Elementary Complexity into the Hyperfinite II_1 Factor

Marco Pedicini[1] and Mario Piazza[2]

[1] IAC, Consiglio Nazionale delle Ricerche, Roma, Italy
marco@iac.cnr.it
[2] Dipartimento di Filosofia, Università di Chieti-Pescara, Italy
m.piazza@unich.it

Abstract. In this paper, we show how the framework of von Neumann algebras can be applied to model the dynamics of computational processes. Namely, our aim is to gain an understanding of classical computation in terms of the *hyperfinite factor*, starting from the class of Kalmar recursive functions. Our model fits within a vast project of reshaping the unified theory of semantics of computation called *geometry of interaction* along the lines recently sketched by J.Y. Girard.

Key words: von Neumann algebras, hyperfinite II_1 factor, classical Turing machines, elementary functions, implicit computational complexity.

1 Introduction

A von Neumann algebra is an algebra of bounded linear operators on a Hilbert space which is closed under the topology of pointwise convergence. *Factors* can be viewed as the elementary constituents from which all the von Neumann algebras are built, i.e., they are von Neumann algebras whose center consists of scalar multiples of the identity [MvN36,MvN37,MvN43].

In this paper, we show how a von Neumann algebras setting can be used in a natural way to obtain an implicit complexity model for classical Turing machine (TM). In particular, we focus on the unique von Neumann algebra which admits finite dimensional approximations and it is a factor II_1: the so called hyperfinite II_1 factor $\mathcal{R}$. More precisely, the focal point is the construction of an ascending sequence (G_i) of finite cyclic groups such that the size of cardinality of G_i doubles at every step, and whose infinite union is a discrete group G.

Then, we embed the configurations of a TM in the Hilbert space $\ell^2(G)$ of the square summable formal series indexed by elements of G with complex coefficients. Since the number of configurations in a finite part of the tape are bounded, there is a mapping which sends a configuration to a finite dimensional subspace. In this way, we model the transition of the machine by means of endomorphisms on the subspace. The set of endomorphisms constitutes a finite dimensional subalgebra of a von Neumann algebra. Namely, we build an element of $\ell^2(G)$, by superimposition, of all configurations appearing in the computation of the machine. Since DTIME($T(n)$) is contained in DTIME($2^{cT(n)}$), we obtain that any machine embedded in the group von Neumann algebra of G_n, denoted $\mathcal{N}(G_n)$, implicitly belongs to the computational class DSPACE(N) and so to DTIME($2^{c\,N}$). Thus, we associate in a natural way a given complexity class of exponential kind to every element of the ascending sequence of algebras $(\mathcal{N}(G_i))_i$. Since G is locally finite but it fails to be an ICC group, $\mathcal{N}(G)$ is not isomorphic to $\mathcal{R}$. In order to obtain an embedding of TM in $\mathcal{R}$, we build an embedding of $\mathcal{N}(G)$ into $\mathcal{R}$, viewed as the group algebra associated with the group of permutations of integers which fix all but finitely many integers.

Our investigation intersects J.-Y. Girard's recent proposal, conceptually ambitious, to reshape the semantics of computation called *geometry of interaction* (GoI) in the realm of von Neumann algebras [Gir06a]. In previous works, [Gir89,Gir90,Gir95,Gir06b], GoI is built in a $\mathbb{C}^*$-algebra: proofs correspond to bounded operators of the infinite dimension Hilbert space, and the execution formula corresponds to the power series of the operator itself. Since von Neumann algebras are $\mathbb{C}^*$-algebras, in this hyperfinite setting we maintain all the advantages of the standard view of GoI and at the same time we obtain an universal mathematical object: the *unique* $\mathcal{R}$, contained in any II_1 factor. More generally, the ideas of this paper provide pointers towards the construction of a hyperfinite semantics of computation with an implicit complexity bound.

2 Background

Let G be a (discrete) group fixed throughout the article. Let $\mathbb{C}[[G]]$ denote the set of all functions from G to $\mathbb{C}$ expressed as formal sums, that is, a function $a : G \to \mathbb{C}$, $g \mapsto a(g)$, will be written as $\sum_{g\in G} a(g)g$. For each $a \in \mathbb{C}[[G]]$, we define $||a|| := (\sum_{g\in G} |a(g)|^2)^{1/2} \in [0, \infty]$, and $tr(a) := a(1) \in C$.

