Computing Surfaces via pq-Permutations

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Abstract. In algebraic topology, compact 2-dimensional manifolds are usually dealt through a well-defined class of words denoting polygonal presentations. In this paper, we show how to eliminate the useless bureaucracy intrinsic to word-based presentations by considering very simple combinatorial structures called *pq-permutations*. Thanks to their effectiveness, pq-permutations allow to define a rewriting system $\mathcal{P}$ able to compute, in a very easy and intuitive way, the quotient surface associated with any given polygonal presentation. From an algorithmic point of view, this procedure constitutes a remarkable improvement with respect to the classical one afforded by Massey.

1 Introduction

A standard result in algebraic topology establishes that any compact 2-dimensional manifold (usually simply called surface) $\mathscr{S}$ can be univocally determined by a finite set of polygons $W_{\mathscr{S}} = \{w_1, w_2, \ldots, w_n\}$, each one having the edges labeled and oriented (triangulation theorem). The idea is that a surface $\mathscr{S}$ is characterized by a finite set of polygons $W_{\mathscr{S}}$ if, modulo identification of paired edges, the quotient surface induced by $W_{\mathscr{S}}$ is exactly $\mathscr{S}$. Such a set of polygons $W_{\mathscr{S}}$ is said to be a *polygonal presentation* of $\mathscr{S}$ and it is usually presented as a set of words [7, 10]. An effective procedure for computing surfaces from their polygonal presentations can be found in [10]. In this classical reference, Massey affords an algorithm for transforming any given presentation into an equivalent one having perimeter in canonical form: a standard shape in which the basic geometrical information concerning the presented surface is explicitly displayed.

The notion of *q-permutation* has been gradually introduced in a few contributions concerning theoretical computer science and, in particular, the ambit of linear proof theory [6, 8, 1]. The idea leading to q-permutations consists in remarking that the basic information concerning any compact and orientable 2-manifold (possibly with boundary) can be encoded by a very easy mathematical structure formed by a permutation σ paired with a natural number q. Roughly speaking, whereas σ denotes, cycle by cycle, each boundary-component, q works as a counter for the number of tori involved in the connected sum to which the surface at issue is homeomorphic. The notion of q-permutation is clearly rooted in the well-known classification theorem which states that any orientable surface turns out to be homeomorphic to either a connected sum of tori or a sphere (no tori in the connected sum) [10].

A more general structure able to characterize surfaces in general, not only orientable but also non-orientable, is here once again suggested by the classification theorem which ensures that any non-orientable surface is always homeomorphic to a connected sum of projective planes [10]. Thus, whereas the part of our permutative structure encoding the boundary is kept unmodified, we replace our single counter with a couple of natural numbers: the first one for counting, as usual, tori and the second one for indicating projective planes. This kind of enriched structures are here called *pq-permutations* and they should be seen as a way to polish up words from useless bureaucracy, a more perspicuous and efficient way to express word-based presentations.

The specific perspicuousness displayed by pq-permutations is shown to have interesting computational spin-offs. pq-Permutations induce in fact a rewriting system $\mathcal{P}$ which is able clearly "mimic", step by step, the process of forming a surface through identification of paired edges. As a consequence, we have that $\mathcal{P}$ constitutes an algorithmic improvement of Massey's procedure and, more generally, of all the classical word-based treatment of topological surfaces. Moreover, $\mathcal{P}$ is shown to enjoy both the fundamental computational properties of strong normalization and strict strong confluence [2].

2 From Polygonal Presentations to Quotient Surfaces

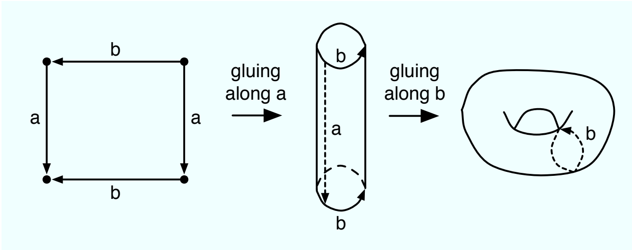
2.1 Polygonal Presentations

It is a well-known achievement in algebraic topology that any surface $\mathscr{S}$ can be completely characterized by a finite set of polygons forming an its *polygonal presentation* [10, 7]. In particular, a presentation $W_{\mathscr{S}}$ of a surface $\mathscr{S}$ consists in a finite set of polygons $\{w_1, \ldots, w_n\}$ whose perimeters are constituted by labelled and oriented edges, such that:

- no more than two edges can have the same label;
- the quotient of $W_{\mathscr{S}}$, modulo identification of paired edges, is the surface $\mathscr{S}$.

Since fixed a clockwise or an anticlockwise orientation, any polygon w turns out to be completely determined by its perimeter, namely by a cycle of oriented edges. Edges having orientation opposite to the fixed one, are indicated by raising them at the minus one power. Thus, polygonal presentations are usually written as sets of words on an alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$, where $\mathcal{A} = \{a, b, c, \ldots\}$ and $\mathcal{A}^{-1} = \{a^{-1}, b^{-1}, c^{-1}, \ldots\}$, considered up to circular permutations. In the sequel of this paper we will adopt the simplified notation x and $\bar{x}$ ($x \in \mathcal{A} \cup \mathcal{A}^{-1}$), for meaning that the pair of edges labeled with x have opposite orientations. The bar-operation ($\bar{\ }$) is clearly an involution without fix point, namely, for any $x \in \mathcal{A} \cup \mathcal{A}^{-1}$, $\bar{\bar{x}} = x$ and $x \neq \bar{x}$.
We recall some basic polygonal presentations: *sphere*: $\boldsymbol{a\bar{a}}$; *torus*: $\boldsymbol{ab\bar{a}\bar{b}}$ (see Figure 1); *projective plane*: $\boldsymbol{aa}$; *Klein bottle*: $\boldsymbol{ab\bar{a}b}$.

Fig. 1. The polygon $ab\bar{a}\bar{b}$ becomes a torus.



Theorem 1 (classification theorem). *Any compact connected surface (possibly with boundary) is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum*[1] *of tori, or a finite connected sum of projective planes (possibly with boundary). The sphere and connected sums of tori are orientable surfaces, whereas connected sums of projective planes are non-orientable.*

Notation. $W, U, V, \ldots$ denote sets of words, whereas we adopt small letters $w, u, v, q, \ldots$ for indicating single words. If $w = a_1 a_2 \ldots a_n$, then $\bar{w} = \bar{a}_n \bar{a}_{n-1} \ldots \bar{a}_1$; and, if $W = \{w_1, w_2, \ldots, w_n\}$, then $\bar{W} = \{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n\}$. Polygonal presentations consisting in a sigleton $W_{\mathscr{S}} = \{w\}$ are simply indicated with $w_{\mathscr{S}}$.

