A characterization of MALL hypercoherent semantic correctness

Paolo Tranquilli*

Dipartimento di Matematica – Università Roma Tre
Largo S. Leonardo Murialdo, 1 – 00146 Roma – Italy

Laboratoire PPS – Université Paris Diderot - Paris 7
Case 7014 – 75205 Paris – France

`tranquil@mat.uniroma3.it`

Abstract. We give a graph theoretical criterion on Multiplicative Additive Linear Logic (MALL) cut-free proof structures that exactly characterizes those whose interpretation is a hyperclique in Ehrhard's hypercoherent spaces. This criterion is strictly weaker than the one given by Hughes and van Glabbeek characterizing proof nets (i.e. desequentialized sequent calculus proofs). We thus also give the first proof of semantical soundness of hypercoherent spaces with respect to proof nets entirely based on graph theoretical trips, in the style of Girard's proof of semantical soundness of coherent spaces for proof nets of the multiplicative fragment of Linear Logic.

1 Introduction

Proof nets (PN) are the syntax of choice for unit-free multiplicative linear logic (MLL, [6]). The robustness of such a syntax consists in its ability to quotient proofs of MLL modulo inessential rule commutation in a canonical way. Each proof net represents in fact an equivalence class of sequential proofs, and such equivalence is validated by numerous semantic models. This is obtained by building proofs in a more general syntax, *proof structures* (PS), among which one may characterize the ones that come from sequent calculus proofs via a host of well established *correctness* criterions, where *correctness* here means *sequentializability*. The most famous ones are the long trip one due to Girard [6], and the Danos-Regnier one [4] of switching acyclicity and connectedness.

One of the reasons for such a successful account of MLL can be tracked to the very birth of Linear Logic. Since the beginning there was a tight pairing between the logical system and the semantic model that brought the intuitions necessary for its discovery: coherent spaces. One of the most spectacular clues of this is the interpretation of PNs in coherent spaces via the notion of *experiment*. As PNs live inside the more general world, also the interpretation is in fact defined on PSs in general, yielding simply a set[1].

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[1] In fact one may regard this interpretation as living in the category **Rel** of sets and relations, though this becomes less clear in the presence of the exponential modality !.

Clearly the first thing to check is the semantic soundness of such an interpretation: are PNs interpreted as objects of coherent spaces, i.e. cliques? If $[\![\ ]\!]$ stands for such an interpretation, chosen by assigning a coherent space to each type literal, the following theorem addresses such a question.

Theorem 1 (Girard, [6]). *If π on a sequent Γ is switching acyclic, then for any interpretation $[\![\ ]\!]$ we have that $[\![\pi]\!]$ is a clique in $[\![\Gamma]\!]$.*

As the sole role of switching connectedness is to invalidate the mix rule, which is accepted by coherent spaces, one drops it from the requirements.

There is now another question one can ask. As it makes sense to interpret a PS, it also makes sense to ask when such an interpretation is a clique. Such *semantic correctness*, in the case of MLL, turns out to be equivalent to the sequentializability one, as one has the following, reverse theorem.

Theorem 2 (Retoré, [16]). *If $[\![\pi]\!]$ is a clique in $[\![\Gamma]\!]$ for any interpretation $[\![\ ]\!]$, then π is switching acyclic.*

This strong pairing begins to break when one extends the system with units, or exponentials, or additives, which are the main concern of this work. On one side, the problem of providing unit-free multiplicative additive linear logic (MALL) a canonical syntax extending the good properties of the MLL one proved to be a longstanding question. A partial answer was given by Girard in [7] and a more satisfactory one was developed by Hughes and van Glabbeek in [8], a work which is one of our starting points. PSs are in this framework represented as sets of purely multiplicative structures, usually referred to as *slices* (see for example [9]), identified in this context by *linkings* (see Section 2 for more details). Again [8] provides a geometrical criterion, which we call the HvG one (page 14) characterizing sequentializable structures.

On the other hand, one would also like to extend the good semantic pairing of MLL to MALL. Coherent spaces are known to not provide the same results for MALL PSs as for MLL. In fact there is a PS, the Gustave one, which is the proof theoretical counterpart of the Gustave function G in the stable model of PCF. In the same way as G is an unsequentializable stable function, the Gustave PS which we will show in Figure 1 at page 7 is an incorrect structure which is interpreted by a clique, so that no analog of Theorem 2 is possible for MALL and coherent spaces.

The Gustave function G is however rejected by Bucciarelli and Ehrhard's strongly stable model [3], and starting from it Ehrhard developed in [5] a new model of LL extending the coherent one: the *hypercoherent spaces* (Section 2.2). One may then turn to such a model hoping for a better account of MALL. Semantic soundness clearly holds if one passes through the sequentialization theorem of [8], though a more direct proof might be desirable (we will in fact give it, by combining Proposition 17 and Theorem 11). As for the analog of Theorem 2, the Gustave PS is indeed rejected, but one stumbles anyway upon another counterexample [12], which we show in Figure 2 on page 7. It has been conjectured in [12] that such fracture between MALL syntax and hypercoherent semantic is due to the intrinsic unconnectedness of the counterexample.

Conjecture 3 (Pagani). If θ is a proof structure, and $\forall\lambda \in \theta\colon \lambda$ is switching connected, and $[\![\theta]\!]$ is a hyperclique for any interpretation $[\![\ ]\!]$, then θ is correct for sequentializability.

We decided to "factorize" the conjecture by first finding the criterion for semantic correctness, which we call *hypercorrectness* (Definition 5). This criterion exactly characterizes the structures which have a hyperclique as interpretation. This approach has much similarity to the work of Pagani on visible acyclic nets [11,13] in the framework of exponential LL. More from a distance, a similarity can be established with what happened in the study of models of PCF: once it was clear that Scott-continuous functions, or even stable ones, were not fully abstract for PCF, two directions were taken. One was to refine the models (from continuity to stability and from stability to strong stability), while the other, similar to what we do here, was to find which languages were fully abstract for these same models (parallel PCF for the continuous one [15] and stable PCF for the stable one [14]). One difference is that in our work and that of [11,13] one really finds a discerning *geometrical* criterion (something that has sense because of the presence of generally "incorrect" objects, PSs) corresponding to an *algebraic* one, apparently distant (hypercliques here, finitary relations in [13]). In MALL the other approach is the direction taken in [2], where a proof of full completeness is given by refining hypercoherent spaces via an operation of double glueing.

Returning to the conjecture, we set out to prove

1. for θ proof structure, θ is hypercorrect iff $[\![\theta]\!]$ is a hyperclique for any interpretation;
2. for θ proof structure with $\forall\lambda \in \theta\colon \lambda$ switching connected, θ is hypercorrect iff θ is correct for sequentializability.

We address here point 1, proving both sides of the equivalence in Theorems 11 and 15, and leave point 2 as a further conjecture. The criterion uses the notion of &-*oriented cycles*: contrary to what happens in sequentalizability criterions the orientation of paths counts. There are already many hints of such behaviour when relating to semantics. Apart from [11], in [2] for double glued hypercoherent spaces and [1] for concurrent games the result of full completeness is obtained by employing cycles where the orientation is decided by *jumps*, though the framework there is the one of Girard's non canonical proof nets. More recently, investigation on games semantics in [10] has as well brought to the fore an oriented interpretation of the acyclicity criterion in proof nets.

Outline. In Section 2 we define the standard notions appearing in this work. Next, in Section 3, we define hypercorrectness and prove the characterization. Finally in Section 4 we present some contour information and results.

2 The framework

We will here introduce the main actors involved in this work: *MALL proof structures*, *hypercoherence spaces* and *experiments*.

Given a denumerable set of type variables $\mathcal{V}$, unit-free MALL formulas are generated by the grammar

$$\mathcal{F} ::= \mathcal{V} \mid \mathcal{V}^{\perp} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \parr \mathcal{F} \mid \mathcal{F} \oplus \mathcal{F} \mid \mathcal{F} \,\&\, \mathcal{F},$$

with the linear negation $(\ )^{\perp}$ defined as usual by De Morgan dualities $(A \otimes B)^{\perp} := A^{\perp} \parr B^{\perp}$ and $(A \oplus B)^{\perp} := A^{\perp} \,\&\, B^{\perp}$. Connectives $\otimes/\parr$ are said to be **multiplicative**, while $\oplus/\&$ are **additive**. A sequent Γ is a multiset of formulas $A_1, \dots, A_n$.

We will identify a formula with its graph-theoretical representation as a syntactical tree, which has a distinguished root node (the **conclusion** of the formula), logical connectives as intermediate nodes (called **links**), and atomic formulas (of the form α or $\alpha^{\perp}$) as leaves. The term "node" will therefore indicate any of these parts, while among edges we will call the one above the root **terminal** and the ones above a given link **premises** to that link. Every edge has a subformula corresponding to it, and it is called its **type**. Different occurrences of nodes or edges will be noted by lowercase Latin letters. Two leaves are dual if their atomic formulas are dual. Sequents are likewise identified with their representation as syntactical forests, and their conclusions are the conclusions of the formulas in them. The tree structure naturally induces an (arborescent) order on links and edges, which we will denote by $\leq$, with conclusions being minimal. For nodes a, b connected by an edge e in Γ we will write $a \xrightarrow{e} b$ (resp. $a \xleftarrow{e} b$) if e is a premise of b (resp. a). We will omit any of a, b, e if it is of no importance, so that for example $\xrightarrow{e} b$ means simply "e is a premise of b".

