Permutative Logic*

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Abstract. Recent work establishes a direct link between the complexity of a linear logic proof in terms of the exchange rule and the topological complexity of its corresponding proof net, expressed as the minimal rank of the surfaces on which the proof net can be drawn without crossing edges. That surface is essentially computed by sequentialising the proof net into a sequent calculus which is derived from that of linear logic by attaching an appropriate structure to the sequents. We show here that this topological calculus can be given a better-behaved logical status, when viewed in the variety-presentation framework introduced by the first author. This change of viewpoint gives rise to permutative logic, which enjoys cut elimination and focussing properties and comes equipped with new modalities for the management of the exchange rule. Moreover, both cyclic and linear logic are shown to be embedded into permutative logic. It provides the natural logical framework in which to study and constrain the topological complexity of proofs, and hence the use of the exchange rule.

1 Introduction

In order to study proofs as topological objects, notably proofs of linear logic [7], one is naturally led to view proof nets as surfaces on which the usual proofs are drawn without crossing edges [5, 14, 13]. Recent work by Métayer [14] establishes a direct link between the complexity of a linear logic proof in terms of the exchange rule and the topological complexity of its corresponding proof net, expressed as the minimal rank of the compact oriented surfaces with boundary on which the proof net can be drawn without crossing edges and with the conclusions of the proof on the boundary. For instance, cyclic linear logic proofs [19] are drawn on disks since they are purely non-commutative, and the standard proof of $\vdash (A \otimes B) \multimap (B \otimes A)$ is drawn on a torus with a single hole. In general, exchange rules introduce handles or disconnect the boundary.

Gaubert [6] shows that that surface can be computed by sequentialising the proof net into a sequent calculus, proposed by the third author, which is derived from that of linear logic by incorporating an appropriate structure to the

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sequents. Indeed, the above surfaces turn out to be oriented, and it is standard that any oriented compact surface is homeomorphic to a connected sum of tori (see, e.g., [12]). On the other hand, the conclusions of the proofs are drawn on disjoint oriented circles, hence the appropriate structure in [6] is that of a permutation (product of disjoint cycles) together with a natural number (number of tori), actually a complete topological invariant of the surface.

Interestingly, these structures and the operations which are performed on them constitute an instance of the variety-presentation framework introduced in [3]: the varieties we consider in the present paper are the structures used in [6], our presentations are simply varieties with a distinguished point, and both are related by simple axioms, which sort of generalise the properties of partial orders and order varieties in non-commutative logic [2].

We show that the calculus in [6] can be given a better-behaved logical status, when viewed in this framework. This change of viewpoint gives rise to permutative logic, PL for short, where connectives are presentations together with a polarity (positive or negative): the usual pair $\otimes, \parr$ of linear logic is naturally extended with new modalities $\#, \flat$ for (dis)connecting cycles and new constants $h, \hbar$ for the management of handles. The sequents of PL are varieties and the sequent calculus comes with structural rules, also considered in Melliès' planar logic [13]. The sequent calculus of PL enjoys cut elimination and the focussing [15, 4] property; these properties do not hold in [6] because the two par rules are not reversible. Moreover, both cyclic and linear logic are shown to be embedded into PL.

Unlike [14, 6] which enable to quantify the exchange and topological complexities of a proof, PL provides control mechanisms and is the natural logical framework in which to study and constrain these complexities. We believe in particular that PL should be of interest to concurrent programming and computational linguistics, two fields in which these issues matter [18, 8, 10, 11, 1].

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2 Surfaces and permutations

2.1 Q-permutations

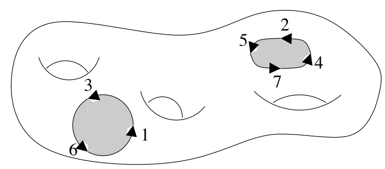
Surfaces (with or without boundary) are connected 2-dimensional topological manifolds, and it is standard that any compact surface is homeomorphic to a connected sum of tori and projective planes. In the case of orientable compact surfaces, the above homeomorphism is simply with a connected sum of tori. For instance, the sphere corresponds to a sum of 0 torus, etc. For a classical textbook on algebraic topology, we refer the reader to, e.g., [12].