A detailed proof for Theorem 1 can be found in [10], where Massey provides an algorithm for rewriting any given 1-polygon presentation into an equivalent one (i.e. denoting the same surface) having perimeter in so-called canonical form. The advantage of dealing with presentations in *canonical form* consists in the fact that they make easily understood the fundamental information concerning the presented surface.

Definition 1 (canonical forms). *Words of the shape* $a_1 b_1 \bar{a}_1 \bar{b}_1 \ldots a_n b_n \bar{a}_n \bar{b}_n$ *and* $a_1 a_1 \ldots a_n a_n$ *are respectively abbreviated with* $\texttt{tor}_n$ *and* $\texttt{pjp}_n$*. The following three canonical shapes*

$$a\bar{a}x_1 u_1 \bar{x}_1 \ldots x_q u_q \bar{x}_q \qquad \texttt{tor}_n x_1 u_1 \bar{x}_1 \ldots x_q u_q \bar{x}_q \qquad \texttt{pjp}_n x_1 u_1 \bar{x}_1 \ldots x_q u_q \bar{x}_q$$

respectively denote a sphere, a connected sum of n tori and a connected sum of n projective planes, in all cases with the boundary decomposed into q components: $u_1, u_2, \ldots, u_q$.

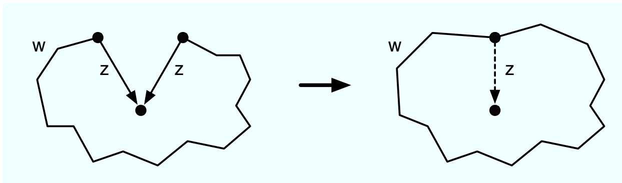
2.2 Massey's Algorithm

We consider the problem of computing the connected surface $\mathscr{S}$ associated with a given polygon $w_{\mathscr{S}}$; the connectness of $\mathscr{S}$ allows to consider the simplest case

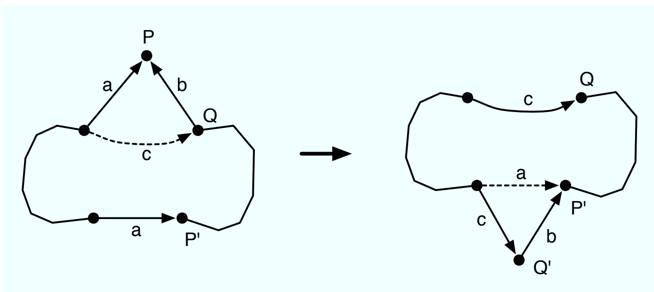
[1] Roughly speaking, the connected sum operation consists in connecting two surfaces with a tube after cutting out holes in the surfaces where the tubes are attached.

of a 1-polygon presentation: disconnected surfaces can be easily recovered by singularly considering connected components. The problem of computing $\mathscr{S}$ corresponds to the problem of rewriting $w_{\mathscr{S}}$ into an equivalent polygon $w'_{\mathscr{S}}$ having canonical form. We summarise below the procedure provided by Massey for proving the classification theorem [10]; it essentially concerns surfaces without boundary: bordered 2-manifolds will be recovered through a particular *escamotage*. Recall that, in algebraic topology, surfaces are usually considered modulo "reversible cuts", in other words: we can always cut a surface provided that we leave on the two new edges in this way obtained the information (label + orientation) needed for recomposing them without ambiguities.

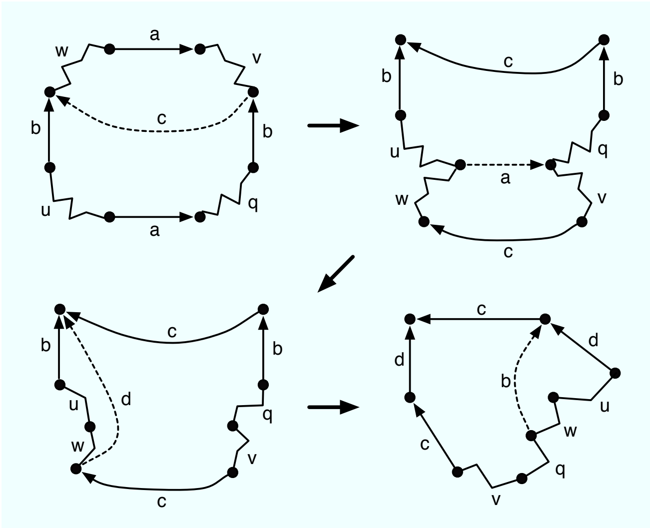
- **Step 1: eliminating redundant edges.** Adjacent paired opposite edges have to be identified before applying each one of the following four steps.



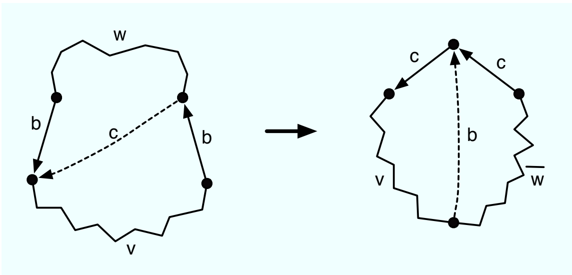
- **Step 2: forming an unique equivalence class.** We say that two vertices P and Q are *equivalent* if, and only if, they are to be identified (for instance, in the polygon on the right, P and P' turn out to be equivalent). Suppose that P and Q belong to two different equivalence classes, namely $[P] \neq [Q]$; we can make one point of $[P]$ migrate into $[Q]$ by cutting along c and gluing along b. By successive migrations, we can easily obtain an unique equivalence class.



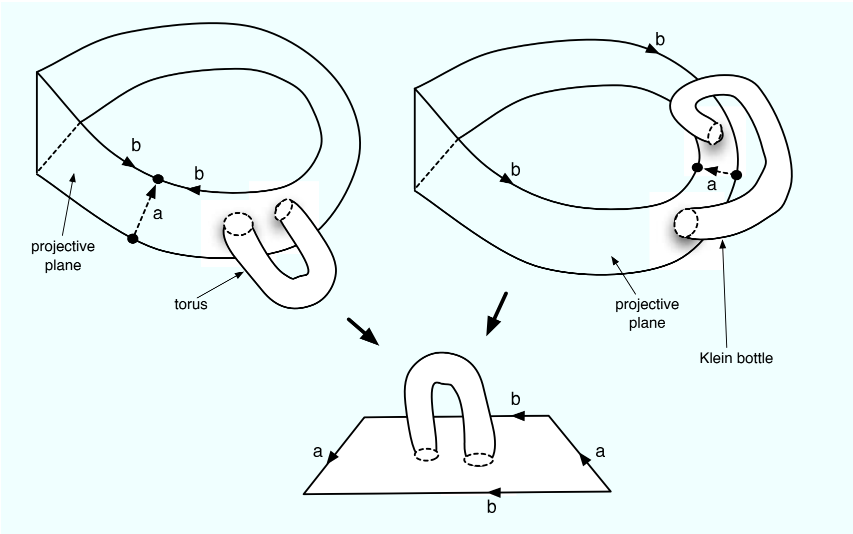
- **Step 3: storing a torus.** A torus (namely a segment $cd\bar{c}\bar{d}$) can be explicitly achieved by cutting and gluing as indicated below.