2.1 MALL proof structures

We will now define cut-free MALL PSs, mostly following [8].

In the following let us fix a sequent Γ. An **axiom** is an unordered pair of dual leaves of Γ. Any set of axioms λ naturally defines a subforest of Γ which we denote by $\Gamma \upharpoonright \lambda$, by taking $\left(\bigcup \lambda\right)\downarrow$, the set of leaves in axioms of λ down-closed with respect to $\leq$, i.e. the subforest of Γ obtained by taking edges and links which have an axiom in λ above them. In $\Gamma \upharpoonright \lambda$ connectives are either binary or unary. We call λ a **linking** (on Γ) if axioms in λ are pairwise disjoint and $\Gamma \upharpoonright \lambda$ contains all conclusions of Γ, no unary multiplicative connectives $\otimes/\parr$ and no binary additive connectives $\oplus/\&$. The **slice** $\mathcal{G}_{\lambda}$ associated to a linking λ is the graph obtained from $\Gamma \upharpoonright \lambda$ by adding a new node for every axiom $\{a, b\}$ of λ with edges to the leaves a and b. By extending the notation, also these new nodes in $\mathcal{G}_{\lambda}$ are called axioms, and the new edges are premises to the leaves. The order $\leq$ is extended to $\mathcal{G}_{\lambda}$ by setting the axiom nodes and edges as greater than the leaves they connect (axioms are maximal).

Given Λ a set of linkings, we define $\Gamma \upharpoonright \Lambda := \bigcup_{\lambda \in \Lambda} \Gamma \upharpoonright \lambda$, where superposition is trivially defined as all is inside Γ. We define the sets $\&2(\Lambda)$ as the set of binary & connectives in $\Gamma \upharpoonright \Lambda$. For two linkings $\lambda_1, \lambda_2 \in \Lambda$ we use the notation $\lambda_1 \overset{w}{-} \lambda_2$ if either $\lambda_1 = \lambda_2$, or $\&2(\{\lambda_1, \lambda_2\}) = \{w\}$, and the notation $\lambda_1 \overset{w}{\not-} \lambda_2$ (λ_1 and λ_2 toggle w uniquely) if the equality does not hold.

A **&-resolution** G of Γ is a subforest of Γ obtained by erasing from it one branching (whether left or right) from each & in Γ. A linking λ **is on** a &-resolution G if $\Gamma \upharpoonright \lambda \subseteq G$, i.e. all axioms in λ are on leaves of G.

Definition 4 (Proof structures). *A PS on a sequent Γ is a set θ of linkings such that for every &-resolution G of Γ there exist a unique $\lambda \in \theta$ on G (**resolution** condition).*

2.2 Hypercoherent spaces

The first denotational semantics of Linear Logic were coherent spaces, [6], which in fact were the mathematical notion that gave the first intuitions for Linear Logic. Much later, Ehrhard introduces in [5] a refinement, the *hypercoherent spaces*, which we briefly present here.

A **hypercoherent space** $\mathcal{X}$ is given by a pair $(|\mathcal{X}|, \overset{\frown}{\smile}_{\mathcal{X}})$ where

- $|\mathcal{X}|$ is a set called the **web** of $\mathcal{X}$.
- $\overset{\frown}{\smile}_{\mathcal{X}}$, called the **hypercoherence** of $\mathcal{X}$, is a *predicate* $\overset{\frown}{\smile}_{\mathcal{X}} \subseteq \mathcal{P}^*_{<\omega}(|\mathcal{X}|)$, the finite non-empty subsets of the web of $\mathcal{X}$, which is *reflexive* in the sense that it contains the set of singletons $\mathcal{P}_{=1}(|\mathcal{X}|)$.

The hypercoherent space as subscript of the relation is omitted if no confusion is possible. Apart from $\overset{\frown}{\smile}$, one defines the following relations, from which $\overset{\frown}{\smile}$ can be in turn recovered: strict hypercoherence $\frown := \overset{\frown}{\smile} \setminus \mathcal{P}_{=1}(|\mathcal{X}|)$, hyperincoherence $\asymp := \mathcal{P}^*_{<\omega}(|\mathcal{X}|) \setminus \frown$ and strict hyperincoherence $\smile := \mathcal{P}^*_{<\omega}(|\mathcal{X}|) \setminus \overset{\frown}{\smile}$. The **hypercliques** of $\mathcal{X}$ are

$$\mathcal{H}(\mathcal{X}) := \{\, h \subseteq |\mathcal{X}| \mid \forall s \subseteq^*_{<\omega} h \colon\ \overset{\frown}{\smile} s \,\},$$

where $s \subseteq^*_{<\omega} h$ means s a finite non-empty subset of h.

All connectives of linear logic have a corresponding operation on hypercoherent spaces. We define here all of them but the exponential one which is of no interest here.

Dual: $\left|\mathcal{X}^\perp\right| := |\mathcal{X}|$, and $\overset{\frown}{\smile}_{\mathcal{X}^\perp} := \asymp_{\mathcal{X}}$.

Multiplicatives: $|\mathcal{X} \otimes \mathcal{Y}| = |\mathcal{X} \parr \mathcal{Y}| := |\mathcal{X}| \times |\mathcal{Y}|$, and given $s \subseteq^*_{<\omega} |\mathcal{X}| \times |\mathcal{Y}|$ we set

$$\overset{\frown}{\smile}_{\mathcal{X}\otimes\mathcal{Y}}\, s \iff \overset{\frown}{\smile}_{\mathcal{X}}\, \pi_0(s) \text{ and } \overset{\frown}{\smile}_{\mathcal{Y}}\, \pi_1(s),$$
$$\frown_{\mathcal{X}\parr\mathcal{Y}}\, s \iff \frown_{\mathcal{X}}\, \pi_0(s) \text{ or } \frown_{\mathcal{Y}}\, \pi_1(s),$$

with π_0 and π_1 the usual left and right projections.

Additives: $|\mathcal{X}_0 \oplus \mathcal{X}_1| = |\mathcal{X}_0 \,\&\, \mathcal{X}_1| := |\mathcal{X}_0| + |\mathcal{X}_1|$, the disjoint sum. We denote an element of such a disjoint sum as $x.i$, with $i = 0$ or $i = 1$ and $x \in |\mathcal{X}_i|$. Given $s \subseteq^*_{<\omega} |\mathcal{X}_0| + |\mathcal{X}_1|$, let $s_i := \{\, x \in |\mathcal{X}_i| \mid x.i \in s \,\}$. Then we set

$$\overset{\frown}{\smile}_{\mathcal{X}_0\oplus\mathcal{X}_1}\, s \iff s_i = \emptyset \text{ and } \overset{\frown}{\smile}_{\mathcal{X}_{1-i}}\, s_{1-i} \text{ for } i = 0 \text{ or } 1,$$
$$\overset{\frown}{\smile}_{\mathcal{X}_0\&\mathcal{X}_1}\, s \iff \text{either } s_0 \neq \emptyset \text{ and } s_1 \neq \emptyset, \text{ or } s_i = \emptyset \text{ and } \overset{\frown}{\smile}_{\mathcal{X}_{1-i}}\, s_{1-i} \text{ for } i = 0 \text{ or } 1.$$

Note therefore that if s_0 and s_1 are both non-empty, one automatically has $\frown_{\mathcal{X}_0\&\mathcal{X}_1} s$ and $\smile_{\mathcal{X}_0\oplus\mathcal{X}_1} s$ regardless of the elements of s, as it cannot be a singleton.

Clearly the operations defined above respect De Morgan's duality.

2.3 Experiments

The notion of experiment was developed by Girard in [6] to give a way to directly interpret multiplicative proof nets in coherent semantics, without passing through sequent calculus. The remainder of this section will be devoted to defining experiments on (cut-free) slices and PSs.

Suppose given an interpretation $[\![\ ]\!]$ on type variables, i.e. a mapping from type variables to hypercoherent spaces. It can be easily extended to all formulas A by induction, chasing down all connectives and applying the corresponding operation on hypercoherent spaces. Then the interpretation of a sequent $\Gamma = A_1, \dots A_n$ is $[\![\Gamma]\!] := \bigparr_{i=1}^{n} [\![A_i]\!]$. We disregard any problem of bracketing, and consider the web of $[\![\Gamma]\!]$ as made up of n-uples.

Given a linking λ on Γ, an **experiment** e on λ (notation $e : \lambda$) is a function that assigns to each axiom ℓ in λ of type $\alpha/\alpha^\perp$ an element $e(\ell) \in \left|[\![\alpha]\!]\right|$. This function can then be extended by induction to every edge f of type A in $\mathcal{G}_\lambda$, so that $e(d) \in \left|[\![A]\!]\right|$:

- if A is atomic, f has an axiom $\ell \in \lambda$ above it, and one sets $e(f) := e(\ell)$;
- if A is multiplicative, f has above it a $\otimes/\parr$ link with both of its premises f_0 and f_1, and one sets $e(f) := \big(e(f_0), e(f_1)\big)$;
- if A is additive, f has above it a $\oplus/\&$ with only one of its premises f_i ($i = 0$ for left, 1 o.w.), and one sets $e(f) := e(f_i).i$.

