell})}{\vdash \Gamma^{p\ell}, (A^{p\ell}, \flat B^{p\ell})}\,\flat}{\vdash \Gamma^{p\ell}, (A^{p\ell} \parr \flat B^{p\ell})}\,\parr
$$

$$
\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \quad \overset{p\ell}{\longmapsto} \quad \frac{\# \dfrac{\vdash_0 \Gamma^{p\ell}, (A^{p\ell})}{\vdash_0 \Gamma^{p\ell}, (\# A^{p\ell})} \qquad \vdash_0 (B^{p\ell}), \Delta^{p\ell}}{\vdash_0 \Gamma^{p\ell}, (\# A^{p\ell} \otimes B^{p\ell}), \Delta^{p\ell}}\,\otimes
$$

$$
\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\,\& \quad \overset{p\ell}{\longmapsto} \quad \frac{\vdash_0 \Gamma^{p\ell}, (A^{p\ell}) \qquad \vdash_0 \Gamma^{p\ell}, (B^{p\ell})}{\vdash_0 \Gamma^{p\ell}, (A^{p\ell} \& B^{p\ell})}\,\&
$$

$$
\frac{\vdash \Gamma, A}{\vdash \Gamma, A, ?B}\,\text{weak.} \quad \overset{p\ell}{\longmapsto} \quad \frac{\dfrac{\vdash_0 \Gamma^{p\ell}, (A^{p\ell})}{\vdash_0 \Gamma^{p\ell}, (?A^{p\ell}, ?B^{p\ell})}\,\text{weak.}}{\vdash_0 \Gamma^{p\ell}, (?A^{p\ell}), (?B^{p\ell})}\,\text{divide}
$$

($\Leftarrow$) It is sufficient to remark that any PLL proof $\pi \vdash_0 \Gamma^{p\ell}$ can be turned into an LL proof $\pi' \vdash \Gamma$ simply by removing all the superfluous information: structural rules together with permutative decorations.

3.3 An Alternative Approach to Additives

In order to perform a &-rule involving two premises having different structures, we have to relax sequents since we arrive to a compromise, a common form allowing the application of the basic &-rule. This process of "approaching" sequents throughout structural rules is formalized by the set of the *nearest common stops* introduced in Definition 7. Before introducing this notion some technics concerning chains of structural transformations are required.

Notation. Let α and β be two q-permutations such that $\alpha \succeq \beta$. With $\mathscr{C} : \alpha \rightsquigarrow_{d/m} \beta$ we denote a chain of q-permutations rewriting α into β, such that each step of $\mathscr{C}$ corresponds to an application of either divide or merge rule.

Definition 6 (minimal chain). *A chain is said to be minimal, if it consists in a minimal number of steps.*

Procedure 4 (computing chains) [14] *Let α and β be two q-permutations such that $\alpha \succeq \beta$ We can obtain a chain $\mathscr{C} : \alpha \rightsquigarrow_{d/m} \beta$ simply by arbitrarily applying the following three specific versions of divide and merge rules. τ denotes the permutation of β.*

$$\text{If } \tau(a) = b: \qquad \frac{\{\Sigma, (a, \Gamma, b, \Delta)\}, p}{\{\Sigma, (a, b, \Delta), (\Gamma)\}, p}\ \textit{divide(1);}$$

$$\text{if } \Gamma \text{ is a cycle of } \tau\text{:} \qquad \frac{\{\Sigma, (\Gamma, \Delta)\}, p}{\{\Sigma, (\Gamma), (\Delta)\}, p}\ \textit{divide(2);}$$

$$\text{if } \tau(a) = b\text{:} \qquad \frac{\{\Sigma, (\Gamma, a), (b, \Delta)\}, p}{\{\Sigma, (\Gamma, a, b, \Delta)\}, p+1}\ \textit{merge.}$$

Example 3. Procedure 4 is here applied in order to produce a chain $\mathscr{C} : \alpha \rightsquigarrow_{d/m} \beta$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\alpha = \{(a,b,c,d,e)\}, 0}{\{(a,d,e),(b,c)\}, 0}\ \text{divide(1)}}{\{(a,d),(e),(b,c)\}, 0}\ \text{divide(2)}}{\{(a,d),(e,b,c)\}, 1}\ \text{merge}}{\beta = \{(a,d),(e,b),(c)\}, 1}\ \text{divide(2)}$$