– **Step 4: storing a projective plane.** We can explicitly achieve a projective plane (namely a segment cc) by cutting along c and gluing along b.

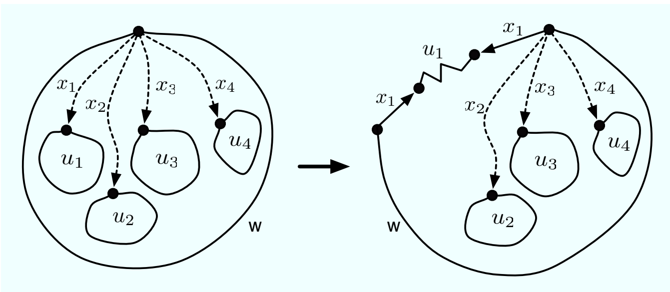


– **Step 5: applying a basic homeomorphism.** An algorithm based on the previous four steps may provide polygons in pre-canonical form, namely having explicit tori mixed with explicit projective planes: $\texttt{tor}_n\texttt{pjp}_m$. So, for obtaining final canonical forms, Massey recurs to the basic homeomorphism between the connected sum of a projective plane with a torus and the connected sum of three projective planes. This homeomorphism is proved by showing that the connected sum of a projective plane with a torus (figure on the left) and the connected sum of a projective plane with a Klein bottle (figure on the right) can be reported to the same surface by cutting them along a. We recall that a Klein bottle is homeomorphic to the connected sum of two projective planes. Therefore, a polygon having perimeter $\texttt{tor}_n\texttt{pjp}_m$ is equivalent to a polygon having canonical form $\texttt{pjp}_{2n+m}$.



Remark 1. At first sight, Step 2 may seem to have not a precise task in the mechanism of Massey's procedure, so we precise that its specific role consists in preventing to have deadlock configurations as, for instance, $aacbb\bar{c}$. This polygon indicates the connected sum of two projective planes – i.e. it is equivalent to $aabb$ –, but we do not know how to eliminate the redundant information afforded c-edges. If we explicit the vertices $aPaQcRbSbT\bar{c}U$, it easy to check that $\{U, P, Q\}$ and $\{T, S\}$ form two distinct equivalence classes.

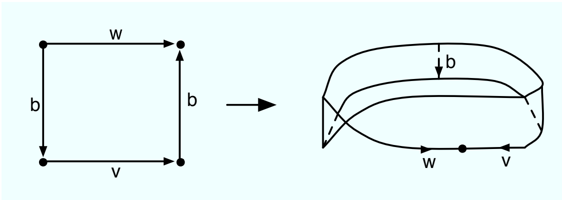
Recovering boundaries. As already seen, Massey's algorithm essentially deals with non-bordered surfaces. In case of surfaces with boundary, we can assume as a starting point a polygon W including all boundary components in its interior (it is a corollary of the triangularisation theorem [10]). So, at first, we transform W into a polygon W' in canonical form; then, we "extract" on the perimeter all the boundary-components as indicated below. In this way, each connected piece of boundary u_i will be explicitly achieved as a segment $x_i u_i \bar{x}_i$.



2.3 A Rewriting System on Words

Definition 2 (rewriting system). *A rewriting system $\mathcal{R}$ consists of a set of terms $\{t_1, t_2, \ldots\}$ closed with respect to a set of transformations $\{r_1, r_2, \ldots, r_n\}$.*

Fig. 2. Intuitive explanation of the Möbius rule.



Notation. In the specific jargon of term rewriting, an application of a single rule is called *step of reduction*. Consider a generic rewriting system $\mathcal{R}$: we write $t \rightarrow_{r_i} t'$ and $t \rightarrow_{\mathcal{R}} t'$ for meaning that t' is obtained from t respectively by applying the (single) specific transformation r_i and a (single) generic transformation of $\mathcal{R}$. $t \rightsquigarrow_{\mathcal{R}} t'$ indicates that t' is obtained from t throughout a sequence of reduction steps [2]. We write $t \rightsquigarrow^*_{\mathcal{R}} t'$ for meaning that t' is not further rewritable; t' is said to be a *normal form* for t.

Definition 3. *The rewriting system $\mathcal{W}$ is defined by taking polygonal presentations as terms together with the following six rules:*

- glue: $W, wa, \bar{a}v \rightarrow W, wv$
- split: $W, wv \rightarrow W, wa, \bar{a}v$
- cutting-out: $W, wa\bar{a} \rightarrow W, w$
- pump: $W, w \rightarrow W, wa\bar{a}$
- invert: $W, w \rightarrow W, \bar{w}$
- shift: $W, wxu\bar{x}v \rightarrow W, wvx\sigma(u)\bar{x}$*, where σ is a cyclic permutation.*

The set of rules just listed is a slight variant of that one already proposed in [5]: in particular, the primary list has been here closed under inversion of rules (e.g., *pump* is nothing else but the leftward reading of *cutting-out*). This kind of closure allows to state that, if $W \rightsquigarrow_{\mathcal{W}} W'$, then $W' \rightsquigarrow_{\mathcal{W}} W$, which is a very natural property for the specific topological context we consider in these pages.

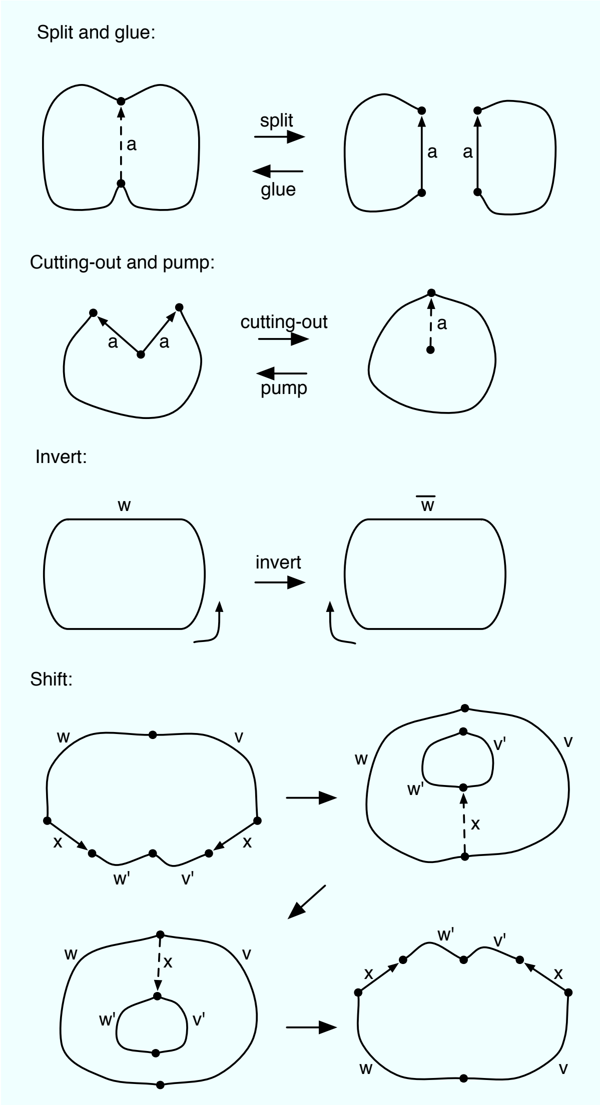
Lemma 1. *The following rule is admissible in $\mathcal{W}$: $W, wava \rightarrow_{\text{Mobius}} W, w\bar{v}aa$.*