If c is a conclusion of Γ with a terminal edge f above it, we set $e(c) := e(f)$. If $c_1, \dots, c_n$ are the conclusions of Γ, then the **result** of the experiment e on λ is defined as $e(\lambda) := \big(e(c_1), \dots, e(c_n)\big) \in \left|[\![\Gamma]\!]\right|$. An experiment e on a PS θ is an experiment on any of its linkings λ, with $e(\theta) := e(\lambda)$. The interpretation of a PS is then given as

$$[\![\theta]\!] := \{\, e(\theta) \mid e \text{ experiment on } \theta \,\} \subseteq \left|[\![\Gamma]\!]\right|.$$

Given experiments $e_1, \dots, e_n$ on θ, if an edge d is in all $\mathcal{G}_{\lambda_i}$ where $e_i : \lambda_i$, then it makes sense to ask whether $\frown\{e_i(d)\}$ holds, obviously by taking as space the interpretation of the type of d.

2.4 Examples

The Gustave PS γ is presented in Figure 1, its five linkings shown one above the other. It is an unsequentializable structure, as all terminal $\oplus$s are binary, so no final $\oplus$ rule may be applied in sequent calculus. In fact HvG criterion (page 14) rejects such structure. While the interpretation of γ in coherent spaces is a clique, as coherence is checked on at most two slices at a time, $[\![\gamma]\!]$ in hypercoherent spaces is not a hyperclique.

Figure 2 shows the counterexample to hypercoherent semantic correctness being equivalent to sequentializability. The PS δ, whose linkings are shown in Figure 2(a), is not sequentializable as the final rule must be $\otimes$, however it cannot

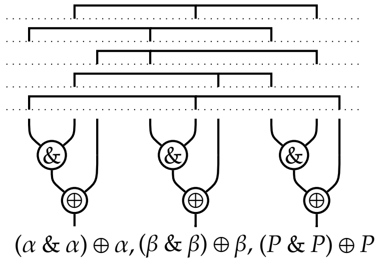
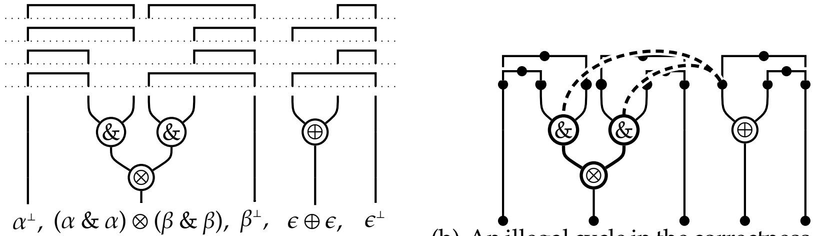


Fig. 1: The Gustave PS γ. P is short for $\alpha^\perp \otimes \beta^\perp$, and the linkings shown are a short graphical representation of the trivial and multiplicatively correct linking on $\alpha, \beta, \alpha^\perp \otimes \beta^\perp$.

split the $\epsilon \oplus \epsilon, \epsilon^\perp$ part of the context as it depends on both &s. Such a dependency is registered by *jumps*, which give an illegal cycle in such a structure, as shown in Figure 2(b). Notice that the cycle traverses the &s in opposite directions. The interpretation $[\![\delta]\!]$ is a hyperclique because of the way binary &s entail strict coherence whatever comes above them.



(a) The linkings of δ

(b) An illegal cycle in the correctness graph $\mathcal{G}_\delta^{\mathrm{HvG}}$ (defined on page 14)

Fig. 2: The proof structure δ: an unsequentializable structure such that $[\![\delta]\!]$ is a hyperclique.

3 The criterion

In this section we will define the criterion and then show the main results.

3.1 Hypercorrectness

We will define *correctness graphs* in the style of [8], with a substantial difference though. While jumps in [8] are drawn from the axioms, here we will draw them from the places where slices begin to differ from bottom to top. In any case a discussion on equivalent forms of this criterion will be made in Section 4.

Given a set of linkings Λ, the **pre-correctness graph** $\mathcal{G}'_\Lambda$, is obtained by superposing all slices of Λ, i.e. $\mathcal{G}'_\Lambda := \bigcup_{\lambda\in\Lambda} \mathcal{G}_\lambda$. The $\Gamma \upharpoonright \lambda$ part of each slice is inside $\Gamma \upharpoonright \Lambda$, so in fact $\mathcal{G}'_\Lambda$ is obtained by adding axioms to it. Superposition (i.e. identification) of axiom nodes and edges happens if and only they connect the same leaves. An edge or a node in $\mathcal{G}'_\Lambda$ is said to be **total** (for Λ) if it is in all slices, i.e. in $\bigcap_{\lambda\in\Lambda} \mathcal{G}_\lambda$, **partial** otherwise. An **additive contraction**, or simply contraction, is a total non-& node with partial premises, and their set is noted as contr(Λ). Contractions are in fact binary ⊕s and total leaves under partial axioms.

The **correctness graph** $\mathcal{G}_\Lambda$ is obtained from $\mathcal{G}'_\Lambda$ by adding new edges, called **jumps**, from a node $c \in \mathrm{contr}(\Lambda)$ to $w \in \&2(\Lambda)$ whenever

$$\exists \lambda_1, \lambda_2 \in \Lambda \mid \lambda_1 \overset{w}{\nsim} \lambda_2 \text{ and } c \in \mathrm{contr}(\{\lambda_1, \lambda_2\}).$$

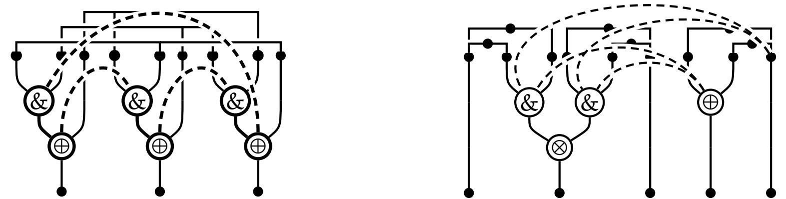
A jump j from c to w is denoted $c \overset{j}{\rightsquigarrow} w$. Jumps are considered partial, and premises to the & they jump to. Let tot(Λ) (resp. part(Λ)) denote the set of total (resp. partial) edges in $\mathcal{G}_\Lambda$.

A **path** ϕ in $\mathcal{G}_\Lambda$ is a finite non-repeating sequence e_i of edges such that e_i and e_{i+1} are *adjacent*, i.e. share a node, and such that also every shared node is not repeated. The order matters, i.e. paths are *oriented*. The *source* (resp. *target*) of ϕ is the unshared node of the first (resp. last) edge in ϕ. A **cycle** is a non-empty path whose source and target coincide. We identify ϕ with the edges and nodes it traverses, so that we may write $w \in \phi$ for a node w. Paths may also be denoted with the concatenated notations for premises and jumps, as for example in $\xrightarrow{e}\rightarrow x \leftarrow\overset{j}{\rightsquigarrow} w$. Note how some node or edge names may be omitted, and recall that jumps are considered also as premises, so that in the example e may be a jump. Also arrowheads will be omitted (as in $x \overset{e}{—} y$) if we do not want to specify whether the path is going upwards or downwards. For $e \in \phi$, write $\downarrow e \in \phi$ (resp. $\uparrow e \in \phi$) if e is traversed going down (resp. up), i.e. if d is traversed towards (resp. from) the node it is premise of. A path **bounces** on a node x if it contains a segment of shape $\rightarrow x \leftarrow$ or $\leftarrow x \rightarrow$. Cycles are to be considered bouncing on their source/target if their first and last edges are both immediately above or below it. A path or cycle is **switching** if it never bounces on a ⅋ or &.

Finally, a switching path ϕ is said to be **&-oriented** if exits partial on &s only and enters partial on contractions only, i.e. for every $\overset{e}{—} x \overset{f}{—}$ in ϕ, if $e \in \mathrm{part}(\Lambda)$ and $f \in \mathrm{tot}(\Lambda)$ (resp. viceversa) then $x \in \&2(\Lambda)$ (resp. $x \in \mathrm{contr}(\Lambda)$). Furtherly, two paths ϕ and ψ are said to be **bounce-compatible** if each time they totally bounce together, they do so in the same direction, i.e. whenever ϕ and ψ both bounce on the same total tensor or axiom x, taversing its adjacent edges a, b, then a, b appear in the same order in ϕ and ψ. A union of paths is said to be bounce-compatible if its paths are pairwise bounce-compatible.

Definition 5 (Hypercorrectness). *A proof structure θ is hypercorrect if for every $\Lambda \subseteq \theta$ and every bounce-compatible non-empty union of &-oriented cycles in $\mathcal{G}_\Lambda$, there is $w \in \&2(\Lambda)$ such that $w \notin S$.*

Note that for any λ, as the whole $\mathcal{G}_{\{\lambda\}} = \mathcal{G}_{\lambda}$ is total and lacks binary &s, this criterion entails the absence of switching cycles, i.e. multiplicative correctness of every linking. Revisiting the examples shown in Figures 1 and 2, we show in Figures 3(a) and 3(b) respecively one of their correctness graphs.



(a) The correctness graph of three linkings of the Gustave PS. Only three out of six jumps are shown. Note that the cycle could also be formed by going through axioms. However the cycle shown is strictly &-oriented (page 9). Axiom nodes are omitted.

(b) The correctness graph $\mathcal{G}_{\delta}$. The only way to form a cycle would be to bounce on the tensor, but that would not be a &-oriented one.

Fig. 3: Two examples of correctness graphs. The first one shows the rejection of the Gustave PS by the criterion, while the second structure is hypercorrect. Leaf nodes and axiom nodes are marked by •s.