We consider here oriented compact surfaces *with decomposed boundary*, i.e., triples (S, X, ι) where S is a compact surface with boundary and a given orientation, X is a finite set and $\iota : X \to \partial S$ is an injective map from X into to the boundary ∂S of S, such that any hole (i.e., connected component of the boundary) contains at least one distinguished point (i.e., a point in the image of ι). Since holes are circles topologically, the last condition says exactly that X induces a cell decomposition of ∂S (into one or several edges).

These surfaces with decomposed boundary are the objects of a category, which is simply a subcategory of the category of pairs of CW-complexes considered for instance in relative homology: a morphism (resp. isomorphism) from (S, X, ι) to (S', X, ι') is an orientation-preserving continuous map (resp. homeomorphism) $f : S \to S'$ such that $f(\iota(x)) = \iota'(x)$ for each $x \in X$.

Now, an oriented compact surface with decomposed boundary (S, X, ι) induces a cyclic order on each subset of X which is the inverse image by ι of a hole of S. By taking the product of these disjoint cycles, we obtain a permutation $\sigma \in \mathfrak{S}(X)$. On the other hand, S comes with a natural number d called the genre of S, the number of tori (handles) in the connected sum forming S. This leads to the following definition of a q-permutation (where q is meant to remind that a quantity, here a natural number, is attached to the permutation).

Definition 1 (q-permutation). *A* q-permutation *is a triple* (X, σ, d) *where* X *is a finite set,* σ *is a permutation on* X *and* d *is a natural number.*



Hence, to each oriented compact surface with decomposed boundary (S, X, ι) having d handles is associated the q-permutation (X, σ, d) with σ defined as above. For instance, the surface with decomposed boundary illustrated in the above figure induces the q-permutation $(X, \{(1, 3, 6), (2, 5, 7, 4)\}, 3)$ on $X = \{1, \ldots, 7\}$. It is clear that (X, σ, d) is invariant under isomorphism: the number of handles is a topological invariant, and so is the cyclic order on each hole because orientation is preserved. We actually have a complete invariant: (S, X, ι) is isomorphic to (S', X, ι') if, and only if, the associated q-permutations are equal. In the sequel, all the operations we define can be interpreted either in terms of q-permutations or in terms of oriented compact surfaces with decomposed boundary up to isomorphism.

2.2 The variety-presentation framework of q-permutations

Q-permutations form a variety-presentation framework as defined in [3]. We give here the ingredients of the variety-presentation framework of q-permutations, i.e., the support set operator, the promotion, composition and decomposition operators and the relaxation relation.[1]

We assume given an arbitrary countably infinite set $\mathcal{P}$, the elements of which are called places, and a distinguished element $\mathbf{0} \notin \mathcal{P}$. Now, a variety (resp. a presentation) is simply a q-permutation on a finite subset of $\mathcal{P} \cup \{\mathbf{0}\}$ which does not contain (resp. contains) $\mathbf{0}$. This is consistent with the usual view of presentations as varieties with a distinguished place, which is generic to all variety-presentation frameworks.

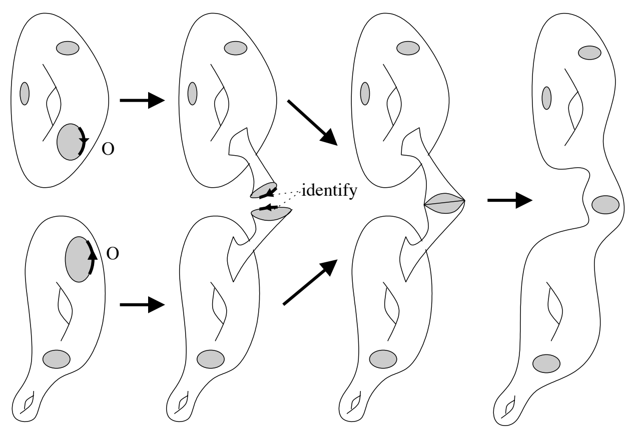
[1] In fact, we adopt a slight variant in the presentation w.r.t. [3] as to the status of places and of the support set operator.