Define

$$\ell^2(G) := \{a \in \mathbb{C}[[G]] : ||a|| < \infty\}.$$

We view $\mathbb{C} \subset \mathbb{C}G \subset \ell^2(G) \subset \mathbb{C}[[G]]$. There is a well-defined external multiplication map

$$\ell^2(G) \times \ell^2(G) \to \mathbb{C}[[G]], \quad (a, b) \mapsto a \cdot b,$$

where, for each $g \in G$, $(a \cdot b)(g) := \sum_{h\in G} a(h)b(h^{-1}g)$; this sum converges in $\mathbb{C}$, and, moreover, $|(a \cdot b)(g)| \leq ||a||\,||b||$, by the Cauchy-Schwarz inequality. The external multiplication extends the multiplication of $\mathbb{C}G$.

Then $\ell^2(G)$ is a Hilbert space with the scalar product defined by

$$\langle \sum_{g\in G} \lambda_g g, \sum_{g\in G} \mu_g g\rangle = \sum_{g\in G} \lambda_g \overline{\mu}_g$$

where $\overline{\lambda}$ denotes the complex conjugate of λ.

The *group von Neumann algebra of* G, denoted $\mathcal{N}(G)$, is defined as the algebra of G-equivariant bounded operators from $\ell^2(G)$ to $\ell^2(G)$:

$$\mathcal{N}(G) := \{\alpha : \ell^2(G) \to \ell^2(G) \mid ||\alpha|| < +\infty, \text{ for any } g, h \in G,\ g\alpha(h) = \alpha(gh)\}.$$

In other words, α is equivariant by the group action. This definition can be also rephrased as: $\mathcal{N}(G)$ is the ring of bounded $\mathbb{C}G$-endomorphisms of the right $\mathbb{C}G$-module $\ell^2(G)$; see [Lüc02, 1.1]. We view $\mathcal{N}(G)$ as a subset of $\ell^2(G)$ by the map $\alpha \mapsto \alpha(1)$, where 1 denotes the identity element of $\mathbb{C}G \subset \ell^2(G)$ and moreover,

$$\mathcal{N}(G) = \{a \in \ell^2(G) | a \cdot \ell^2(G) \subset \ell^2(G)\}. \tag{2.1}$$

The action of $\mathcal{N}(G)$ on $\ell^2(G)$ is given by the external multiplication. Notice that $\mathcal{N}(G)$ contains $\mathbb{C}G$ as a subring and that there exists an induced *trace map* $tr : \mathcal{N}(G) \to \mathbb{C}$, i.e., $tr(\alpha) := tr(\alpha(1)) = \alpha(1)(1)$. Of course, $\mathcal{N}(G)$ is a von Neumann algebra. For more details on von Neumann algebras we refer the reader to standard monographs, for example [Lüc02].

2.1 General construction of the Hyperfinite II_1 factor

Let us recall that the fundamental result of Murray and von Neumann ensures that the hyperfinite II_1 factor is essentially unique (precisely, that there is one isomorphism class, [MvN43]). This means that its construction can be achieved in many different ways.

In particular, the discrete group G is obtained by an approximation procedure, i.e., by giving an ascending sequence of finite subgroups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n \subset \ldots$$

such that $G = \bigcup_{i=0}^{\infty} G_i$. In this case we say G is *locally finite*. This condition is not sufficient to make $\mathcal{N}(G)$ a factor. Now, we need to establish the following lemma.