3.2 Hypercorrectness implies hypercoherence

We will devote this section to the proof of Theorem 11, the analog of Theorem 1.

Let us fix in the following θ a PS on a sequent Γ. A set of linkings Λ is said to be **saturated** if for every $\lambda \in \theta \setminus \Lambda$, $\Lambda \cup \{\lambda\}$ has more binary &s than Λ. A &-oriented path or cycle ϕ is **strictly &-oriented** if it always descends on partial edges, i.e. if $e \in \phi$, $e \in \mathrm{part}(\Lambda)$, then $\downarrow e \in \phi$. Note that this implies not passing any partial axioms. The following are two basic lemmas needed for our proofs later.

Lemma 6. *For Λ saturated, every $c \in \mathrm{contr}(\Lambda)$ has a jump $c \rightsquigarrow$ in $\mathcal{G}_{\Lambda}$.* *→tech.app.*

Lemma 7. *If θ is hypercorrect and $\Lambda \subseteq \theta$ is saturated, then every non-empty bounce-compatible union S of strictly &-oriented cycles has a jump out of it, i.e. $\exists w \in \&2(\Lambda)$ and $c \in S \cap \mathrm{contr}(\Lambda)$ such that $c \rightsquigarrow w \in \mathcal{G}_{\Lambda}$.* *→tech.app.*

The following is the main lemma opening us the way for Theorem 11.

Lemma 8. *Let θ be a hypercorrect PS on a sequent Γ, $e_1, \dots, e_n$ experiments on θ, such that $\smile\{e_i(c)\}$ on a conclusion c. Then there exist in $\mathcal{G}_{\theta}$ a strictly &-oriented path ϕ starting from c and ending on a conclusion c' such that $\frown\{e_i(c')\}$.*

Proof. Consider Λ the minimal saturated set of linkings containing those on which experiments e_i are taken. By minimality binary &s are the same. From now on all paths will be taken in $\mathcal{G}_\Lambda \subseteq \mathcal{G}_\theta$. We will give a precise algorithm which will build the path ϕ. The base step of such an algorithm is the non-deterministic function next, taking as inputs a direction ϵ which can be $\uparrow$, $\downarrow$ and an edge $d \in \mathcal{G}_\Lambda$ such that

1. if $d \in \mathrm{part}(\Lambda)$ then $\epsilon = \downarrow$;
2. if $d \in \mathrm{tot}(\Lambda)$ and $\epsilon = \uparrow$, then $\smile\{e_i(d)\}$;
3. if $d \in \mathrm{tot}(\Lambda)$ and $\epsilon = \downarrow$, then $\frown\{e_i(d)\}$.

The output will be a direction ϵ' and an adjacent edge d' with the same properties and such that dd' is a path with $\epsilon d, \epsilon' d' \in dd'$. Let us define next by the three cases described above.

1. Let $\overset{d}{\to} x$. If $x \in \mathrm{part}(\Lambda)$, then $x \overset{d'}{\to}$ with $d' \in \mathrm{part}(\Lambda)$, and let $\mathrm{next}(\downarrow d) := \downarrow d'$. If $x \in \mathrm{tot}(\Lambda)$, then either $x \in \&2(\Lambda)$, in which case $x \overset{d'}{\to}$ and $\mathrm{next}(\downarrow d) := \downarrow d'$, or $x \in \mathrm{contr}(\Lambda)$. By Lemma 6, there is $x \overset{d'}{\leadsto}$, and we set $\mathrm{next}(\downarrow d) := \downarrow d'$.
2. Let $\overset{d}{\leftarrow} x$. If $x \in \mathrm{contr}(\Lambda)$, then proceed as the above case, by setting $\mathrm{next}(\uparrow d)$ to a jump from x. Otherwise let us define next by cases on the nature of x:
   - **axiom:** x is total, and $\overset{d}{\leftarrow} x \overset{d'}{\to}$. Set $\mathrm{next}(\uparrow d) := \downarrow d'$. The property is preserved as the value of the experiments on the two edges is the same and their types are dual;
   - **leaf or unary additive:** there is a unique $x \overset{d'}{\leftarrow}$, $d' \in \mathrm{tot}(\Lambda)$ with the same incoherence of d, so we set $\mathrm{next}(\uparrow d) := \uparrow d'$;
   - **binary with:** this case is impossible, as if $\overset{d}{\leftarrow} x$ then automatically $\frown\{e_i(d)\}$, as &s binary in Λ are also binary for the linkings where the experiments are taken;
   - **par:** we have $\overset{d_0}{\to} x \overset{d_1}{\leftarrow}$ the two premises of x, and as $\smile \{e_i(d)\}$ and $\{e_i(d_j)\} = \pi_j\{e_i(d)\}$, we have $\smile\{e_i(d_j)\}$ on both, and may set $\mathrm{next}(\uparrow d) := \uparrow d_j$ for any of the two js;
   - **tensor:** we have $\overset{d_0}{\to} x \overset{d_1}{\leftarrow}$, and as $\frown\{e_i(d)\}$, one of the two projections $\{e_i(d_j)\}$ must be strictly hyperincoherent, and we may set $\mathrm{next}(\uparrow d) := \uparrow d_j$ with such a j.
3. Let $\overset{d}{\to} x$. We have that x and all its adjacent edges are total, so x cannot be an axiom, a contraction or a binary &. Again, let us proceed by cases.
   - **leaf or unary additive:** $x \overset{d'}{\to}$, and trivially we can set $\mathrm{next}(\downarrow d) := \downarrow d'$;
   - **par:** $\overset{d}{\to} x \overset{d'}{\to}$, and as $\frown\{e_i(d)\}$, then $\frown\{e_i(d')\}$, and we set $\mathrm{next}(\downarrow d) = \downarrow d'$;
   - **tensor:** let $\overset{d}{\to} x \overset{d'}{\leftarrow}$, d' the other premise of x, and $x \overset{d''}{\to}$; if $\frown\{e_i(d'')\}$, then set $\mathrm{next}(\downarrow d) := \downarrow d''$; otherwise, necessarily $\smile\{e_i(d')\}$, and we may set $\mathrm{next}(\downarrow d) := \uparrow d'$.

A path is **admissible** if it is built by an iteration of next, with its first edge either a terminal one or in turn an output of next.

Fact 9. *If ϕ and ψ are admissible then their composition $\phi :: \psi$ is admissible, and all admissible paths are strictly &-oriented and bounce-compatible between them. In particular, an admissible path ending on one of its nodes forms a strictly &-oriented cycle.* *→tech.app.*

Another non-deterministic function we will use is jmp, which takes as input a union S of admissible cycles (therefore a bounce-compatible union of &-oriented cycles) and gives $\downarrow j$, where j is a jump out of S as existing by Lemma 7. Note that that all jumps are selectable by next: they are therefore admissible, and may be appended to an admissible path preserving the property.

Finally, let W and S be variables for sequences of binary &s and unions of admissible cycles. W_j (resp. S_j) will denote the j-th element of W (resp. S), with W starting from 1 and S from 0, and both ending in k (we will always use k for the size of W). The algorithm will build an admissible ϕ so that at all times W are the &s in ϕ which are not in any cycle of S. In a way W_i will be "in between" S_{i-1} and S_i (W_i will be generated by jmp(S_{i-1})). Also, the algorithm will make it so that &s in W do not appear in $\bigcup S_j$ and viceversa, and that all &s touched at some time by ϕ will be either in W or in $\bigcup S_j$.

The need for such a structure may be hinted by a schematic example[2]. The aim is that starting from the conclusion of the hypothesis the path eventually ends on another one. Suppose that following next we end up in a cycle. Applying jmp to it, we can backtrack and jump to a & w outside it and keep going. Now suppose the path cycles again, intersecting itself *after* w. If we applied jmp to the union of both cycles, it may answer the same jump to w it told before, and it would be useless. In such a case we have to apply jmp to the second cycle only. If then at a certain point we end up on ϕ before w then we may collapse the three cycles into a union and apply jmp to it without risking a useless answer: we may say that w is somehow "burnt" in this process (it gets erased from W).

Going back to the preliminary description of the algorithm, every time ϕ arrives to a node $x \notin \phi$, we store into it ϕ itself as it is at that moment, calling it the *history* of x. We are now ready to present the whole algorithm. Recall that by hypotheses there is a conclusion c such that $\smile\{e_i(c)\}$ and we can apply next to $\uparrow f$ where f is the terminal edge above c. The target of ϕ is $\mathrm{t}(\phi)$.

1. Start by setting $\phi := f$, $\epsilon d := \uparrow f$, $W := \langle\rangle$, $S := \langle\emptyset\rangle$ ($k := 0$).
2. Repeat. . .
   (a) If $\mathrm{t}(\phi) \in \bigcup S_j$ then $\mathrm{t}(\phi) \in \chi$ with χ a cycle. Let ψ be the smallest portion of χ that starting from x crosses ϕ again. $\psi = \langle\rangle$ if $\mathrm{t}(\phi) \in \phi$, and $\psi = \chi$ if χ does not intersect ϕ elsewhere. Set $\phi := \phi :: \psi$ (note that the following condition will be automatically satisfied).
   (b) If $\mathrm{t}(\phi) \in \phi$ then let χ be the cycle thus formed, and do the following steps. . .
      i. Let i be such that W_i is the last W_j strictly before $\mathrm{t}(\phi)$ in ϕ if one exists, $i := 0$ otherwise (note χ contains all W_j with $j > i$).