Theorem 5. [14] *If $\mathscr{C}$ is a chain afforded by Procedure 4, then it is minimal.*

If we ignore the superfluous information concerning indexes, the divide\merge rewriting system can be seen as directly working on permutations. In this way, any chain of q-permutations $\mathscr{C} : \alpha \rightsquigarrow_{d/m} \beta$, where $\alpha = (X, \sigma, p)$ and $\beta =$

(X,τ,q), is implicitly a chain of permutations $\sigma \rightsquigarrow_{d/m} \tau$ too. Moreover, remark that, unlike chains of q-permutations, any chain of permutations $\sigma \rightsquigarrow_{d/m} \tau$ can be reversed into a chain $\tau \rightsquigarrow_{d/m} \sigma$ such that, if $\sigma \rightsquigarrow_{d/m} \tau$ is minimal, then $\tau \rightsquigarrow_{d/m} \sigma$ is minimal too.

Theorem 6. [14] *Any chain of permutations implicit into a minimal chain of q-permutations, is minimal too.*

Definition 7 (nearest common stops). *Let α and β be two q-permutations sharing the support. A q-permutation ξ belongs to the set of the* nearest common stops *of α and β, denoted with $\mathtt{ncs}(\alpha,\beta)$, if, and only if, $\alpha,\beta \succeq \xi$ and $rk(\xi)$ is minimal.*

Proposition 2. *For any pair of q-permutations α and β sharing the support, we have:*

1. $\mathtt{ncs}(\alpha,\beta) = \mathtt{ncs}(\beta,\alpha)$;
2. $\mathtt{ncs}(\alpha,\beta) \neq \varnothing$;
3. *if ξ, $\xi' \in \mathtt{ncs}(\alpha,\beta)$, then they are incomparable;*
4. *if $\alpha \succeq \beta$, then $\mathtt{ncs}(\alpha,\beta) = \{\beta\}$.*

Now we aim to provide an effective procedure able to reach elements in any set $\mathtt{ncs}(\alpha,\beta)$. For α and β such that $\alpha \succeq \beta$, we know that $\mathtt{ncs}(\alpha,\beta) = \{\beta\}$. The next theorem deals with the case in which α and β are incomparable.

Theorem 7. *Consider two incomparable q-permutations $\alpha = (X,\sigma,p)$ and $\beta = (X,\tau,q)$, and a third one ξ obtained as follows.*

> *According to Procedure 4, we start rewriting α in order to reconstruct the permutation τ expressed by β: we call ξ the first q-permutation we meet such that it relaxes β.*

We have that $\xi \in \mathtt{ncs}(\alpha,\beta)$.

Proof. Consider three q-permutations $\alpha = (X,\sigma,p)$, $\beta = (X,\tau,q)$ and ξ obtained from α and β according to the claim of the theorem. Suppose by absurd that $\xi \notin \mathtt{ncs}(\alpha,\beta)$ and consider any $\theta \in \mathtt{ncs}(\alpha,\beta)$ (by Proposition 2.2, we know that $\mathtt{ncs}(\alpha,\beta) \neq \varnothing$). Now consider the chain $\mathscr{C} : \alpha \rightsquigarrow_{d/m} \beta'$, where $\beta' = (X,\tau,q+k)$, computed in order to obtain ξ. By the fact that $\theta \in \mathtt{ncs}(\alpha,\beta)$ and $rk(\theta) < rk(\xi)$, there exist two chains $\mathscr{C}_1 : \alpha \rightsquigarrow_{d/m} \theta$ and $\mathscr{C}_2 : \beta \rightsquigarrow_{d/m} \theta$ respectively shorter than $\mathscr{C}'_1 : \alpha \rightsquigarrow_{d/m} \xi$ and $\mathscr{C}'_2 : \beta \rightsquigarrow_{d/m} \xi$. So, we have a chain of permutations $\sigma \rightsquigarrow_{d/m} \tau$ shorter than that one implicit in $\mathscr{C}$ which is, by Theorem 6, absurd.