Proof. The mechanism of this rule is intuitively explained in Figure 2. Nevertheless, for being more precise, we show that $W, wava \rightsquigarrow_{\mathcal{W}} W, w\bar{v}aa$:

$$W, wava \rightarrow_{\text{split}} W, waz, \bar{z}va \rightarrow_{\text{inv.}} W, waz, \bar{a}\bar{v}z \rightarrow_{\text{glue}} W, w\bar{v}zz =_{\text{rename}} W, w\bar{v}aa.$$

Lemma 2. *Segments indicating tori or projective planes behave as central elements, namely they can be freely moved inside words.*

Table 1. Geometrical visualization of the rules in $\mathcal{W}$.



Proof. The proof consists in detailing the following two chains:

$$W, waav \rightsquigarrow_{\mathcal{W}} W, wvaa \qquad \text{and} \qquad W, wab\bar{a}\bar{b}v \rightsquigarrow_{\mathcal{W}} W, wvab\bar{a}\bar{b}.$$

By the leftward reading of the chain used to prove the previous lemma, we have the admissibility of $w\bar{v}aa \rightarrow_{\text{Mob.}^{-1}} wava$; thus, we can write:

$$W, waav \rightarrow_{\text{Mob.}^{-1}} W, a\bar{w}av \rightarrow_{\text{inv.}} W, \bar{v}\bar{a}w\bar{a} \rightarrow_{\text{Mob.}} W, \bar{v}\bar{a}\bar{a}\bar{w} \rightarrow_{\text{inv.}} W, waav.$$

For what concerns the other chain, we have:

$$W, wab\bar{a}\bar{b}v \rightarrow_{\text{shift}} W, wavb\bar{a}\bar{b} \rightarrow_{\text{shift}} W, w\bar{b}avb\bar{a} \rightarrow_{\text{shift}}$$
$$\rightarrow_{\text{shift}} W, w\bar{a}\bar{b}avb \rightarrow_{\text{shift}} W, wvb\bar{a}\bar{b}a =_{\text{rename}} W, wvab\bar{a}\bar{b}.$$

Definition 4. *Two polygonal presentations W and V are said to be equivalent, $W \sim V$, if they present the same surface.*

Theorem 2. *If W and W' are two presentations such that $W \rightarrow_{\mathcal{W}} W'$, then $W \sim W'$.*

Proof. We sketch an intuitive version of the proof (the reader can find more details in [10]). All the rules listed in Definition 3 are geometrically explained in Table 1. As already recalled, in algebraic topology surfaces can be considered modulo "reversible" cuts. The two rules of *split* and *pump* (together with their relative inverses *glue* and *cutting-out*) exactly express this idea. *Invert* rule just says that the perimeter of a polygon can be read following both the possible orientations (clockwise or anticlockwise) without changing the presented surface. The rule of *shift* is the most meaningful one. The idea is that a segment of perimeter u included between paired opposite letters, $xu\bar{x}$, can always be "carried inside" the polygon by identifying the x-edges (see the last figure in Table 1). Since u is an "hole" inside the polygon, it can be once again "extracted" on the perimeter by performing a new cut on the surface. *Shift* rule expresses the fact that this new cut can be performed from an arbitrary vertex on the perimeter to an arbitrary vertex on u.

Lemma 3. *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

Proof. In terms of words, connected sum is nothing else but concatenation, so the connected sum of a torus with a projective plane with boundary can be presented by a polygon having perimeter $\texttt{tor}_1\texttt{pjp}_1 = ab\bar{a}\bar{b}cc$. Then, we rewrite our word as follows:

$$ab\bar{a}\bar{b}cc \rightarrow_{\text{shift}} acb\bar{a}\bar{b}c \rightarrow_{\text{Lemma1}} aba\bar{b}cc \rightarrow_{\text{Lemma1}} \bar{b}aa\bar{b}cc \rightarrow_{\text{Lemma2}} \bar{b}\bar{b}aacc,$$

namely $\texttt{pjp}_3$.

3 Pq-Permutations

If we consider a surface $\mathscr{S}$ as the final result of identifying paired edges in a set of polygons forming an its topological presentation, we have that each boundary-component will be formed by at least one edge. Let $\partial\mathscr{S}$ be the set of labels occurring on the boundary of $\mathscr{S}$; since fixed an orientation, we can notice that $\mathscr{S}$ induces a cyclic order on each one of the subsets of $\partial\mathscr{S}$ corresponding to boundary-components; in other words, we obtain a permutation on $\partial\mathscr{S}$. The idea leading to the notion of pq-permutation is that the basic information concerning any surface $\mathscr{S}$ can always be encoded by a very easy mathematical structure consisting in a permutation σ (denoting, cycle by cycle, the boundary $\partial\mathscr{S}$) together with a couple of natural numbers $\langle p, q\rangle$ respectively counting tori and projective planes in the connected sum to which $\mathscr{S}$ is homeomorphic.

Notation. pq-permutations are denoted with small Greek letters $\alpha, \beta, \ldots$; big Greek letters $\Sigma, \Xi, \Psi, \ldots$ denote sets of pq-permutations. When letters $W, V, U, \ldots$ and $w, v, u, \ldots$ appear in pq-permutations they respectively stand for sets of cyles and series of elements (i.e. $w = a_1, a_2, \ldots, a_n$). The permutation having empty support indicated with ϵ. If $\mathcal{A} = \{a, b, c, d, \ldots\}$, then $\bar{\mathcal{A}} = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \ldots\}$.