Definition 2 (support set, promotion, void presentation). *For any q-permutation* $\mu = (X, \sigma, d)$*, its* support set *is defined by* $|\mu| = X \cap \mathcal{P}$*. Any place* $x \in \mathcal{P}$ *can be associated with a presentation, called its* promotion*, which is the q-permutation* $(\{\mathbf{0}, x\}, \chi_{\mathbf{0},x}, 0)$*, where* $\chi_{a,b}$ *denotes the transposition exchanging* a *and* b*. By abuse of notation, it will be denoted by* x *so that* $|x| = \{x\}$*. Finally, the* void *presentation* $\bigcirc$ *is the q-permutation* $(\{\mathbf{0}\}, \emptyset, 0)$*; obviously* $|\bigcirc| = \emptyset$*.*

The topological interpretation of the promotion of x (resp. of the void presentation) is a disk the border of which is labelled by $\mathbf{0}$ and x (resp. $\mathbf{0}$ alone).

Definition 3 (composition). *Let* $\omega = (X, \sigma, d)$ *and* $\tau = (Y, \theta, e)$ *be presentations such that* $|\omega| \cap |\tau| = \emptyset$ *(i.e.,* $X \cap Y = \{\mathbf{0}\}$*). Then* $\omega * \tau$ *is the variety* (Z, ξ, f) *where*

- $Z = (X \cup Y) \setminus \{\mathbf{0}\}$,
- *if* $\sigma_1, \ldots, \sigma_p, (\mathbf{0}, \gamma)$ *are the disjoint cycles of* σ *and* $\theta_1, \ldots, \theta_q, (\mathbf{0}, \delta)$ *are the disjoint cycles of* θ *(here,* γ *and* δ *are ordered lists of places), then the disjoint cycles of* ξ *are*
  - *either* $\sigma_1, \ldots, \sigma_p, \theta_1, \ldots, \theta_q, (\gamma, \delta)$ *when* γ *or* δ *is non-empty*
  - *or* $\sigma_1, \ldots, \sigma_p, \theta_1, \ldots, \theta_q$ *when both* γ *and* δ *are empty, i.e. when* $\sigma(\mathbf{0}) = \theta(\mathbf{0}) = \mathbf{0}$,
- $f = d + e$.



The permutation ξ above is obtained by gluing at $\mathbf{0}$ the orbits of $\mathbf{0}$ in σ and θ. In terms of surfaces, the composition operator is the amalgamated sum of the two surfaces over a small interval around $\mathbf{0}$ on the boundary. Standard topology of surfaces ensures that the result is indeed an oriented surface. This can be visualised in the above figure. The number of holes in the output surface is the sum of the numbers of holes in the input ones, decreased by one (the two holes containing $\mathbf{0}$ have been merged into one) or two (if the two holes containing $\mathbf{0}$ contain no other distinguished point).

Definition 4 (decomposition). *Let* $\alpha = (X, \sigma, d)$ *be a variety and* $x \in |\alpha| = X$*. The presentation* $(\alpha)_x$ *is defined as the triple* $(X \setminus \{x\} \cup \{\mathbf{0}\}, \sigma', d)$ *where* σ' *is obtained from* σ *by replacing* x *by* $\mathbf{0}$*.*

The topological interpretation of this operation is quite straightforward: it does not change the surface, simply relabels x as $\mathbf{0}$.