---

[2] Unfortunately, examples where one can really see the finesse of the algorithm are too complex and big to give them in detail.

ii. $S_i := \bigcup_{j=i}^{k} S_j \cup \chi$, and erase from W and S all subsequent elements (in fact, set $k := i$).

iii. $\epsilon d := \mathrm{jmp}(S_i) = \mathrm{jmp}(S_k)$, and let $c \overset{d}{\rightsquigarrow} w$ (note that $w \notin S_k$). Set ϕ to the history of c, and then append d to it.

(c) . . . else, do the following.

i. If $\mathrm{t}(\phi) \in \&2(\Lambda)$, then set $W := W :: \mathrm{t}(\phi)$ and $S := S :: \emptyset$ (and in fact $k := k + 1$).

ii. $\epsilon d := \mathrm{next}(\epsilon d)$ and $\phi := \phi :: d$.

3\. . . . until $t(\phi)$ is a conclusion.

Fact 10. *The algorithm shown above always terminates.* →*tech.app.*

Proof (sketch). One shows that the following measure strictly decreases for lexicographic ordering:

$$\mu := \left(\# \&2(\Lambda) - \# \&2\left(\bigcup S_j\right) - k, \# \&2(\Lambda) - \# \&2(S_k \cup \{\mathrm{t}(\phi)\}), |\mathcal{G}_\Lambda| - \left|\phi\right|\right)$$

where $\&2(T) := \&2(\Lambda) \cap T$ and the size $|\ |$ counts the edges. The component μ_1 decreases strictly in step 2(c)i, else μ_2 does it in step 2(b)iii, else μ_3 does it in step 2(c)ii.

Therefore the lemma is proved: if $\xrightarrow{f'} c'$ is the conclusion on which ϕ ends, then $\downarrow f' \in \phi$, and by the properties of next, $\smile\{e_i(f')\}$. □

Theorem 11. *If θ is a hypercorrect PS on a sequent Γ, then $\llbracket\theta\rrbracket$ is a hyperclique in $\llbracket\Gamma\rrbracket$ for every interpretation $\llbracket\ \rrbracket$.*

Proof. Let $\llbracket\ \rrbracket$ be any interpretation, and let $c \subseteq^{*}_{<\omega} \llbracket\theta\rrbracket$. By definition $c = \{\, e_1(\theta), \dots, e_n(\theta) \,\}$. Suppose $\neq c$, i.e. c is not a singleton. Then there is a conclusion c of Γ such that $\neq\{e_i(c)\}$. Either $\frown\{e_i(c)\}$ which implies $\frown\{e_i(\theta)\}$, or else $\smile\{e_i(\theta)\}$, which by above Lemma 8 entails the existence of another conclusion c' with $\frown\{e_i(c')\}$ which also implies $\frown\{e_i(\theta)\}$. In any case, coherence of c is proved, and therefore $\llbracket\theta\rrbracket$ is a hyperclique. □

3.3 Hyperincorrectness implies hyperincoherent

This section will prove Theorem 15, the analog of Retoré's theorem. This will be done using the following lemma, a sort of dual to Lemma 8.

Lemma 12. *Let θ be a PS, c_1 and c_2 two of its conclusions, $\Lambda \subseteq \theta$, and $\phi_1, \dots, \phi_k$ pairwise bounce-compatible and &-oriented paths in $\mathcal{G}_\Lambda$ such that every ϕ_i either is a cycle or a path starting from c_1 and ending in c_2, with at least one of the second kind and $\&2(\Lambda) \subseteq \bigcup_j \phi_j$. Then there exist an interpretation $\llbracket\ \rrbracket$ and experiments $e_1, \dots, e_n$ such that $\smile\{e_i(c_1)\}$, and $\asymp\{e_i(c)\}$ for every conclusion $c \neq c_1, c_2$.*

Proof. The interpretation we define is $\llbracket\ \rrbracket_{\mathcal{X}}$, which maps all literals to a space $\mathcal{X}$. We give a sketch on how to define such a space and the experiments.

Fact 13. *There is a hypercoherent space* $\mathcal{X}$ *and experiments* $e_1, \dots, e_n$ *relative to* $[\![\ ]\!]_{\mathcal{X}}$ *with* $n = \max(\#\Lambda, 2)$ *such that*

(E1) for each total axiom ℓ *such that* $\exists \phi_j \colon \ell \in \phi_j$*, if* $\uparrow a \in \phi_j$ *is one of the axiom edges of* ℓ *then* $\smile\{e_i(a)\}$*;*
(E2) for each other total axiom $=\{e_i(\ell)\}$*;*
(E3) for each contraction leaf x*, if* f *is the edge below it then* $\smile\{e_i(f)\}$*.* *→tech.app.*

Proof (sketch). The aim is to define an experiment e_i on each λ_i (one sets $\lambda_1 = \lambda_2$ in the degenerate case $\#\Lambda = 1$). E1 can be easily achieved by bounce-compatibility if $\mathcal{X}$ contains a strict coherent pair and a stric incoherent one, by making the experiments give one or the other depending on the direction of the path wrt duality. The problems come from E3, as there may be partial axioms linking two contractions. These are solved by building an ad-hoc space $\mathcal{X}$ having as web such partial axioms plus three distinguished points `c`, `i`, `n` (for coherent, incoherent and neutral).

Fact 14. *From preperties E1–3 of the experiments of Fact 13 we can deduce the following ones:*

(P1) for every $d \in \mathrm{tot}(\Lambda)$*, if* $\exists d' \geq d$ *and* j *such that* $d' \in \phi_j$*, then* $\neq \{e_i(d)\}$*, i.e. it is not a singleton;*
(P2) for every $d \in \mathrm{tot}(\Lambda)$*, if* $\forall j \colon \downarrow d \notin \phi_j$*, i.e.* d *is not traversed downward by any* ϕ_j*, then* $\asymp\{e_i(d)\}$*.* *→tech.app.*

Proof (sketch). The proof of P2 is done by an easy induction on the type of the edge, by regarding what happens above it. In the tensor case bounce compatibility plays a central role in order to apply i.h. Binary additive cases are trivial: for & the hypothesis never applies, for $\oplus$ the thesis always applies.

These two properties immediately entail the result, as if f_1 is the terminal edge above c_1, then by hypotheses $\forall j \colon \downarrow f_1 \notin \phi_j$ and $\exists j \mid f \in \phi_j$, so by P1 and P2 combined we have $\smile\{e_i(c_1)\}$. Again by hypotheses for every $c \neq c_1, c_2$, if f is the terminal edge above c we have $\downarrow f \notin \phi_j$ for any j, so that P2 gives the rest of the result. ☐

With the above lemma at hand, we can easily prove the second main theorem of this work. Note how we weaken the hypothesis without asking the resolution condition (Definition 5).

Theorem 15. *If* θ *is a set of linkings, and for every* $[\![\ ]\!]$ *we have that* $[\![\theta]\!]$ *is a hyperclique, then* θ *is hypercorrect.* *→tech.app.*

Proof (sketch). One shows that if θ is invalidated by a union S of cycles in $\mathcal{G}_\Lambda$ then one can build hyperincoherent experiments on Λ, by an induction on the number of links in $\mathcal{G}_\Lambda$. One deconstructs $\mathcal{G}_\Lambda$ one terminal link at a time, until one arrives to break S by taking out a $\otimes$. This makes the structure fall into the hypotheses of Lemma 12, and the result easily follows by the law of hypercoherence on $\otimes$.

4 Compendium

Equivalent criterions. We define the **partial contractions** as the set $\mathrm{pcontr}(\Lambda) := \bigcup_{\lambda,\mu\in\Lambda} \mathrm{contr}(\{\lambda,\mu\})$, and the graph $\mathcal{G}^{\mathrm{p}}_{\Lambda}$ with jumps from $\mathrm{pcontr}(\Lambda)$ with the same rule. In fact $\mathrm{contr}(\Lambda) \subseteq \mathrm{pcontr}(\Lambda)$ and $\mathcal{G}^{\mathrm{p}}_{\Lambda} = \bigcup_{\lambda,\mu\in\Lambda} \mathcal{G}_{\{\lambda,\mu\}}$, with jumps identified iff they have same target and same source.

Proposition 16. *Hypercorrectness (Definition 5) is equivalent to having any number of its parts substituted in the following ways.*

1. *bounce-compatibility can be strengthened with plain compatibility, i.e.* ϕ *and* ψ *are compatible iff whenever* $e \in \phi \cap \psi$ *then* ϕ *and* ψ *traverse* e *in the same direction;*
2. *&-orientedness can be strengthened with strict &-orientedness (defined on page 9);*
3. *the condition asking the presence of* $w \in \&2(\Lambda)$ *outside* S *can be strengthened to be the presence of* $w \in \&2(\Lambda) \cap \mathrm{tot}(\Lambda)$ *outside* S*,*
4. *the graph* $\mathcal{G}^{\mathrm{p}}_{\Lambda}$ *can replace* $\mathcal{G}_{\Lambda}$*.* *→tech.app.*

The presence of all these variants is somewhat disquieting. Our guess:

- in the presence of cuts some of these equivalent forms might not apply (see bottom paragraph of this section);
- it might be that one should not employ unions of cycles but single ones.

In fact the second point is conjectured for the sequentializability criterion in [8, Single Switching Cycle Conjecture, 4.22].