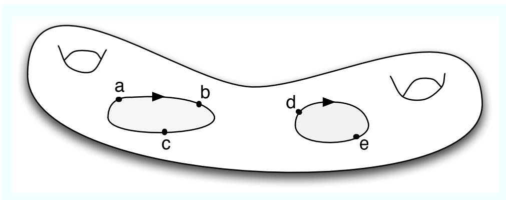
Definition 5 (q-permutation). *A pq-permutation α is an ordered quadruple (X, σ, p, q) such that:*

- *X is a finite multiset from $\mathcal{A} \cup \bar{\mathcal{A}}$ in which any letter – considered up to its orientation – occur at most twice;*
- *σ is a permutation on X;*
- *p and q are positive integers.*

pq-Permutations are here simply written as indexed permutations:

$$\alpha = \{(w_1), (w_2), \ldots, (w_n)\}_{\langle p,q\rangle}.$$

Example 1. The oriented surface illustrated below induces the pq-permutation $\{(a, b, c), (d, e)\}_{\langle 2,0\rangle}$.



Example 2. For taking an example of a non-orientable surface, a Klein bottle without boundary will induce the pq-permutation $\epsilon_{\langle 0,2\rangle}$ (it is in fact homeomorphic to the connected sum of two projective planes [10]).

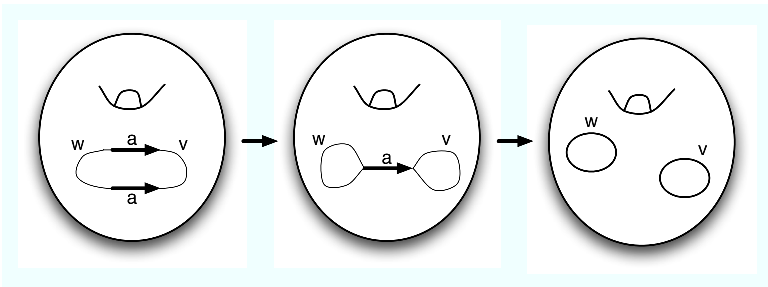
Remark 2. pq-Permutations should be seen a way of making the structure of canonical words more perspicuous by avoiding useless bureaucracy. In particular segments of the shapes $\mathtt{tor}_p$ and $\mathtt{pjp}_q$ – respectively used for storing tori and projective planes – are discarded through the two indices $\langle p, q\rangle$, whereas the part concerning the boundary is considered modulo *shift* rule: $x_1 u_1 \bar{x}_1 x_2 u_2 \bar{x}_2 \ldots x_r u_r \bar{x}_r$ becomes the set of cycles $\{(u_1), (u_2), \ldots, (u_r)\}$.

It is now clear that the structure of pq-permutations provides an invariant for considering surfaces modulo isomorphisms, namely modulo homeomorphisms preserving orientation, alternative to that one provided by words.

Definition 6. *We define the rewriting system $\mathcal{P}$ by taking sets of pq-permutations as terms together with the following six rules:*

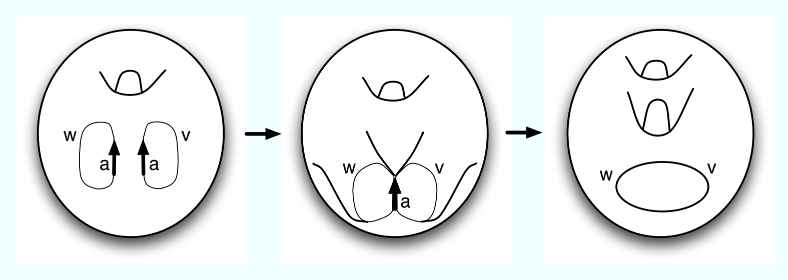
- gluing: $\Sigma, \{W, (w, a)\}_{\langle p,q\rangle}, \{V, (v, \bar{a})\}_{\langle p',q'\rangle} \rightarrow \Sigma, \{W, V, (w, v)\}_{\langle p+p',q+q'\rangle}$
- invert: $\Sigma, \{(w_1), \ldots, (w_n)\}_{\langle p,q\rangle} \rightarrow \Sigma, \{(\bar{w}_1), \ldots, (\bar{w}_n)\}_{\langle p,q\rangle}$
- cylinder: $\Sigma, \{W, (w, a, v, \bar{a})\}_{\langle p,q\rangle} \rightarrow \Sigma, \{W, (w), (v)\}_{\langle p,q\rangle}$
- torus: $\Sigma, \{W, (w, a), (\bar{a}, v)\}_{\langle p,q\rangle} \rightarrow \Sigma, \{W, (w, v)\}_{\langle p+1,q\rangle}$
- Möbius: $\Sigma, \{W, (w, a, v, a)\}_{\langle p,q\rangle} \rightarrow \Sigma, \{W, (w, \bar{v})\}_{\langle p,q+1\rangle}$
- Klein: $\Sigma, \{W, (w, a), (a, v)\}_{\langle p,q\rangle} \rightarrow \Sigma, \{W, (w, \bar{v})\}_{\langle p,q+2\rangle}$
- sieve: $\Sigma, \{W\}_{\langle p,q\rangle} \rightarrow \Sigma, \{W\}_{\langle 0,2p+q\rangle}$.

Fig. 3. Cylinder rule.



Gluing and *invert* rules are nothing else but the conterpart of their homonymous rules in $\mathcal{W}$. *Cylinder* expresses the fact that the effect of identifying two opposite edges occurring on the same piece of boundary, is that one of decomposing this boundary-component into two components (Figure 3). As far as the *torus* rule is concerned, if opposite paired edges occur on two different boundary-components, their identification forms a new handle on the surface, namely we achieve one more torus in the connected sum (Figure 4). The *Möbius* rule comes straightforwardly from Lemma 1, whereas the *Klein* rule should be interpreted as

Fig. 4. Torus rule.



a kind of "non-orientable torus" whose effect, as its own name suggests, consists in producing a Klein bottle (two more projective in the connected sum). Finally, the rule of *sieve* just expresses the basic homeomorphism stated in Lemma 3.

Definition 7 (weak and strong normalization properties). *A rewriting system $\mathcal{R}$ enjoys the* weak normalization property *if, for every term $t \in \mathcal{R}$, there exists a rewriting sequence able to transform t into a normal form. If any rewriting strategy is able to carry t into a normal form, our system is said to be* strongly normalizing.

Remark 3. According to the previous definition, we remark that a pq-permutation α is in normal form if $|\alpha|$ does not contain paired edges and at least one of the two indices is null (three admitted situations: $\langle p, 0\rangle$, $\langle 0, q\rangle$ and $\langle 0, 0\rangle$).

Theorem 3. *The rewriting system $\mathcal{P}$ strongly normalizes.*

Proof. For proving this property, one usually attaches a convenient size to terms and shows that it decreases at each single step of reduction. In case of pq-permutations, we associate to each α a size $[\alpha] = i - j$, where i is the number of paired edges occurring in $|\alpha|$ and j the number of stored tori (namely, the first index of α). Now it is sufficient to remark that, if $\alpha \rightarrow_{\mathcal{P}} \alpha'$, then $[\alpha'] < [\alpha]$.