Lemma 1. *$\bigcup_{i=0}^{\infty} \mathcal{N}(G_i)$ is weakly dense in the von Neumann group algebra of $G = \bigcup_{i=0}^{\infty} G_i$.*

Proof. First we prove that any $a \in \mathcal{N}(G_i)$ is indeed in $\mathcal{N}(G)$. In fact, if $a \in \ell^2(G_i)$, then, by equation 2.1, $a \cdot \ell^2(G_i) \in \ell^2(G_i)$. To prove also that $a \in \mathcal{N}(G)$, we must show that $a \cdot b \in \mathcal{N}(G)$, for any $b \in \mathcal{N}(G)$; since $a \cdot b \in \mathbb{C}[G]$, we show that $a \cdot b$ is bounded. In fact, by definition

$$||a \cdot b|| = \left(\sum_{g \in G} |(a \cdot b)(g)|^2 \right)^{1/2} = \left(\sum_{g \in G} \left| \sum_{h \in G_i} a(h) b(h^{-1} g) \right|^2 \right)^{1/2},$$

and in order to prove $||a \cdot b|| \leq c$ for some c, we have

$$\begin{aligned} ||a \cdot b||^2 \leq ||a||^2 \left(\sum_{h \in G_i} \sum_{g \in G} |b(h^{-1} g)|^2 + (\sum_{h_1 \neq h_2 \in G_i} \sum_{g \in G} |b(h_1^{-1} g)| |b(h_2^{-1} g)| \right) \leq \\ \leq ||a||^2 (|G_i| ||b||^2 + \sum_{h_1 \neq h_2 \in G_i} 2||b||^2) = \\ = ||a||^2 (|G_i| + 2|G_i|(|G_i| - 1)) ||b||^2. \end{aligned} \tag{2.2}$$

Specifically, in order to prove inequality (2.2), we consider

$$\sum_{g\in G} |b(h_1^{-1}g)||b(h_2^{-1}g)| = \sum_{g\in G} |b(g)||b(h_2^{-1}h_1 g)| \leq 2\sum_{g\in G} |b(g)|^2 = 2||b||^2$$

in fact, for any $g \in G$ and for any fixed pair $h_1, h_2 \in G_i$, we define

$$c(g) = \begin{cases} |b(g)|^2 & \text{if } |b(g)| > |b(h_2^{-1}h_1 g)|, \\ |b(h_2^{-1}h_1 g)|^2 & \text{if } |b(g)| \leq |b(h_2^{-1}h_1 g)|. \end{cases}$$

Since any term of $\sum_{g\in G} |b(g)|^2$ appears in $\sum_{g\in G} c(g)$ at most twice we have the following inequality

$$\sum_{g\in G} |b(g)||b(h_2^{-1}h_1 g)| \leq \sum_{g\in G} c(g) \leq 2\sum_{g\in G} |b(g)|^2 = 2||b||^2.$$

Since $\mathcal{N}(G)$ is a von Neumann algebra, any pointwise convergent sequence (α_k) of operators in $\mathcal{N}(G_i)$ converges to an element in $\mathcal{N}(G)$. The reason is that if $\alpha_k \in \mathcal{N}(G_i)$, then $\alpha_k \in \mathcal{N}(G)$ for any k, which is dense by definition and so $\phi \in \mathcal{N}(G)$. □

An immediate consequence of the Lemma 1, is that $\mathcal{N}(G)$ is an AFD (approximately finite dimensional) von Neumann algebra, that is an algebra A such that there is a family A_i of finite dimensional subalgebras whose union $\bigcup_{i=0}^{\infty} A_i$ is σ-weakly dense in A.

A common way to obtain an isomorphic copy of $\mathcal{R}$ is based on the notion of ICC group:

Definition 2.1. *A discrete group G is an* ICC group *if every non-trivial conjugacy class $C(h) = \{ghg^{-1} | g \in G\}$, for $h \neq 1_G$, is infinite.*

This property is sufficient to obtain a II_1 factor, but not to obtain a *hyperfinite* one (as it is the case when we consider free groups generated by $n > 1$ generators which are ICC groups but not locally finite):

Proposition 2.1. *If G is an ICC group, then $\mathcal{N}(G)$ is a II_1 factor.*

Proposition 2.2. *If G is a locally finite ICC group, then $\mathcal{N}(G)$ is a hyperfinite II_1 factor.*

The classical example of a locally finite ICC group is the group S_X of permutations $\pi : X \to X$ of a countable set X such that $\pi(x) = x$ for all $x \in X$ but a finite subset of X. By Proposition 2.2, it follows that $\mathcal{N}(S_X)$ is a hyperfinite II_1 factor [Bla06].