Comparison with sequentializability. For $\Lambda \subseteq \theta$, let $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$ be the correctness graph of Λ as defined in [8]. The only difference is where jumps are drawn from. In $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$ one adds to $\mathcal{G}'_{\Lambda}$ a jump $a \rightsquigarrow w$ for every a leaf such that there are $\lambda \overset{w}{\not\smile} \mu$ with an axiom $\ell \in \lambda \setminus \mu$ above a. Then the MIX-sequentializability criterion shown in [8] is

$$\text{(HvG)} \qquad \forall \Lambda \subseteq \theta \colon \exists w \in \&2(\Lambda) \mid\ w \text{ is in no switching cycle of } \mathcal{G}^{\mathrm{HvG}}_{\Lambda}.$$

We can directly infer hypercorrectness from the HvG criterion.

Proposition 17. *Every sequentializable PS is hypercorrect.* *→tech.app.*

Proof (sketch). Without passing via the Sequentialization theorem of [8], interpreting with a hyperclique and applying 15, one can give a more direct proof by translating each cycle in $\mathcal{G}_{\Lambda}$ into one in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$ containing the same &s. This is done by substituting jumps in $\mathcal{G}_{\Lambda}$ with certain paths mounting up the edges above the contraction, until arriving to a leaf jumping to the same & in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$.

There is also a restating of the HvG criterion using jumps of $\mathcal{G}^{\mathrm{p}}_{\Lambda}$, though the not so trivial proof of equivalence is beyond our scope here, and will be detailed in future work. This may lead to employ "cleaner" correctness graphs, having in general fewer jumps, and could possibly open the way for a richer syntax (non-η-expanded proof nets, second order and/or exponential boxes).

Cut reduction. The study of the computational significance of hypercorrectness is left for future work. The main point is to give a good definition of jumps in the presence of cuts and prove stability under cut reduction. Already in the context of the HvG criterion, the latter is very delicate. One will probably have to tweak the criterion via its equivalent versions, and also proving the Single Switching Cycle Conjecture might help. Clearly this issue is now the most important one for this criterion. One expects, as happens for visible acyclicity in [13] or for the PCF variants of [15,14], that such a semantic correctness is not just a static characterization but also has a dynamic content, possibly shedding light on new computational aspects of both syntax and semantics.

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Technical appendix

Proofs of Section 3.2

Given a subset $\Lambda \subseteq \theta$ and $w \in \&2(\Lambda)$, the set Λ^w denotes those linkings in Λ that *do not* choose right on w, i.e. $\lambda \in \Lambda$ such that the right premise of w is not in $\Gamma \upharpoonright \lambda$. The following are properties of saturated sets and Λ^w we use, already pointed out in [8]. Let Λ be a saturated set of linkings. Then

(S1) Λ^w is saturated;
(S2) for every $\lambda \in \Lambda$ there exists a unique $\lambda_w \in \Lambda^w$ with $\lambda \overset{w}{-} \lambda_w$;
(S3) for every $\lambda, \mu \in \Lambda$, if $\lambda \overset{x}{-} \mu$ then $\lambda_w \overset{x}{-} \mu_w$.

The following lemma is used in both subsequent proofs.

Lemma 18. *For Λ saturated, $w \in \&2(\Lambda)$, $c \in \mathrm{contr}(\Lambda)$, e any partial edge of c in $\mathcal{G}_\Lambda$, if $e \notin \mathcal{G}_{\Lambda^w}$ then $c \rightsquigarrow w$ in $\mathcal{G}_\Lambda$.*

Proof. Let us first settle the case in which e is not a jump. Suppose first c is a leaf, so $\overset{e}{\leftarrow} a$ are an axiom edge and node. As they disappear in $\mathcal{G}_{\Lambda^w}$, there must be $\lambda \in \Lambda \setminus \Lambda^w$ so that (identifying axiom nodes with axiom pairs) $a \in \lambda$. By S2, take λ_w: we have $\lambda_w \overset{w}{\not-} \lambda$, and as c is total λ_w has an axiom over c which cannot be a, so $c \in \mathrm{contr}(\{\lambda, \lambda_w\})$ and $c \rightsquigarrow w$. Almost the same reasoning can be done for c a binary plus.

Now suppose e is a jump $c \overset{e}{\rightsquigarrow} x$. There are $\lambda \overset{x}{\not-} \mu$ with $c \in \mathrm{contr}(\{\lambda, \mu\})$, and if we take λ_w and μ_w we know by S3 that $\lambda_w \overset{x}{-} \mu_w$. Now, as e is not in $\mathcal{G}_{\Lambda^w}$, c cannot be a contraction in $\{\lambda, \lambda_w\}$ and $\{\mu, \mu_w\}$, so these two pairs have the same edges above c (whether it is a $\oplus$ or a leaf), and therefore c is a contraction in $\{\lambda_w, \mu_w\}$. Having a contraction implies also inequality, so $c \rightsquigarrow w$. Here again it is important that as c is total, no linking can avoid making a choice on it. ☐

Lemma 6. *For Λ saturated, every $c \in \mathrm{contr}(\Lambda)$ has a jump $c \rightsquigarrow$ in $\mathcal{G}_\Lambda$.*

Proof. Let us reason by induction on the cardinality of $\&2(\Lambda)$.

If $\#\&2(\Lambda) = 0$ then $\Lambda = \{\lambda\}$ and there exists no contraction.

Otherwise, consider any $w \in \&2(\Lambda)$ and Λ^w. If $c \in \mathrm{contr}(\Lambda^w)$ we may apply induction hypothesis (as by S1 Λ^w is saturated) and conclude (as $\mathcal{G}_{\Lambda^w} \subseteq \mathcal{G}_\Lambda$). If not, then necessarily we are in the hypotheses of Lemma 18, so that $c \rightsquigarrow w$. ☐

Lemma 7. *If θ is hypercorrect and $\Lambda \subseteq \theta$ is saturated, then every non-empty bounce-compatible union S of strictly &-oriented cycles has a jump out of it, i.e. $\exists w \in \&2(\Lambda)$ and $c \in S \cap \mathrm{contr}(\Lambda)$ such that $c \rightsquigarrow w \in \mathcal{G}_\Lambda$.*

Proof. By induction on $\#\&2(\Lambda)$. If $\#\&2(\Lambda) = 0$ there cannot be any cycle.

Otherwise, by hypercorrectness, there is $w \in \&2(\Lambda)$, $w \notin S$. Consider Λ^w: if S still exists in $\mathcal{G}_{\Lambda^w}$ then we may apply induction hypothesis, as Λ^w is saturated by S1. If not, there is $e \in S$ such that $e \notin \mathcal{G}_{\Lambda^w}$, so e is necessarily partial. Consider the cycle ϕ containing e, then it contains $\xrightarrow{e}$ by strictness. Backtracking on ϕ

from e means to go up in the partial part of $\mathcal{G}_\Lambda$ through edges that also are not in $\mathcal{G}_{\Lambda^w}$. By strictness, as no axiom can be traversed, one arrives to backtrack on a jump $c \overset{j}{\rightsquigarrow}\subseteq \phi$, which also cannot be in $\mathcal{G}_{\Lambda^w}$. By Lemma 18, $c \rightsquigarrow w$, ending the proof. ☐

Fact 9. *If ϕ and ψ are admissible then their composition $\phi :: \psi$ is admissible, and all admissible paths are strictly &-oriented and bounce-compatible between them. In particular, an admissible path ending on one of its nodes forms a strictly &-oriented cycle.*

Proof. Take ϕ and ψ admissible and composable paths, with ϵd and $\epsilon' d'$ their last and first edge respectively, sharing the node x. Clearly d' cannot be terminal (otherwise composition would be impossible), therefore $\epsilon' d' = \mathrm{next}(\epsilon'' d'')$. However also $\epsilon' d' = \mathrm{next}(\epsilon d)$ by eventually making a different choice in the definition of next: this can be seen case by case, as given x the possible inputs to it and outputs out of it are in fact fixed by partiality and hypercoherence, regardless of the actual input of next is. So, as the begininng of ψ is also next of the end of ϕ, the composition is admissible.

Now, the fact that admissible paths are switching and strictly &-oriented can be directly deduced from the definition of next: no bounce is done on ⅋s and &s, the partial part can only be entered with jumps from contractions, and exited only on binary &s, and partial edges are traversed downward by definition. Every total bounce is either on an axiom or on a tensor, in the latter case only when the experiments are strictly incoherent on it. In both cases, the direction of the bounce is fixed a priori by the coherence-incoherence of the experiments on the two edges, so admissible paths are also bounce-compatible. In fact, by regarding all cases and checking all hypercoherences, one may see that all admissible paths are compatible, i.e. traverse all common edges in the same direction.