Example 4. Consider the following chain performed in order to compute an element $\xi \in \mathtt{ncs}(\alpha,\beta)$, where $\alpha = \{(a,b,c,d)\},0$ and $\beta = \{(a,d,c),(b)\},0$. The first line we meet such that it relaxes β is the third one and so $\xi = \{(a,d,c,b)\},1$.

$$\cfrac{\cfrac{\cfrac{\alpha = \{(a,b,c,d)\},0}{\{(a,d),(b,c)\},0}\ \text{divide}}{\xi = \{(a,d,c,b)\},1}\ \text{merge}}{\beta' = \{(a,d,c),(b)\},1}\ \text{divide}$$

At this point, we have at disposal a complete technical background for providing a version of the &-rule, denoted with [&], which enables to mix two premises having different structures by compacting and optimizing structural rules.

Definition 8. *We write* $\alpha\,[a'/a]$ *for the q-permutation obtained from* α *by replacing an element* a *of its support with another one* a'*. The* [&]*-rules is here introduced by indicating sequents as q-permutations.*

$$\frac{\vdash \alpha \qquad\qquad \vdash \beta}{\vdash \xi \in \mathtt{ncs}(\alpha\,[A\&B/A], \beta\,[A\&B/B])}\,[\&],\ \textit{where}\ |\alpha|\backslash\{A\} = |\beta|\backslash\{B\}.$$

Example 5. Below we propose a concrete application of the [&]-rule together with an its "extracted" version.

$$\frac{\vdash_0 (a, b, c, f_1) \qquad \vdash_0 (b), (c, a, f_2)}{\vdash_1 (c, b, a, f_1\&f_2)}\,[\&] \quad \cong$$

$$\cong \quad \frac{\dfrac{\dfrac{\vdash_0 (a, b, c, f_1)}{\vdash_0 (a, f_1), (b, c)}\,\text{divide}}{\vdash_1 (c, b, a, f_1)}\,\text{merge} \qquad \dfrac{\vdash_0 (b), (c, a, f_2)}{\vdash_1 (c, b, a, f_2)}\,\text{merge}}{\vdash_1 (c, b, a, f_1\&f_2)}\,\&.$$

Thanks to the main result provided in the next section (Theorem 8), we can easily notice that cut-elimination is preserved by replacing the basic version of the &-rule with that one just provided in Definition 8. Remark that, unlike the basic &, the [&] connective cannot be catalogued as a negative one because of the fact that different conclusions may be in accordance with the same pair of premises.

4 Cut Elimination and Isomorphisms

4.1 Cut Elimination

Theorem 8. *Any PL proof* $\pi \vdash_p \Sigma$ *can be rewritten into a PL proof* $\pi' \vdash_p \Sigma$ *without cuts.*

Proof. Here we extend the proof already provided in [5] for the limited case of multiplicatives. Our proof is organized in two steps. At first we remark that cut-elimination for LL [9] implies that any PLL proof $\pi \vdash \alpha$ can be reduced into a PLL proof $\pi' \vdash \beta$ without cuts, such that $|\alpha| = |\beta|$. In other words, cut-elimination preserves multisets of formulas. The second step consists in showing that cut-elimination preserves permutative structures too. It is easy to check; we illustrate below just some key cases of symmetric reductions.

– Contraction/promotion.

$$\dfrac{\text{contr.}\,\dfrac{\vdash_p \Sigma,(\Gamma,?A),(\Delta,?A)}{\vdash_p \Sigma,(\Gamma,?A),(\Delta)} \qquad \dfrac{\vdash_0 ?\Xi,(?\Lambda,A^\perp)}{\vdash_0 ?\Xi,(?\Lambda,!A^\perp)}\,!}{\vdash_p \Sigma,?\Xi,(\Gamma,?\Lambda),(\Delta)}\,\text{cut} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash_p \Sigma,(\Gamma,?A),(\Delta,?A) \qquad \dfrac{\vdash_0 ?\Xi,(?\Lambda,A^\perp)}{\vdash_0 ?\Xi,(?\Lambda,!A^\perp)}\,!}{\vdash_p \Sigma,?\Xi,(\Gamma,?\Lambda),(\Delta,?A)}\,\text{cut} \qquad \dfrac{\vdash_0 ?\Xi,(?\Lambda,A^\perp)}{\vdash_0 ?\Xi,(?\Lambda,!A^\perp)}\,!}{\vdash_p \Sigma,?\Xi,?\Xi,(\Gamma,?\Lambda),(\Delta,?\Lambda)}\,\text{cut}}{\vdash_p \Sigma,?\Xi,(\Gamma,?\Lambda),(\Delta)}\,\text{contr.}$$