Definition 8 (confluence, strict strong confluence). *A rewriting system $\mathcal{R}$ is said to be* confluent *if, for any three terms $a, b, c \in \mathcal{R}$ such that $a \rightsquigarrow_{\mathcal{R}} b$ and $a \rightsquigarrow_{\mathcal{R}} c$, there exists a fourth term $d \in \mathcal{R}$ such that $b \rightsquigarrow_{\mathcal{R}} d$ and $c \rightsquigarrow_{\mathcal{R}} d$. $\mathcal{R}$ enjoys the* strict strong confluence *property if, in the definition of confluence, the arrow "$\rightsquigarrow$" can be replaced everywhere by the single step arrow "$\rightarrow$".*

Lemma 4. *If we consider pq-permutations modulo* sieve *rule, then $\mathcal{P}$ is strictly strongly confluent.*

Proof. With $\alpha \rightarrow_a \alpha'$ we mean that the pq-permutation α' has been obtained from α by identifying a-edges. By considering all the possible cases it is easy to see that, if $\alpha \rightarrow_a \beta$, $\alpha \rightarrow_b \gamma$, $\beta \rightarrow_b \delta$ and $\gamma \rightarrow_a \delta'$, then $\delta = \delta'$.

Example 3. We exemplify below the idea of strict strong confluence. If we have

$\{(a,w),(a,v,c),(\bar{c},u)\}_{\langle 0,0\rangle} \rightarrow_{\text{Klein}} \{(w,\bar{c},\bar{v}),(\bar{c},u)\}_{\langle 0,2\rangle}$ and
$\{(a,w),(a,v,c),(\bar{c},u)\}_{\langle 0,0\rangle} \rightarrow_{\text{torus}} \{(a,w),(a,v,u)\}_{\langle 1,0\rangle}$, then
$\{(w,\bar{c},\bar{v}),(\bar{c},u)\}_{\langle 0,2\rangle} \rightarrow_{\text{Klein}} \{(w,\bar{u},\bar{v})\}_{\langle 0,4\rangle}$ and
$\{(a,w),(a,v,u)\}_{\langle 1,0\rangle} \rightarrow_{\text{Klein}} \{(w,\bar{u},\bar{v})\}_{\langle 1,2\rangle} \cong_{(\text{sieve})} \{(w,\bar{u},\bar{v})\}_{\langle 0,4\rangle}$.

Remark 4. Strict strong confluence implies both confluence and the uniqueness of normal forms (namely, any pq-permutation has exactly one normal form). It means, that $\mathcal{P}$ is a deterministic system, not only in terms of outputs, but also in terms of computations. Strict strong confluence extends in fact determinism to computational processes by asserting their equivalence modulo permutation of rules (in case of pq-permutations, modulo permutations of identified edges).

Definition 9. *We associate with any pq-permutation α a word w_α defined as follows:*

$$\alpha = \{(w_1),\ldots,(w_n)\}_{\langle p,q\rangle} \mapsto w_\alpha = \mathtt{tor}_p\mathtt{pjp}_q x_1 w_1 \bar{x}_1 \ldots x_n w_n \bar{x}_n.$$

If $\Sigma = \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, then $W_\Sigma = \{w_{\alpha_1},w_{\alpha_2},\ldots,w_{\alpha_n}\}$; so, the equivalence relation "$\sim$" can be extended to sets of pq-permutations in a very natural way: $\Sigma \sim \Xi$ if, and only if, $W_\Sigma \sim W_\Xi$.

Theorem 4. *Given two pq-permutations α and β, if $\alpha \rightarrow_{\mathcal{P}} \beta$, then $\alpha \sim \beta$.*

Proof. The proof consists in showing that any chain of pq-permutations $\Xi \rightsquigarrow_{\mathcal{P}} \Xi'$ has a precise counterpart in terms of words $W_\Xi \rightsquigarrow_{\mathcal{W}} W_{\Xi'}$ and, in particular, if Ξ' is in normal form, then $W_{\Xi'}$ is in canonical form. Just a preliminary remark on notation: when a set of cycles $W = \{(u_1),\ldots,(u_n)\}$ occurring in a pq-permutation is "translated" into a word, its notation is kept unchanged but it is meant to be $W = x_1 u_1 \bar{x}_1 \ldots x_n u_n \bar{x}_n$. Thus, it is clear that a segment like W can be freely moved inside a word throughout a series of *shift* rules.

- **Gluing:** The set $\Sigma, \{W,(w,a)\}_{\langle p,q\rangle}, \{V,(v,\bar{a})\}_{\langle p',q'\rangle}$ becomes
  $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q W x_1 w a \bar{x}_1, \mathtt{tor}_{p'}\mathtt{pjp}_{q'} V x_2 v \bar{a} \bar{x}_2$. Then we have:
  $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q W x_1 w a \bar{x}_1, \mathtt{tor}_{p'}\mathtt{pjp}_{q'} V x_2 v \bar{a} \bar{x}_2 \rightarrow_{\text{glue}}$
  $\rightarrow_{\text{glue}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q W x_1 w \bar{x}_2 \mathtt{tor}_{p'}\mathtt{pjp}_{q'} V x_2 v \bar{x}_1 \rightsquigarrow_{\text{Lemma2}}$
  $\rightsquigarrow_{\text{Lemma2}} W_\Sigma, \mathtt{tor}_p\mathtt{tor}_{p'}\mathtt{pjp}_q\mathtt{pjp}_{q'} W x_1 w \bar{x}_2 V x_2 v \bar{x}_1 \sim$
  $\sim W_\Sigma, \mathtt{tor}_{p+p'}\mathtt{pjp}_{q+q'} W x_1 w \bar{x}_2 V x_2 v \bar{x}_1 \rightsquigarrow_{\text{shift}}$
  $\rightsquigarrow_{\text{shift}} W_\Sigma, \mathtt{tor}_{p+p'}\mathtt{pjp}_{q+q'} W V x_1 w \bar{x}_2 x_2 v \bar{x}_1 \rightarrow_{\text{cut}}$
  $\rightarrow_{\text{cut}} W_\Sigma, \mathtt{tor}_{p+p'}\mathtt{pjp}_{q+q'} W V x_1 w v \bar{x}_1$;
  in terms of pq-permutations: $\Sigma, \{W,V,(w,v)\}_{\langle p+p',q+q'\rangle}$.
- **Invert:** easy.
- **Cylinder:** $\Sigma, \{W,(w,a,v,\bar{a})\}_{\langle p,q\rangle} \rightarrow \Sigma, \{W,(w),(v)\}_{\langle p,q\rangle}$. Two cases.