Finally, if an admissible path ϕ ends on a node $x \in \phi$, and d and d' are respectively the last edge of ϕ and the first one after x, then they, taken as singletons oriented in the same direction of ϕ, are composable admissible paths. Thus dd' is admissible, therefore it is switching, and the segment of ϕ after x is a (striclty &-oriented) switching cycle. ☐

The algorithm

1. Start by setting $\phi := f$, $\epsilon d := \uparrow f$, $W := \langle\rangle$, $S := \langle\emptyset\rangle$ ($k := 0$).
2. Repeat. . .
   (a) If $\mathrm{t}(\phi) \in \bigcup S_j$ then $\mathrm{t}(\phi) \in \chi$ with χ a cycle. Let ψ be the smallest portion of χ that starting from x crosses ϕ again. $\psi = \langle\rangle$ if $\mathrm{t}(\phi) \in \phi$, and $\psi = \chi$ if χ does not intersect ϕ elsewhere. Set $\phi := \phi :: \psi$ (note that the following condition will be automatically satisfied).
   (b) If $\mathrm{t}(\phi) \in \phi$ then let χ be the cycle thus formed, and do the following steps. . .
      i. Let i be such that W_i is the last W_j strictly before $\mathrm{t}(\phi)$ in ϕ if one exists, $i := 0$ otherwise (note χ contains all W_j with $j > i$).

        ii. $S_i := \bigcup_{j=i}^{k} S_j \cup \chi$, and erase from W and S all subsequent elements (in fact, set $k := i$).
        iii. $\epsilon d := \mathrm{jmp}(S_i) = \mathrm{jmp}(S_k)$, and let $c \overset{d}{\rightsquigarrow} w$ (note that $w \notin S_k$). Set ϕ to the history of c, and then append d to it.
    (c) …else, do the following.
        i. If $\mathrm{t}(\phi) \in \&2(\Lambda)$, then set $W := W :: \mathrm{t}(\phi)$ and $S := S :: \emptyset$ (and in fact $k := k + 1$).
        ii. $\epsilon d := \mathrm{next}(\epsilon d)$ and $\phi := \phi :: d$.
3. …until $t(\phi)$ is a conclusion.

Fact 10. *The algorithm shown above always terminates.*

Proof. Let us now prove termination of this algorithm. We do it by presenting the following strictly decreasing measure:

$$\mu := \left(\#\,\&2(\Lambda) - \#\,\&2(\bigcup S_j) - k, \#\,\&2(\Lambda) - \#\,\&2(S_k \cup \{\mathrm{t}(\phi)\}), |\mathcal{G}_\Lambda| - \left|\phi\right|\right)$$

where $\&2(T) := \&2(\Lambda) \cap T$ and the size $|\ |$ counts the edges. Let μ_1, μ_2 and μ_3 denote the three components. Then

1. If step 2(c)i applies, then clearly μ_1 decreases by one, otherwise it remains constant. In fact, other changes to W and S are made only in the block following step 2b. There the cycle χ is such that it contains no new & with respect to W and $\bigcup S_j$. As χ contains all &s W_j with $j > i$, when we move it to the pile of S_js and erase all W_js with $j > i$ (so that $k := i$) we in fact keep μ_1 constant.
2. If μ_1 remains constant, then the union S_k can only increase, and μ_2 can only change in steps 2a and 2b. It cannot change in step 2(c)ii, as if it happens it means that the old $\mathrm{t}(\phi)$ was a binary &, therefore step 2(c)i had to apply and μ_1 had to decrease. Now, there are two cases possible. If at the beginning of the repeat cycle $\#\,\&2(S_k \cup \{\mathrm{t}(\phi)\}) = \#\,\&2(S_k)$, then μ_2 may decrease in steps 2a and 2(b)ii and will surely decrease in step 2(b)iii as $\mathrm{t}(\phi)$, the & selected by jmp, is not in S_k. If at the beginning $\#\,\&2(S_k \cup \{\mathrm{t}(\phi)\}) = \#\,\&2(S_k) + 1$, then at the start $\mathrm{t}(\phi) \in \&2(\Lambda)$ and $\mathrm{t}(\phi) \notin S_k$. In fact μ_2 may increase in step 2a (the new $\mathrm{t}(\phi)$ may be in S_k or not a &), however regaining in step 2(b)ii (as the old $\mathrm{t}(\phi)$ is inside the cycle χ fused into S_k), and then again surely strictly decreasing in step 2(b)iii.
3. If μ_1 and μ_2 remain constant, then necessarily no steps in 2a, 2b or 2(c)i apply, and step 2(c)ii strictly decreases μ_3. □

Proofs of Section 3.3

Fact 13. *There is a hypercoherent space $\mathcal{X}$ and experiments $e_1, \dots, e_n$ relative to $[\![\ ]\!]_{\mathcal{X}}$ with $n = \max(\#\Lambda, 2)$ such that*

(E1) for each total axiom ℓ such that $\exists \phi_j : \ell \in \phi_j$, if $\uparrow a \in \phi_j$ is one of the axiom edges of ℓ then $\smile\{e_i(a)\}$;

(E2) for each other total axiom $=\{e_i(\ell)\}$*;*
(E3) for each contraction leaf x*, if* f *is the edge below it then* $\smile\{e_i(f)\}$*.*

Proof. If we take contraction leaves as nodes and axioms between them in Λ as edges we form a bipartite unoriented graph A. Bipartition is set by the duality of the atomic types of the contractions. Given a contraction leaf x, let $A(x)$ be the set of edges of A in x, and let $E(A)$ be the set of all edges of A. Clearly A may contain also isolated nodes, i.e. contractions x not connected to other contractions, where $A(x) = \emptyset$. It is important that the only other case in which $x \neq y$ and $A(x) = A(y)$ is when x and y are connected by a single axiom and are not connected to anything else ($A(x) = A(y)$ is a singleton). If in fact $A(x)$ is not a singleton, then there can be only one node (x itself) to which all $\ell \in A(x)$ are connected, otherwise superposition identifes axioms. Let $\mathcal{X}$ be the hypercoherent space given by

- web $|\mathcal{X}| := E(A) + \{\mathtt{c}, \mathtt{i}, \mathtt{n}\}$ (which stand for coherent, incoherent and neutral);
- hypercoherence, given $s \subseteq^{*}_{<\omega} |\mathcal{X}|$, $\neq s$, defined by

$$\begin{aligned} \frown s :\Longleftrightarrow\ & \mathtt{c} \in s \quad \text{or} \\ & s = A(x) \quad \text{for } x \text{ contraction leaf of type } \alpha^{\perp} \text{ for any } \alpha. \end{aligned}$$

Note that $\mathtt{i} \in s, \mathtt{c} \notin s$ implies $\smile s$. Now define the experiments by the following cases.

1. If ℓ is total and $\exists j \mid \xleftarrow{a} \ell \xrightarrow{b} \subseteq \phi_j$ (i.e. ϕ_j first goes up a and then goes down b) then if a is of type α (resp. $\alpha^{\perp}$) set $e_1(\ell) := \mathtt{i}$ (resp. $\mathtt{c}$) and $e_i(\ell) := \mathtt{n}$ for $i > 1$. Experiments are well defined here because of bounce-compatibility.
2. If $\ell \in \lambda_i$ is partial and is above two contraction leaves (therefore $\ell \in E(A)$), set $e_i(\ell) := \ell$.
3. If $\ell \in \lambda_i$ is partial and is above only one contraction leaf x of type α (resp. $\alpha^{\perp}$), then if $A(x) = \emptyset$ and $i = 1$ set $e_1(\ell) := \mathtt{n}$, else set $e_i(\ell) := \mathtt{i}$ (resp. $\mathtt{c}$).
4. In every other case, for $\ell \in \lambda_i$ set $e_i(\ell) = \mathtt{n}$.

Now let us prove that these definitions satisfy the requirements.

E1 is a direct consequence of point 1 above, as $\frown\{\mathtt{c}, \mathtt{n}\}$ and $\smile\{\mathtt{i}, \mathtt{n}\}$. A total axiom ℓ of E2 falls into case 4 of the definition, so $\{e_i(\ell)\} = \{\mathtt{n}\}$. Now take a contraction x, with f the edge below it of type α (resp. $\alpha^{\perp}$). There are two cases. One is that x is not connected to any other contraction leaf (i.e. $A(x) = \emptyset$), in which case $\{e_i(f)\} = \{\mathtt{i}, \mathtt{n}\}$ (resp. $\{\mathtt{c}, \mathtt{n}\}$) by point 3, and we have strict hyperincoherence. If $A(x) \neq \emptyset$ it is easy to see that

$$A(x) \subseteq \{e_i(f)\} \subseteq A(x) \cup \{\mathtt{i}\}$$

(resp. $\{\mathtt{c}\}$), where the last point may be included or not depending on $A(x)$ being all the axioms above x or not. Note such a point must be included if $A(x)$ is a singleton (no contraction leaf can have a single axiom on it). In case the type is α: if $\mathtt{i} \in \{e_i(f)\}$, as $\mathtt{c} \notin \{e_i(f)\}$ we have $\smile\{e_i(f)\}$, and the same if $\mathtt{i} \notin \{e_i(f)\}$ (i.e. $\{e_i(f)\} = A(x)$), as the non-singleton $A(x)$ cannot be equal to any $A(y)$ for y of type $\alpha^{\perp}$. If the type is $\alpha^{\perp}$ then we have more directly strict hyperincoherence whether $\mathtt{i} \in \{e_i(f)\}$ or $\{e_i(f)\} = A(x)$.

Fact 14. *The experiments of Fact 13 have the following properties:*

(P1) *for every $d \in \mathrm{tot}(\Lambda)$, if $\exists d' \geq d$ and j such that $d' \in \phi_j$, then $\neq \{e_i(d)\}$, i.e. it is not a singleton;*
(P2) *for every $d \in \mathrm{tot}(\Lambda)$, if $\forall j : \downarrow d \notin \phi_j$, i.e. d is not traversed downward by any ϕ_j, then $\asymp\{e_i(d)\}$.*

Proof. Let us prove the two properties. For P1, if $d' \in \phi_j$ for a j, one can go up d' following ϕ_j and find a maximal $d'' \geq d' \geq d$ with $d'' \in \phi_j$. If d'' is partial, then there must be either a binary additive or a contraction leaf between d and d'': in the first case, the resulting experiment cannot be a singleton by construction on additives, and also in the second one, because of property E3. If d'' is total, then there are only three cases possible for it to be maximal. Two of them are that it is a contraction from which ϕ_j jumps or a binary & ϕ_j jumps to (jumps are outside $\leq$), and these by the same arguments as above give $\neq \{e_i(d)\}$. Last case is that d'' is the edge of a total axiom, to which property E1 gives a non-singleton that tracked down to d again gives $\neq \{e_i(d)\}$.