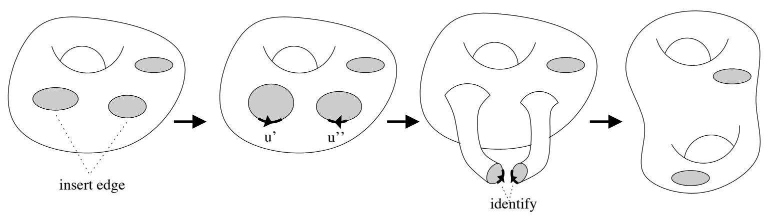
Definition 5 (relaxation). *The* relaxation *relation is the smallest reflexive transitive relation* $\preccurlyeq$ *on q-permutations such that:*

- divide*:* $(X, \sigma, d) \preccurlyeq (X, \theta, d)$*, where* σ *is obtained from* θ *by dividing one cycle* (γ, δ) *into two:* (γ) *and* (δ)*,*
- merge*:* $(X, \sigma, d + 1) \preccurlyeq (X, \theta, d)$*, where* σ *is obtained from* θ *by merging two cycles* (γ) *and* (δ) *into one* (γ, δ)*,*
- degenerate merge*:* $(X, \sigma, d + 1) \preccurlyeq (X, \sigma, d)$*.*

The degenerate merge rule is in fact obtained by taking (γ) or (δ) empty in the merge rule, but we make it a separate case since the cycles of a permutation are, by definition, non-empty. Since both divide and merge increment the rank $2d + p - 1$ of a q-permutation (X, σ, d) where σ is a permutation with p cycles, we have:

Proposition 1. *Relaxation is a partial order on q-permutations.*

The topological interpretation of relaxation is simply an amalgamated sum: given an oriented compact surface with decomposed boundary (S, X, ι), take two intervals u' and u'' on ∂S and not containing any distinguished point of S. Orient u' in the direction induced by S and u'' in the opposite direction and identify the oriented edges thus obtained. When u' and u'' are on the same connected component (hole) of ∂S, this is a divide; otherwise, this is a merge, and results in a new handle, as is illustrated below:



Theorem 1 (variety-presentation framework). *Q-permutations, together with the above operators, satisfy the axioms of variety-presentation frameworks.*

Proof — These axioms, recalled in Appendix A, are almost trivial, and result from direct application of the definitions. The Composition axiom for example essentially expresses that edge identifications in a surface can be performed in any order. □

Following [3], q-permutations, as any variety-presentation framework, define a coloured logic in which connectives are presentations together with a polarity, and sequents are varieties. We explicit that logic, called Permutative Logic (PL), in Section 3.

A note on the categorical interpretation of q-permutations. It is worth observing that q-permutations on initial segments of $\mathbb{N}^*$ are also the morphisms of a traced symmetric tensor category [9] where objects are natural numbers, tensor is the sum, and the trace is determined by the feedback $\mathrm{tr}_{n,1}(\sigma, d) : n \to n$

on a single wire (for a permutation $\sigma : n+1 \to n+1$), which is defined as follows: $\mathrm{tr}_{n,1}(\sigma, d) = (\sigma\restriction_{1,\ldots,n}, d+1)$ if $\sigma(n+1) = n+1$; otherwise $\mathrm{tr}_{n,1}(\sigma, d) = (\sigma', d)$, with $\sigma'(i) = \sigma(i)$ when $\sigma(i) \leq n$, and $\sigma'(i) = \sigma(n+1)$ when $\sigma(i) = n+1$. This category is essentially obtained from the category of tangles with same number of inputs and outputs by forgetting over- and under-crossings. Hence, the trace determines a kind of "restriction operator" on q-permutations, but not the one we are interested in here, which is motivated by the topological interpretation of q-permutations and can be computed as in any variety-presentation framework using the composition, decomposition operations and the void presentation by:

$$\alpha\restriction_D = (((\alpha)_{x_1} * \bigcirc)_{x_2} * \bigcirc \ldots)_{x_n} * \bigcirc \quad \text{where } \; |\alpha| \setminus D = \{x_1, \ldots, x_n\}$$

The restriction of a variety (X, σ, d) to a set $Y \subseteq X$ is clearly the variety (Y, τ, d) where the cycles of τ are those of σ from which the elements outside Y are removed. The topological interpretation of restriction is simply the composition of $\iota : X \to \partial S$ with the inclusion map $Y \subseteq X$. For instance, the restriction of the above-mentioned q-permutation $(X, \{(1,3,6),(2,5,7,4)\}, 3)$ to $Y = \{1,\ldots,5\}$ is $(Y, \{(1,3),(2,5,4)\}, 3)$. If id_X denotes the identity on X, $X = \{1,2,3\}$ and $Y = \{1,2\}$, then $(\mathrm{id}_X, 0)\restriction_Y = (\mathrm{id}_Y, 0)$ whereas $\mathrm{tr}_{3,1}(\mathrm{id}_X, 0) = (\mathrm{id}_Y, 1)$.