- Weakening/promotion.

$$\dfrac{\text{weak.}\,\dfrac{\vdash_p \Sigma,(\Gamma)}{\vdash_p \Sigma,(\Gamma,?A)} \qquad \dfrac{\vdash_0 ?\Xi,(?\Delta,A^\perp)}{\vdash_0 ?\Xi,(?\Delta,!A^\perp)}\,!}{\vdash_p \Sigma,?\Xi,(\Gamma,?\Delta)}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\vdash_p \Sigma,(\Gamma)}{\vdash_p \Sigma,?\Xi,(\Gamma,?\Delta)}\,\text{weak.}$$

- $\&/\oplus$.

$$\dfrac{\&\,\dfrac{\vdash_p \Sigma,(\Gamma,A) \qquad \vdash_p \Sigma,(\Gamma,B)}{\vdash_p \Sigma,(\Gamma,A\&B)} \qquad \dfrac{\vdash_q \Xi,(\Delta,B^\perp)}{\vdash_q \Xi,(\Delta,B^\perp\oplus A^\perp)}\,\oplus_R}{\vdash_{p+q} \Sigma,\Xi,(\Gamma,\Delta)}\,\text{cut} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\vdash_p \Sigma,(\Gamma,B) \qquad \vdash_q \Xi,(\Delta,B^\perp)}{\vdash_{p+q} \Sigma,\Xi,(\Gamma,\Delta)}\,\text{cut}.$$

4.2 Some Isomorphisms

Definition 9 (η-expansion). *We inductively define the function η which associates to each PL formula F a PL proof $\eta(F)$, η-expansion of F.*

- *For every atom A:* $\eta(A) = \eta(A^\perp) = \dfrac{}{\vdash_0 (A, A^\perp)}\,ax.$
- $\eta(1) = \dfrac{}{\vdash_0 (1)}\,1$
- $\eta(h) = \dfrac{}{\vdash_1 (h)}\,h$
- $\eta(\top) = \dfrac{}{\vdash_p \Sigma,(\Gamma,\top)}\,\top$
- *For every formula F: $\eta(F) = \eta(F^\perp)$ and, if $F = \psi(F_1,\dots,F_n)$ with ψ positive n-ary connective, then:*

$$\eta(F): \quad \dfrac{\dfrac{\eta(F_1) \qquad \dots \qquad \eta(F_n)}{\psi(F_1,\dots,F_n),F_1^\perp,\dots,F_n^\perp}\,\psi}{\psi(F_1,\dots,F_n),\psi^\perp(F_1^\perp,\dots,F_n^\perp)}\,\psi^\perp$$

Definition 10 (isomorphism). *Consider the proofs* $\lambda \circ \pi \vdash_0 (A^{\perp}, A)$ *and* $\pi \circ \lambda \vdash_0 (B^{\perp}, B)$ *obtained from the two proofs* $\pi \vdash_0 (A^{\perp}, B)$ *and* $\lambda \vdash_0 (B^{\perp}, A)$ *respectively by cutting* B *with* $B^{\perp}$ *and* A *with* $A^{\perp}$. $A \dashv\vdash B$ *is said to be an isomorphism if, and only if,* $\pi \circ \lambda = \eta(A)$ *and* $\lambda \circ \pi = \eta(B)$.

Theorem 9 (multiplicative isomorphisms). *The following equivalences are isomorphisms:*

$$(A \parr B) \parr C \dashv\vdash A \parr (B \parr C) \qquad \flat\flat A \dashv\vdash \flat A \qquad A \parr \flat B \dashv\vdash \flat B \parr A$$
$$A \parr \perp \dashv\vdash A \qquad \flat\perp \dashv\vdash \perp \qquad \flat(A \parr \flat B) \dashv\vdash \flat A \parr \flat B$$
$$\perp \parr A \dashv\vdash A \qquad \flat\hbar \dashv\vdash \hbar \qquad \flat(A \parr B) \dashv\vdash \flat(B \parr A)$$

Proof. We detail below just the case of $A \parr \perp \dashv\vdash A$.