We prove P2 by induction on the type of d, and as usual reasoning by cases. Let x be the node directly above d.

Atomic formula: if d is an axiom edge, then the axiom is total, and properties E1 if $\exists j \mid d \in \phi_j$ (necessarily with $\uparrow d \in \phi_j$) and E2 otherwise make it so that $\{e_i(d)\}$ is assigned either a hyperincoherent set or a singleton respectively. If x is a leaf, then either x is a contraction, and we are settled by property E3 as the thesis of P2 always applies, or it is under a total axiom and we can proceed as above in this same point.

Par: suppose the par x has premises f_0 and f_1, necessarily total. As no path ϕ_j can bounce on x, $\forall j : \downarrow d \notin \phi_j$ implies the same for f_0 and f_1. Applying induction hypothesis gives hyperincoherence on both and therefore hyperincoherence on d.

Tensor: suppose x has premises f_0 and f_1. If the hypothesis $\forall j : \downarrow d \notin \phi_j$ applies for both f_0 and f_1 then i.h. gives us hyperincoherence on both that implies hyperincoherence on d. Otherwise, suppose that for one of the two, say f_0, there is h with $\downarrow f_0 \in \phi_h$. Because of the hypothesis on d this path must bounce on the tensor and go up f_1 (implying $\neq \{e_i(f_1)\}$ by P1). By bounce-compatibility $\forall i : \downarrow f_1 \notin \phi$, which together with i.h. gives us $\smile\{e_i(f_1)\}$ and therefore $\smile\{e_i(d)\}$.

Unary additive: straightforward application of i.h.

Binary with: by the hypotheses of the lemma such x is in some ϕ_j, and by &-orientedness $\downarrow d \in \phi_j$, so the hypothesis of P2 never applies.

Binary plus: by definition of hypercoherence, the thesis of P2 is always true.

□

Theorem 15. *If θ is a set of linkings, and for every $[\![\ ]\!]$ we have that $[\![\theta]\!]$ is a hyperclique, then θ is hypercorrect.*

Proof. Suppose θ is not hypercorrect. Without loss of generality, as we require the theorem to be valid for any set of linkings, and no hyperclique can contain a non-hyperclique set, we may say that the subset witnessing the failure of &-orientedness is θ itself. So in $\mathcal{G}_\theta$ there exists a bounce-compatible non-empty union S of &-oriented cycles with $\&2(\theta) \subseteq S$. Let us show by induction on the number of links of $\mathcal{G}_\theta$ that there exist an interpretation $\llbracket\ \rrbracket$ and $e_1, \dots, e_n$ experiments such that $\smile e_i(\theta)$, which implies $\llbracket\theta\rrbracket$ is not a hyperclique.

- If there is no link, then no cycle is possible, so this case never applies.
- If there is a terminal unary additive, and Γ' is obtained by erasing it from Γ, then θ is still a set of linkings on Γ' and clearly S still is in $\mathcal{G}_\theta$ which has a link less. Applying induction hypothesis yields the result, as unary additives exactly preserve hypercoherence, so that putting back in the link still gives hyperincoherence with the same experiments.
- If there is a terminal $\parr$, no cycle in S can pass it as it should bounce on it. If Γ' is obtained by erasing it from Γ we have that θ is still a hyperincorrect PS on Γ', and the new $\mathcal{G}_\Lambda$ has less links, i.h. applies, and experiments hyperincoherent on conclusions of Γ' are so also on Γ as pars preserve hyperincoherence of sequents.
- If there is a terminal binary & no cycle can pass it, which entails hypercorrectness and a contradiction, so this case never applies.
- If there is a terminal $\otimes$, we form again Γ' by erasing the tensor from Γ, with θ still a set of linkings on Γ'. Now there are two cases. If S survives, and then we apply i.h. and get the result putting the tensor back in, as hyperincohrence on both premises of a tensor implies hyperincoherence on its conclusion. If S does not survive, it means that some cycles in it were broken. Let f_1 and f_2 be the premises of this tensor. By bounce-compatibility, all cycles broken in this step must, on Γ', be paths that start from the same premise of the tensor, say f_1, and arrive to the other one, f_2. Therefore S induces paths in $\mathcal{G}_\theta$ on Γ' that fall into the hypotheses of Lemma 12. Applying it yields an interpretation $\llbracket\ \rrbracket$ and experiments that are hyperincoherent everywhere else than f_1 and f_2 and strictly hyperincoherent on f_1. Though nothing is said about f_2, this suffices to give strict hyperincoherence on the conclusion of the tensor when we plug it back in, and hyperincoherence of every other conclusion, so that the result is proved.
- Last remaining case, when none of the above applies, is that $\mathcal{G}_\theta$ has only (and at least one) terminal binary $\oplus$s. In this case we take $\llbracket\alpha\rrbracket := 1 = (\{*\}, \{\{*\}\})$ (i.e. an interpretation assigning the multiplicative unit to all literals) and as experiments the only ones possible for $\llbracket\ \rrbracket$ on each linking (which are more than one in order to give binary $\oplus$s). With such experiments, we have singletons on each conclusion without a link above it, and automatically strict incoherence under the $\oplus$s. □

Proofs of Section 4

Proposition 16. *Hypercorrectness (Definition 5) is equivalent to having any number of its parts substituted in the following ways.*

1. *bounce-compatibility can be strengthened with plain compatibility, i.e.* ϕ *and* ψ *are compatible iff whenever* $e \in \phi \cap \psi$ *then* ϕ *and* ψ *traverse* e *in the same direction;*
2. *&-orientedness can be strengthened with strict &-orientedness (defined on page 9);*
3. *the condition asking the presence of* $w \in \&2(\Lambda)$ *outside* S *can be strengthened to be the presence of* $w \in \&2(\Lambda) \cap \mathrm{tot}(\Lambda)$ *outside* S*,*
4. *the graph* $\mathcal{G}^{\mathrm{p}}_{\Lambda}$ *can replace* $\mathcal{G}_{\Lambda}$*.*

Proof. A criterion using point 1 is implied by a criterion not using it, so one has to check only the proofs in Section 3.2. As already noted during the proof of Fact 9, admissible paths are compatible, so the proofs still work.

The strengthening of point 2 is trivially equivalent as in fact one has used strict &-orientedness in the proofs of Section 3.2.

A criterion employing point 3 implies one not employing it, and Lemma 12 in Section 3.3 only requires total &s to be each touched by at least a path.

For point 4, as $\mathcal{G}_{\Lambda} \subseteq \mathcal{G}^{\mathrm{p}}_{\Lambda}$ it is clear that a criterion with $\mathcal{G}^{\mathrm{p}}$ is stronger, so the implication in danger is the one of Section 3.3. One can adapt all proofs there just by substituting partial contraction for contraction everywhere. □

Proposition 17. *Every sequentializable PS is hypercorrect.*

Proof. Suppose that θ is not hypercorrect, i.e. $\exists \Lambda \subseteq \theta$ such that there is a bounce-compatible union S of strictly &-oriented cycles in $\mathcal{G}_{\Lambda}$ (using point 2 of Proposition 16) such that $\&2(\Lambda) \subseteq S$. As &-orientedness and bounce-compatibility do not play any role for criterion HvG, one concentrates on translating each ϕ of S in $\mathcal{G}_{\Lambda}$ into one (or possibly more) cycle ϕ' in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$ containing the same &s. We will substitute each jump j in ϕ with a path ψ_j in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$. So let $c \overset{j}{\rightsquigarrow} w$, with $\lambda, \mu \in \Lambda$, $\lambda \overset{w}{\nsim} \mu$ and $c \in \mathrm{contr}(\{\lambda, \mu\})$. If c is a leaf then trivially j is also in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$, so one sets $\psi_j = j$.

Suppose c is therefore a $\oplus$. By definition one has $c \preceq x_1 \leftarrow \ell_1$ and $c \preceq x_2 \leftarrow \ell_2$ with ℓ_1 (resp. ℓ_2) in $\lambda \setminus \mu$ (resp. viceversa), so that $x_i \rightsquigarrow w$ in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$. Necessarily one of the two partial paths going up paths from c to x_i does not intersect ϕ, as if it was in both, by strictness one would have three edges adjacent to c in ϕ (the premises and the jump). Set ψ_j to be such a path from c to x_i appended with the jump $x_i \rightsquigarrow w$. One sees that none of ψ_j for j jumps in ϕ can intersect each other, as partial trees over contractions (that are total) cannot overlap. Now build ϕ' in $\mathcal{G}^{\mathrm{HvG}}_{\Lambda}$ from ϕ by substituting every jump j with ψ_j, and clearly $\&2(\Lambda) \cap \phi \subseteq \&2(\Lambda) \cap \phi'$. So $\&2(\Lambda) \subseteq \bigcup_{\phi \subseteq S} \phi'$ and θ does not satisfy the HvG criterion. □