For any $n \in \mathbb{N}$, we can consider S_n as the permutation group which moves only n elements of X and $S_X = \bigcup_{n\in\mathbb{N}} S_n$. Moreover, if G is a locally finite discrete group, then for any index $i \in \mathbb{N}$ we can find an index $N(i)$ such that G_i

is subgroup of $S_{N(i)}$ (by the Cayley theorem). This means that any $a \in \ell^2(G_i)$ is naturally embedded in $J_i(a) \in \ell^2(S_X)$:

$$J_i(a)(h) := \begin{cases} a(h) & h \in G_i, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we can analogously define two embeddings $j_i : \ell^2(G_i) \to \ell^2(G)$ and $J : \ell^2(G) \to \ell^2(S_X)$.

Notice that by equivariance these embeddings lift up to embeddings on the corresponding von Neumann group algebras: $j_i : \mathcal{N}(G_i) \to \mathcal{N}(G)$, $J_i : \mathcal{N}(G_i) \to \mathcal{N}(S_X)$ and $J : \mathcal{N}(G) \to \mathcal{N}(S_X)$ respectively. Moreover, for any $\phi \in \mathcal{N}(G)$ we have that $J(\phi) \in \mathcal{N}(S_X)$. Indeed, this follows from pointwise convergence of the sequence $(j_k(\phi_k))$, where $\phi_k \in \mathcal{N}(G_k)$, to $\phi \in \mathcal{N}(G)$. By the fact that embeddings act isometrically on operators we get that the sequence $(J_k(\phi_k))$ pointwise converges to $J(\phi) \in \mathcal{N}(S_X)$.

3 A construction of an AFD von Neumann algebra

In this section, we present an instance of the construction of the G_i such that the Jones index [Jon83], i.e., the relative dimension of successive groups, is $[G_i : G_{i+1}] = 2$. Moreover, we define $G_{i+1} = G_i + \mu_i G_i$ where μ_i is a new generator. The sequence starts at G_0, the trivial group containing the sole identity, $G_1 := C_2$ is the cyclic group of 2 elements, and in general $G_i := C_{2^i}$. Cyclic groups are abelian with one group generator. We give a construction of the multiplication table of G_i in such a way that its group generator coincides with the element $\mu_i = 2^i$. In fact, this construction is recursive and uses an auxiliary permutation matrix $T(i)$, which generalises the twist and gives an isomorphic presentation of C_{2^i}:

$$T(i) := \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } i = 1 \\ \\ \begin{pmatrix} 0 & I \\ T(i-1) & 0 \end{pmatrix} & \text{otherwise.} \end{cases} \tag{3.1}$$

In a similar way we give the construction of the multiplication table of G_i:

$$G_i := \begin{cases} \begin{pmatrix} 0 \end{pmatrix} & \text{if } i = 0, \\ \\ \begin{pmatrix} G_{i-1} & 2^{i-1} + G_{i-1} \\ 2^{i-1} + G_{i-1} & T(i-1).G_{i-1} \end{pmatrix} & \text{otherwise.} \end{cases} \tag{3.2}$$

The main feature of this construction is that any G_i is subgroup of G_{i+1} and it appears directly in its multiplication table as in the top-left corner. Another important property is that the generator of the group is the element 2^i.

In Figure 1, we have depicted with grey tones the multiplication tables of several G_i's, in order to graphically point out the structure of these groups. The first groups are:

$$G_1 = \begin{pmatrix} 0\,1 \\ \underline{1\,0} \end{pmatrix} \qquad G_2 = \begin{pmatrix} 0\,1\,2\,3 \\ 1\,0\,3\,2 \\ \underline{2\,3\,1\,0} \\ 3\,2\,0\,1 \end{pmatrix} \qquad G_3 = \begin{pmatrix} 0\,1\,2\,3\,4\,5\,6\,7 \\ 1\,0\,3\,2\,5\,4\,7\,6 \\ 2\,3\,1\,0\,6\,7\,5\,4 \\ 3\,2\,0\,1\,7\,6\,4\,5 \\ \underline{4\,5\,6\,7\,2\,3\,1\,0} \\ 5\,4\,7\,6\,3\,2\,0\,1 \\ 6\,7\,5\,4\,1\,0\,3\,2 \\ 7\,6\,4\,5\,0\,1\,2\,3 \end{pmatrix}$$