  - v is not the empty word.
    $\Sigma, \{W, (w, a, v, \bar{a})\}_{\langle p,q\rangle}$ becomes $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wxwav\bar{a}\bar{x}$ and so:
    $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wxwav\bar{a}\bar{x} \to_{\text{shift}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wxw\bar{x}av\bar{a}$,
    namely $\Sigma, \{W, (w), (v)\}_{\langle p,q\rangle}$.
  - v is the empty word: instead of a *shift* rule, we apply a *cutting-out*.
- **Torus:** $\Sigma, \{W, (w, a), (\bar{a}, v)\}_{\langle p,q\rangle}$ corresponds to $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wa\bar{x}_1x_2\bar{a}v\bar{x}_2$.
  $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wa\bar{x}_1x_2\bar{a}v\bar{x}_2 \to_{\text{shift}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wv\bar{x}_2a\bar{x}_1x_2\bar{a}$
  $\to_{\text{shift}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wv\bar{a}\bar{x}_2a\bar{x}_1x_2 \to_{\text{shift}}$
  $\to_{\text{shift}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wv\bar{x}_1x_2\bar{a}\bar{x}_2a \sim W_\Sigma, x_2\bar{a}\bar{x}_2a\mathtt{tor}_p\mathtt{pjp}_q Wx_1wv\bar{x}_1 \sim$
  $\sim W_\Sigma, \mathtt{tor}_{p+1}\mathtt{pjp}_q Wx_1wv\bar{x}_1$, in terms of pq-permutations:
  $\Sigma, \{W, (w, v)\}_{\langle p+1,q\rangle}$.
- **Möbius:** $\Sigma, \{W, (w, a, v, a)\}_{\langle p,q\rangle}$ becomes $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wxwava\bar{x}$.
  $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wxwava\bar{x} \to_{\text{Lemma1}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wxw\bar{v}aa\bar{x} \to_{\text{Lemma2}}$
  $\to_{\text{Lemma2}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q aaWxw\bar{v}\bar{x} \sim W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_{q+1} Wxw\bar{v}\bar{x}$,
  namely $\Sigma, \{W, (w, \bar{v})\}_{\langle p,q+1\rangle}$.
- **Klein:** $\Sigma, \{W, (w, a), (a, v)\}_{\langle p,q\rangle}$ becomes $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wa\bar{x}_1x_2va\bar{x}_2$.
  $W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1wa\bar{x}_1x_2va\bar{x}_2 \to_{\text{Lemma1}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1w\bar{v}\bar{x}_2x_1aa\bar{x}_2$
  $\to_{\text{Lemma1}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q Wx_1w\bar{v}\bar{a}\bar{a}\bar{x}_1\bar{x}_2\bar{x}_2 \to_{\text{Lemma2}}$
  $\to_{\text{Lemma2}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q \bar{a}\bar{a}Wx_1w\bar{v}\bar{x}_1\bar{x}_2\bar{x}_2 \to_{\text{Lemma2}}$
  $\to_{\text{Lemma2}} W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_q \bar{a}\bar{a}\bar{x}_2\bar{x}_2Wx_1w\bar{v}\bar{x}_1 \sim W_\Sigma, \mathtt{tor}_p\mathtt{pjp}_{q+2} Wx_1w\bar{v}\bar{x}_1$.
  In terms of pq-permutations: $\Sigma, \{W, (w, v)\}_{\langle p,q+2\rangle}$
- **Sieve:** immediately by applying Lemma 3.

Example 4. As the reader can check below, $\Xi' = \{\{(d), (c)\}_{\langle 1,0\rangle}\}$ is the normal form of $\Xi = \{\{(d, y^{-1}, b^{-1}, z)\}_{\langle 0,0\rangle}, \{(z^{-1}, a, x, c, b, x^{-1}, a^{-1}, y)\}_{\langle 0,0\rangle}\}$:

$$\{(d, y^{-1}, b^{-1}, z)\}_{\langle 0,0\rangle}, \{(z^{-1}, a, x, c, b, x^{-1}, a^{-1}, y)\}_{\langle 0,0\rangle} \to_{\text{glue}}$$
$$\to_{\text{glue}} \{(d, y^{-1}, b^{-1}, a, x, c, b, x^{-1}, a^{-1}, y)\}_{\langle 0,0\rangle} \to_{\text{cyl.}}$$
$$\to_{\text{cyl.}} \{(d, y^{-1}, b^{-1}, a, a^{-1}, y), (c, b)\}_{\langle 0,0\rangle} \to_{\text{torus}}$$
$$\to_{\text{torus}} \{(d, y^{-1}, c, a, a^{-1}, y)\}_{\langle 1,0\rangle} \to_{\text{cyl.}} \{(d, y^{-1}, c, y)\}_{\langle 1,0\rangle} \to_{\text{cyl.}} \{(d), (c)\}_{\langle 1,0\rangle}.$$

By following the instructions provided by the previous proof, we obtain the following chain of words ending with a canonical form.

$$dy^{-1}b^{-1}z,\ z^{-1}axcbx^{-1}a^{-1}y \to_{\text{glue}} dy^{-1}b^{-1}axcbx^{-1}a^{-1}y \to_{\text{shift}}$$
$$dy^{-1}b^{-1}aa^{-1}yxcbx^{-1} \to_{\text{shift}} dy^{-1}x^{-1}b^{-1}aa^{-1}yxcb \to_{\text{shift}}$$
$$\to_{\text{shift}} dy^{-1}cbx^{-1}b^{-1}aa^{-1}yx \to_{\text{shift}} dy^{-1}caa^{-1}yxbx^{-1}b^{-1} \sim$$
$$\sim w_1dy^{-1}caa^{-1}y \to_{\text{cut.}} w_1dy^{-1}cy \sim w_1z^{-1}dzy^{-1}cy.$$

The just-mentioned theorem constitutes the arrival point of this paper: it says that all the transformations included in $\mathcal{P}$ do not affect the geometry of the denoted surface. It means that $\mathcal{P}$ induces an algorithm for computing the quotient surface associated with any given polygonal presentation.