$$\dfrac{\dfrac{1\,\dfrac{}{\vdash_0 (1)} \quad \dfrac{}{\vdash_0 (A^{\perp}, A)}\,\text{ax.}}{\vdash_0 (1 \otimes A^{\perp}, A)}\,\otimes \qquad \dfrac{\dfrac{\dfrac{}{\vdash_0 (A^{\perp}, A)}\,\text{ax.}}{\vdash_0 (A^{\perp}, A, \perp)}\,\perp}{\vdash_0 (A^{\perp}, A \parr \perp)}\,\parr}{\vdash_0 (1 \otimes A^{\perp}, A \parr \perp)}\,\text{cut} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{1\,\dfrac{}{\vdash_0 (1)} \quad \dfrac{}{\vdash_0 (A^{\perp}, A)}\,\text{ax.}}{\vdash_0 (1 \otimes A^{\perp}, A)}\,\otimes}{\vdash_0 (1 \otimes A^{\perp}, A, \perp)}\,\perp}{\vdash_0 (1 \otimes A^{\perp}, A \parr \perp)}\,\parr$$

$$\dfrac{\dfrac{1\,\dfrac{}{\vdash_0 (1)} \quad \dfrac{}{\vdash_0 (A^{\perp}, A)}\,\text{ax.}}{\vdash_0 (A, 1 \otimes A^{\perp})}\,\otimes \qquad \dfrac{\dfrac{\dfrac{}{\vdash_0 (A, A^{\perp})}\,\text{ax.}}{\vdash_0 (A, \perp, A^{\perp})}\,\perp}{\vdash_0 (A \parr \perp, A^{\perp})}\,\parr}{\vdash_0 (A, A^{\perp})}\,\text{cut} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{}{\vdash_0 (A, A^{\perp})}\,\text{ax.}$$

Example 6. $\flat\#A \dashv\vdash \flat A$ constitutes an example of an equivalence which is not an isomorphism too. The reduced proof on the right is not an η-expansion of $\flat\#A$, in fact it presents an application of the $\flat$ rule which interrupts a block of positive rules.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (A, A^\perp)}\,\text{ax.}}{\vdash_0 (\#A), (A^\perp)}\,\#}{\vdash_0 (\#A), (\#A^\perp)}\,\#}{\vdash_0 (\flat\#A, \#A^\perp)}\,\flat
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (A, A^\perp)}\,\text{ax.}}{\vdash_0 (A), (A^\perp)}\,\text{divide}}{\vdash_0 (A), (\flat A^\perp)}\,\flat}{\vdash_0 (A), (\#\flat A^\perp)}\,\#}{\vdash_0 (\flat A, \#\flat A^\perp)}\,\flat
\Big/ \vdash_0 (\flat\#A, \#\flat A^\perp)\ \text{cut}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (A, A^\perp)}\,\text{ax.}}{\vdash_0 (\#A), (A^\perp)}\,\#}{\vdash_0 (\#A), (\flat A^\perp)}\,\flat}{\vdash_0 (\#A), (\#\flat A^\perp)}\,\#}{\vdash_0 (\flat\#A, \#\flat A^\perp)}\,\flat
$$

Theorem 10 (additive and exponential isomorphisms). *All the equivalences listed in Theorem 2 are isomorphisms too.*

5 Future Work

A focussed version of the PLL calculus should be defined by extending that one already existing for PL [5].

Semantical issues (phase and denotational semantics) together with possible topological interpretations of proof-nets with additives are still waiting to be explored. The alternative approach to additives outlined in Subsection 3.3 would be useful in both these directions. In particular, in order to translate proof-nets into topological surfaces [13], we should be able to associate with each link of the net, its corresponding cell. Because of in proof-nets structural rules do not explicitly appear, the problem of associating a cell with a &-link requires to take in account the involved structural rules, exactly what the [&]-rule makes.

Concerning exponentials, the solution proposed in these pages should be considered as a "minimalist" one, i.e. in Section 3.2 we have showed that, in order to embed LL into PLL, it is sufficient to consider exponential formulas as essentially gathered into multisets associated with ordinary permutative sequents. However, it would be worthwhile to define exponentials in a genuine permutative way, really sharing the permutative structure with other formulas: a deductive system in which, for instance, rules for duplicating and absorbing formulas work taking in account their position. In this direction, the main obstacle to overcome consists in defining a deductive system which is still a conservative extension of LL. In our opinion, an in-deep investigation on the relations between permutative modalities and exponentials could be useful. Moreover, in terms of geometry, an extension of our structures including also non-orientable surfaces might offer a wider framework in which this kind of problems could be more properly placed.

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