The underlined element of the table is μ_i and let us consider $\mu_3 = 2^2 = 4$ and its row in the multiplication table. With the aid of this table one may compute the different orbits for the iterated action of μ_i over elements of G_i,

$$O_i(x) := (x, \mu_i \cdot x, \mu_i \cdot \mu_i \cdot x, \ldots, \mu_i^{2^i-1} \cdot x), \qquad \text{for all } x \in G_i$$

for instance $O_3(1) = (1, 5, 3, 7, 0, 4, 2, 6)$

$$(O_3(x))_{0 \le x \le 7} = \begin{pmatrix} 0\,4\,2\,6\,1\,5\,3\,7 \\ 1\,5\,3\,7\,0\,4\,2\,6 \\ 2\,6\,1\,5\,3\,7\,0\,4 \\ 3\,7\,0\,4\,2\,6\,1\,5 \\ 4\,2\,6\,1\,5\,3\,7\,0 \\ 5\,3\,7\,0\,4\,2\,6\,1 \\ 6\,1\,5\,3\,7\,0\,4\,2 \\ 7\,0\,4\,2\,6\,1\,5\,3 \end{pmatrix}.$$

Furthermore, we have also a formula defining the generator of G_i:

$$\mu_i = \bigotimes_{k=0}^{i} I_{2^k} \oplus 2^k. \tag{3.3}$$

Example 1. Let us compute the generators of the G_i's:

$$\begin{aligned} &\mu_0 = (0) \\ &\mu_1 = ((0) + 1) \otimes (0) = (10) \\ &\mu_2 = ((01) + 2) \otimes (10) = (23) \otimes (10) = (2310) \\ &\mu_3 = ((0123) + 4) \otimes (2310) = (4567) \otimes (2310) = (45672310) \end{aligned}$$

we note that for any i we have

$$\mu_{i+1} = (I_{2^i} + 2^i) \otimes \mu_i$$

and by iteratively expanding the above formula we get

$$\mu_{i+1} = (I_{2^i} + 2^i) \otimes \mu_i = (I_{2^i} + 2^i) \otimes (I_{2^{i-1}} + 2^{i-1}) \otimes \mu_{i-1}.$$

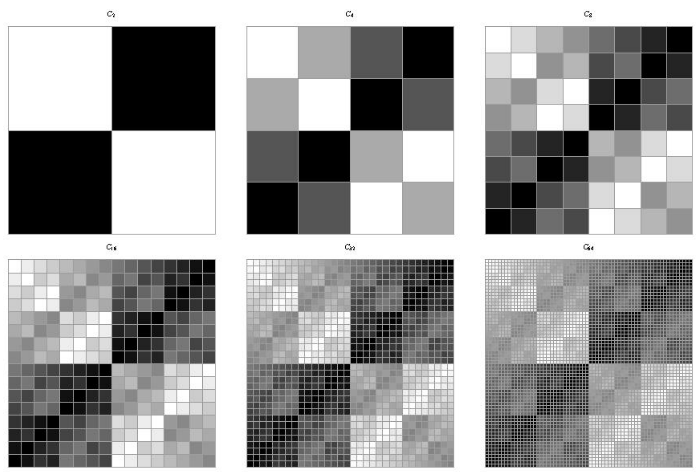

Fig. 1. Group Multiplication Tables for Cyclic groups C_{2^i}, for $i = 1, \ldots, 6$.