2.3 Computing relaxation

A permutation $\sigma \in \mathfrak{S}_k = \mathfrak{S}(\{1,\ldots,k\})$ can be written as a product of transpositions, and the following result is standard:

Lemma 1. *If $\sigma \in \mathfrak{S}_k$, then the smallest number n of transpositions $\tau_1, \ldots, \tau_n$ such that $\sigma = \tau_1 \cdots \tau_n$ is given by $n = k - \sigma^\bullet$, where $\sigma^\bullet$ denotes the number of cycles of σ.*

Observe that the effect of both divide and merge on the permutation σ of a given variety (σ, d) is a composition by a transposition. Indeed, divide amounts to taking a cycle (a, Γ, b, Δ) of σ and split it into the two cycles (a, Γ) and (b, Δ), leading to a permutation θ; conversely, merge amounts to taking two cycles (a, Γ) and (b, Δ) and merge them into a single cycle (a, Γ, b, Δ), leading to a permutation θ: in both cases, $\theta = \chi_{a,b} \circ \sigma$, where $\chi_{a,b}$ denotes the transposition exchanging a and b.

Theorem 2. *Given varieties (σ, d) and (θ, e) with $\sigma, \theta \in \mathfrak{S}_k$, $(\theta, e) \preccurlyeq (\sigma, d)$ if, and only if, $m(\sigma, \theta) \leq e - d$, where:*

$$m(\sigma, \theta) = \frac{k - (\theta\sigma^{-1})^\bullet - \theta^\bullet + \sigma^\bullet}{2}.$$

Proof — By Lemma 1, a sequence of divides and merges, say i divides and j merges, from (σ, d) to (θ, e) gives rise to a decomposition of $\theta\sigma^{-1}$ as a product of at least $k - (\theta\sigma^{-1})^\bullet$ transpositions. On the other hand, each occurrence of divide increments the number of cycles and each occurrence of merge decrements it, so $i - j = \theta^\bullet - \sigma^\bullet$. From $i + j \geq k - (\theta\sigma^{-1})^\bullet$, we deduce $j \geq m(\sigma, \theta)$.

Identities

$$\text{axiom}\ \frac{}{\vdash_0 (A, A^\perp)} \qquad \text{cut}\ \frac{\vdash_d \Sigma, (\Gamma, A) \qquad \vdash_e \Theta, (\Delta, A^\perp)}{\vdash_{d+e} \Sigma, \Theta, (\Gamma, \Delta)}$$

Structural rules

$$\text{divide}\ \frac{\vdash_d \Sigma, (\Gamma, \Delta)}{\vdash_d \Sigma, (\Gamma), (\Delta)} \quad \text{merge}\ \frac{\vdash_d \Sigma, (\Gamma), (\Delta)}{\vdash_{d+1} \Sigma, (\Gamma, \Delta)}$$

Logical rules

$$\text{par}\ \frac{\vdash_d \Sigma, (\Gamma, A, B)}{\vdash_d \Sigma, (\Gamma, A \parr B)} \quad \text{tensor}\ \frac{\vdash_d \Sigma, (\Gamma, A) \qquad \vdash_e \Theta, (\Delta, B)}{\vdash_{d+e} \Sigma, \Theta, (\Delta, \Gamma, A \otimes B)}$$

$$\text{flat}\ \frac{\vdash_d \Sigma, (\Gamma), (A)}{\vdash_d \Sigma, (\Gamma, \flat A)} \qquad \text{sharp}\ \frac{\vdash_d \Sigma, (\Gamma, A)}{\vdash_d \Sigma, (\Gamma), (\# A)}$$

$$\text{hbar}\ \frac{\vdash_{d+1} \Sigma, (\Gamma)}{\vdash_d \Sigma, (\Gamma, \hbar)} \qquad \text{h}\ \frac{}{\vdash_1 (h)}$$

$$\text{bottom}\ \frac{\vdash_d \Sigma, (\Gamma)}{\vdash_d \Sigma, (\Gamma, \perp)} \quad \text{one}\ \frac{}{\vdash_0 (1)}$$