Procedure 5 (computing surfaces) *We aim to compute the quotient surface* $\mathscr{S}$ *associated with the polygonal presentation* $W_{\mathscr{S}} = \{w_1, w_2, \ldots, w_n\}$*. We consider the set of pq-permutations* $\Sigma_{W_{\mathscr{S}}} = \{\alpha_{w_1}, \alpha_{w_2}, \ldots, \alpha_{w_n}\}$ *obtained by translating each polygon* $w_i \in W_{\mathscr{S}}$ *as follows:*

$$w_i = a_1 a_2 \ldots a_k \Rightarrow \alpha_{w_i} = \{(a_1, a_2, \ldots, a_k)\}_{\langle 0,0\rangle}.$$

Then we reduce $\Sigma_{W_{\mathscr{S}}}$ *to its normal form* $\Sigma'_{W_{\mathscr{S}}}$*: Theorem 4 ensures that* $\Sigma'_{W_{\mathscr{S}}}$ *exactly denotes the final surface we are looking for.*

As the reader will be able to notice by looking at the following examples, Procedure 5 turns out to be much more easy and intuitive with respect to the classical algorithm provided by Massey and illustrated in paragraph 2.2. This is essentially due to the fact that pq-permutations admit a set of transformations able to "mimic", step by step, the process of forming a surface $\mathscr{S}$ from an its polygonal presentation $W_{\mathscr{S}}$. From a strict algorithmic point of view, we can remark that, unlike Massey's algorithm, Procedure 5:

- does not require any information about vertices, because it works by only considering edges;
- is able to deal directly with boundary, so we cannot have to pose specific constraints on the starting polygonal presentation;
- provides a very clear combinatorial model of what exactly happens while composing a surface.

Example 5. We show that the connected sum of a torus with a projective plane is homeomorphic to a connected sum of three projective planes. The polygon denoting the surface at issue has perimeter: $ab\bar{a}\bar{b}cc$. According to Procedure 5, we normalise the pq-permutation $\{(a, b, \bar{a}, \bar{b}, c, c)\}_{\langle 0,0\rangle}$ as follows:

$$\{(a, b, \bar{a}, \bar{b}, c, c)\}_{\langle 0,0\rangle} \rightarrow_{\text{cyl.}} \{(b), (\bar{b}, c, c)\}_{\langle 0,0\rangle} \rightarrow_{\text{Mobius}}$$
$$\rightarrow_{\text{Mobius}} \{(b), (\bar{b})\}_{\langle 0,1\rangle} \rightarrow_{\text{torus}} \varnothing_{\langle 1,1\rangle} \rightarrow_{\text{sieve}} \varnothing_{\langle 0,3\rangle}.$$

Or, alternatively:

$$\{(a, b, \bar{a}, \bar{b}, c, c)\}_{\langle 0,0\rangle} \rightarrow_{\text{Mobius}} \{(a, b, \bar{a}, \bar{b})\}_{\langle 0,1\rangle} \rightarrow_{\text{cyl.}}$$
$$\rightarrow_{\text{cyl.}} \{(b), (\bar{b})\}_{\langle 0,1\rangle} \rightarrow_{\text{torus}} \varnothing_{\langle 1,1\rangle} \rightarrow_{\text{sieve}} \varnothing_{\langle 0,3\rangle}.$$

Example 6. We show that the polygon $ab\bar{a}b$ presents a Klein bottle, namely a surface homeomorphic to the connected sum of two projective planes. According to Procedure 5, we normalise the pq-permutation $\{(a, b, \bar{a}, b)\}_{\langle 0,0\rangle}$ as follows:

$$\{(a, b, \bar{a}, b)\}_{\langle 0,0\rangle} \rightarrow_{\text{cyl.}} \{(b), (b)\}_{\langle 0,0\rangle} \rightarrow_{\text{Klein}} \varnothing_{\langle 0,2\rangle}.$$

Or, alternatively:

$$\{(a, b, \bar{a}, b)\}_{\langle 0,0\rangle} \rightarrow_{\text{Mobius}} \{(a, a)\}_{\langle 0,1\rangle} \rightarrow_{\text{Mobius}} \varnothing_{\langle 0,2\rangle}.$$

Example 7. We stress Procedure 5 for showing that the two words

$$a_1a_2\dots a_{2n}\bar{a}_1\bar{a}_2\dots\bar{a}_{2n} \quad \text{and} \quad a_1a_2\dots a_n\bar{a}_1\bar{a}_2\dots\bar{a}_{n-1}a_n$$

constitute an alternative canonical form for the connected sum of respectively n tori and n projective planes (exercises proposed in [10] by Massey).

$$\{(a_1,a_2,a_3,\dots,a_{2n},\bar{a}_1,\bar{a}_2,\bar{a}_3,\dots,\bar{a}_{2n})\}_{\langle 0,0\rangle} \rightarrow_{\text{cyl.}}$$
$$\rightarrow_{\text{cyl.}} \{(a_2,a_3,\dots,a_{2n}),(\bar{a}_2,\bar{a}_3,\dots,\bar{a}_{2n})\}_{\langle 0,0\rangle} \rightarrow_{\text{torus}}$$
$$\rightarrow_{\text{torus}} \{(a_3,\dots,a_{2n},\bar{a}_3,\dots,\bar{a}_{2n})\}_{\langle 1,0\rangle} \rightsquigarrow_{\text{cyl.+torus}} \varnothing_{\langle n,0\rangle}.$$

$$\{(a_1,a_2,\dots,a_{n-1},a_n,\bar{a}_1,\bar{a}_2,\dots,\bar{a}_{n-1},a_n)\}_{\langle 0,0\rangle} \rightarrow_{\text{Mobius}}$$
$$\rightarrow_{\text{Mobius}} \{(a_1,a_2,\dots,a_{n-1},a_{n-1},\dots,a_2,a_1)\}_{\langle 0,1\rangle} \rightsquigarrow_{\text{Mobius}} \varnothing_{\langle 0,n\rangle}.$$

4 Future Work and Applications

Many directions of research are opened, not necessarily in convergent directions.

Some standard achievements in geometry of 2-dimensional manifolds are expected to be recovered by stressing pq-permutations and their algorithmic properties. *In primis*, we guess a new proof for the classification theorem to be obtained by showing that the rule of *sieve* is surperfluous. To be more precise, a polygon presenting an orientable surface should be rewritable by only applying *cylinder* and *torus*, whereas, in case of non-orientable surfaces, the *torus* rule should be shown redundant. Unlike that one afforded in this paper for computing surfaces, a classification-algorithm of this kind might be sufficiently expressive for posing the problem of its P or NP-completeness.

In this paper we have proposed an application of pq-permutations essentially concerning the direction of classification: from polygons to quotient surfaces. Nevertheless, we guess the converse direction (that one of triangulation) to be of interest all the same. We uphold in fact the idea that pq-permutations provide an optimal context for studying the decomposition of surfaces, especially in presence of specific constraints. To take an example, it is clear that a very easy proof of the Jordan curve theorem for closed surfaces can be inductively given by stressing the system $\mathcal{P}$ (in this case our constraint would be that one of connectness).

Finally, we hint at some possible applications in the framework of process calculi applied to biological systems. In Brane Calculi and their variants [3, 4], a topological context is imposed by the fact that membranes are two-dimensional fluids which interact embedded in a three-dimensional fluid. The structure of pq-permutations recall that one of membranes (at least in case of cyclic permutations) and some transformations considered by the system $\mathcal{P}$ would seem to be very close to Cardelli's bitonal interactions.

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