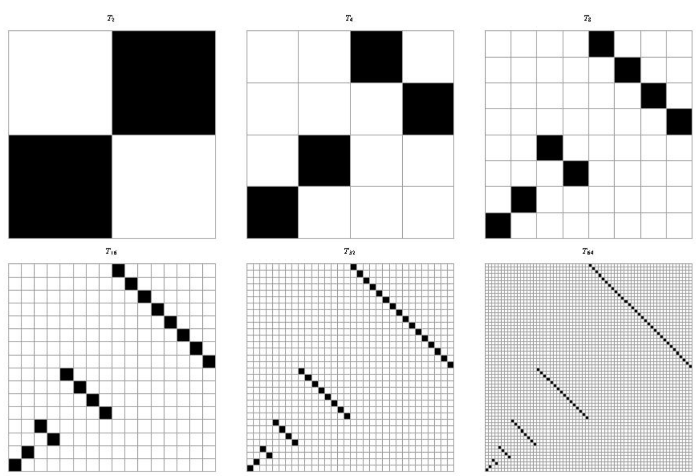

Fig. 2. Matrix T_i, for $i = 1, \ldots, 6$.

4 Deterministic TMs encoded in a von Neumann Algebra

In this section, we present an encoding of TM's in operators acting on von Neumann group algebra $\mathcal{N}(G)$ of the group G, that we have introduced in the previous section. Leaving aside the obvious differences, our contruction is similar to the one given by Nishimura and Ozawa [ON00,NO02]. Without loss of generality, we are concerned with one-way infinite tape machines whose alphabet is reduced to a unique symbol $A = \{1\}$. For any deterministic, one-way infinite tape TM with set of states Q and alphabet A,

$$\mu : Q \times A \to Q' \times A' \times \{-1, +1\},$$

where $A' = A \cup \{\Box\}$ and Q' is $Q' = Q \cup \{q_F^1, q_F^0\}$, we consider an encoding of the transition function acting on the configuration space:

$$C = \{(q, p, f) \mid q \in Q, p \in \mathbb{N}, f : \mathbb{N} \to A'\}.$$

We define subsets C_i of C indexed by integers by considering configurations concerning cells in positions $p \in [0, i]$:

$$C_i = \{(q, p, f) \mid q \in Q, p \in [0, i], f : [0, i] \to A'\}.$$

Let us note that $|C_i| = |Q|\, i\, 2^i$, and that by choosing $s = \log_2(i)$, and $q = \log_2 |Q|$, we have $|C_i| \leq 2^{N_i}$ where $N_i := s + q + i$; this also shows that $C = \bigcup_{i \in \mathbb{N}} C_i$ is denumerable. Let us consider a bijection $\phi : C \to G$. For any $c \in C$, we have one and only one $\phi(c) \in G$ and ϕ induces a correspondence between the configuration space C of μ and $\ell^2(G)$, such that for any configuration $c \in C$ one may define the corresponding element, denoted $\psi(c) \in \ell^2(G)$, in the following way:

$$\psi(c)(g) := \begin{cases} 1 & \text{if } \phi(c) = g, \\ 0 & \textit{otherwise}. \end{cases}$$

We have that $||\psi(c)|| = 1 < +\infty$; notice that $(\psi(c))_{c \in C}$ is automatically an orthonormal basis for a subspace of $\ell^2(G)$. Moreover, for any positive integer i, we consider the restriction $\phi_i : C_i \to G_{N_i}$ of ϕ defined as

$$\phi_i(c) := \phi(c).$$

From this restriction, we obtain a correspondence $\psi_i : C_i \to \ell^2(G_{N_i})$ as above. Let us observe in passing that the dimension of $\ell^2(G_{N_i})$ is 2^{N_i} which is $|C_i|$.

Let be (μ, c_0) a pair where μ is a TM computing a function f in DSPACE$(S(n))$, and c_0 is the initial configuration associated with input $x = (x_1, \ldots, x_n)$ such that $i = ||x||$. Then, we consider the computation of μ starting from c_0 as the finite sequence

$$(c_0, c_1, \ldots c_k) \quad \text{where } k \leq 2^N$$

and we embed this sequence in the element

$$[[(\mu, c_0)]] := \sum_{j=0}^{k} \psi(c_j) \in \ell^2(G).$$

Definition 4.1. *For any TM μ and for any initial configuration c_0, we define the interpretation $[c_j]$ for any configuration c_j appearing in the computation*

$$(c_0, c_1, \dots c_k) \quad \textit{where } k \leq 2^N \textit{ as } [c_j] := \mu_N^j([[(\mu, c_0)]])$$

where μ_N is the operator corresponding to the action of the group generator of G_N.