Table 1. The sequent calculus of permutative logic

Now, $e - d$ is the maximum number of merges in a sequence from (σ, d) to (θ, e). Therefore, if $(\theta, e) \preccurlyeq (\sigma, d)$ and $m(\sigma, \theta) > e - d$, we have $j > e - d$, which is impossible. Conversely, if $m(\sigma, \theta) \leq e - d$, consider a decomposition of $\theta\sigma^{-1}$ as a product of exactly $k - (\theta\sigma^{-1})^\bullet$ transpositions: the number of merges then equals $m(\sigma, \theta)$ and $(\theta, e) \preccurlyeq (\sigma, d)$. □

3 Formulas, sequents, and inference rules

Definition 6 (formula). Formulas *of PL are obtained from a fixed countable set of* negative atoms $p, q, \ldots$ *and their positive duals* $p^\perp, q^\perp, \ldots$*, by means of the binary connectives* $\parr, \otimes$*, the unary connectives* $\flat, \#$*, and the constants* $\hbar, h, \perp, 1$*.*

The involutive duality is given by De Morgan rules:

$$\begin{array}{llll} (A \parr B)^\perp = B^\perp \otimes A^\perp & (\flat A)^\perp = \# A^\perp & \hbar^\perp = h & \perp^\perp = 1 \\ (A \otimes B)^\perp = B^\perp \parr A^\perp & (\# A)^\perp = \flat A^\perp & h^\perp = \hbar & 1^\perp = \perp \end{array}$$

In general, in a variety-presentation framework, there are two n-ary connectives τ^+ and τ^- of opposite polarities for each presentation τ with a normalised support set $\{1, \ldots, n\}$, hence an infinite set of connectives. However, most of the time, this set is redundant, and it is sufficient to restrict to the connectives derived from a finite set of presentations from which all the others can be reconstructed using the operations of the framework (composition, decomposition). More precisely, given Ω a set of presentations, define the set Ω^* of presentations *generated* by Ω to be the smallest set of presentations X containing places, Ω and satisfying: $(\omega * \tau)_x \in X$ for any $\omega, \tau \in X$ and $x \in |\omega| \cup |\tau|$. Furthermore, Ω is said to be *spanning* when Ω^* is the set of all presentations and to be a *basis* when it is spanning and none of its strict subsets is. It is not difficult to check the following for q-permutations:

Proposition 2. *A basis for q-permutations is the set of* 4 *presentations below.*

Presentation τ	Neg. conn. τ^-	Pos. conn. τ^+
$\{(\mathbf{0},1,2)\},0$	$⅋$	$\otimes$
$\{(\mathbf{0}),(1)\},0$	$\flat$	$\#$
$\{(\mathbf{0})\},1$	$\hbar$	h
$\{(\mathbf{0})\},0$	$\bot$	1

Definition 7 (sequent). *A* sequent *is a variety together with a mapping from its support set into the set of formulas, modulo renaming of the support set consistent with the mapping to formulas.*

It is convenient to represent a sequent as a list of lists of formulas, indexed by a natural number, denoted $\vdash_d (\Gamma_1),\ldots,(\Gamma_q)$ where d is the number and $\Gamma_1,\ldots,\Gamma_q$ are the lists of formulas. It corresponds to the presentation (σ, d) where σ is the permutation whose cycles are precisely $(\Gamma_1),\ldots,(\Gamma_q)$, each inner list being taken modulo cyclic exchange and the outer list being taken modulo unrestricted exchange (it is a multiset). Note that if a list Γ_i is empty, it is simply ignored (it does not correspond to a cycle in σ). In other words, we have the implicit equalities:

$$\begin{aligned} \vdash_d \Sigma, \Sigma_1, \Sigma_2, \Sigma' &= \vdash_d \Sigma, \Sigma_2, \Sigma_1, \Sigma' \\ \vdash_d \Sigma, (\Gamma, \Delta) &= \vdash_d \Sigma, (\Delta, \Gamma) \\ \vdash_d \Sigma, () &= \vdash_d \Sigma \end{aligned}$$