By definition, it is clear that $[c_{j+1}] = \mu_N([c_j])$. So, the operator μ_N implements the TM μ in the sense that it makes the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\mu} & C \\ \big\downarrow [.] & & \big\downarrow [.] \\ \ell^2(G) & \xrightarrow{\mu_N} & \ell^2(G). \end{array}$$

This commutativity enable us to prove the following proposition:

Proposition 4.1. *For any deterministic TM μ in $\mathrm{DSPACE}(S(n))$ and for any input x, $||x|| \leq n$ there is an interpretation $[.] : C \to \ell^2(S_X)$ such that there exists an operator $[\mu] \in \mathcal{R}$ such that $[\mu]([c_j]) = [\mu(c_j)]$.*

Proof. Let us consider the interpretation $[.] : C \to \ell^2(G)$ in Definition 4.1, by composition with the embedding $J : \ell^2(G) \to \ell^2(S_X)$. Therefore, we obtain an interpretation $J[.] : C \to \ell^2(S_X)$, where $J[c] := J([c])$. Then, we define $[\mu] := J_{S(N)}[\mu_{S(N)}$ as the embedding via $J_{S(N)} : \mathcal{N}(G_{S(N)}) \to \mathcal{N}(S_X)$ of the operator μ_N associated with the generator of the group $G_{S(N)}$. By Lemma 1, it follows that $[\mu] \in \mathcal{N}(S_X) = \mathcal{R}$. $\square$

5 Space Bounded TMs

Let us denote by $\mathcal{E}$ the class of elementary functions, which was defined by Kalmár [Kal43] as the least class of primitive recursive functions that contains the constant 0, all projections, successor, addition, cut-off subtraction and multiplication, and is closed under composition, bounded sum and bounded product.

Since $\mathcal{E}$ is closed under composition, for each m the m-times iterated exponential $2^{[m]}(x)$ is in $\mathcal{E}$, where $2^{[m+1]}(x) = 2^{2^{[m]}(x)}$ and $2^{[0]}(x) = x$. The elementary functions are exactly the functions computable in elementary time, i.e., the class of functions computable by a TM in a number of steps bounded by some elementary function. Two results are well known concerning the class $\mathcal{E}$:

Proposition 5.1. *1. $\mathcal{E} = \mathrm{DTIME}(\mathcal{E}) = \mathrm{DSPACE}(\mathcal{E})$.*
2. If $f \in \mathcal{E}$, there is a number m such that for all $(x_1, \dots, x_n)$, $f(x_1, \dots, x_n) \leq 2^{[m]}(||(x_1, \dots, x_n)||)$ where $||(x_1, \dots, x_n)|| := \max_{1 \leq i \leq n} |x_i|$.

However, Proposition 5.1.2 tell us that $\mathcal{E}$ does not contain the iterated exponential $2^{[m]}(||x||)$ where the number of iterations m is a variable, since any function in $\mathcal{E}$ has an upper bound where m is fixed. This remark is useful to obtain the following result:

Proposition 5.2. *Given a TM μ computing a Kalmar elementary function $f \in \mathcal{E}$, there exists an integer m such that for every input x, $|x| \leq n$, then the computation of the machine μ starting from the initial configuration associated with x is representable in $\mathcal{N}(G_{2^{[m^*]}(n)})$.*

Proof. By Proposition 5.1.1, we get that since μ computes $f \in \mathcal{E}$ there exists $g \in \mathcal{E}$ such that $f \in \mathrm{DSPACE}(g(n))$. Thus, for every input $(x_1, \ldots, x_n)$ $||x|| \leq n$, the machine μ has halt space $s \leq g(n)$, which by Proposition 5.1.2 implies that there exists m such that $s \leq 2^{[m]}(||x||)$.

For Proposition 4.1, by choosing $m^* = 2^{[m-1]}(n)$, we have that the computation of the machine μ starting from the initial configuration associated with x is representable in $\mathcal{N}(G_{2^{[m^*]}(n)})$. □

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