Modulo these identities, there is a one-to-one correspondence between sequents and their representations as indexed lists of lists. Using this representation, the sequent calculus of PL is given in Table 1. The usual exchange rule of LL is decomposed in PL as follows:

$$\text{divide}\ \frac{\vdash_d \Sigma, (\Gamma, A, B)}{\vdash_d \Sigma, (\Gamma, A), (B)}$$
$$\vdots$$
$$\text{merge}\ \frac{\vdash_d \Sigma, (A, \Gamma), (B)}{\vdash_{d+1} \Sigma, (A, \Gamma, B)}$$
$$\vdots$$
$$\vdash_{d+1} \Sigma, (\Gamma, B, A)$$

Dotted lines here indicate application of the identities on sequents. It is interesting to note that exchange is not involutive, even at the level of sequents, since a handle has been added.

The inference figure for an n-ary connective attached to a presentation τ (normalised so that $|\tau| = \{1, \ldots, n\}$) is directly obtained from the generic pattern of variety-presentation frameworks:

$$\tau^- \; \frac{\omega * \tau(A_1, \ldots, A_n)}{\omega * \tau^-(A_1, \ldots, A_n)} \qquad \tau^+ \; \frac{\omega_1 * A_1 \qquad \cdots \qquad \omega_n * A_n}{\tau(\omega_n, \ldots, \omega_1) * \tau^+(A_1, \ldots, A_n)}$$

Let us detail, for example, how the inference figure for connective $\otimes$ is obtained. In that case, we have $\tau = (\{(\mathbf{0}, 1, 2)\}, 0)$ and the connective is positive. The conclusion of the corresponding inference is therefore $\tau(\omega_B, \omega_A) * A \otimes B$ and its premisses are $\omega_A * A$ and $\omega_B * B$. Now $\tau(\omega_B, \omega_A)$ is defined as presentation τ in which place 1 is substituted by ω_B and place 2 by ω_A. The substitution operation on a presentation in any variety-presentation framework is defined in general by

$$\tau((\omega_z)_{z \in |\tau|}) = (((\tau * x)_{x_1} * \omega_{x_1} \cdots)_{x_n} * \omega_{x_n})_x \tag{1}$$

where x is an arbitrary place outside $|\tau|$ and $x_1, \ldots, x_n$ is an arbitrary enumeration of $|\tau|$ (the axioms of variety-presentations ensure that the result is independent of any choice for x and the enumeration of $|\tau|$). By distinguishing in ω_A and ω_B the cycle containing $\mathbf{0}$, we have that ω_A is of the form $\vdash_d \Sigma, (\Gamma, \mathbf{0})$ and ω_B of the form $\vdash_e \Theta, (\Delta, \mathbf{0})$. By applying (1), we get that $\tau(\omega_B, \omega_A)$ is the sequent $\vdash_{d+e} \Sigma, \Theta, (\Delta, \Gamma, \mathbf{0})$. Hence the inference figure for $\otimes$. Similarly, the inference figure for $\#$ is the positive inference associated with $\tau = (\{(\mathbf{0}), (1)\}, 0)$. Its conclusion is $\tau(\omega) * \#A$ and its premiss $\omega * A$. Now representing ω as $\vdash_d \Sigma, (\Gamma, \mathbf{0})$, we get that $\tau(\omega)$ is $\vdash_d \Sigma, (\Gamma), (\mathbf{0})$, and, composing ω with A and $\tau(\omega)$ with $\#A$ we obtain the result. The other inference figures are obtained in the same way.

3.1 Basic properties

Let $A_1, \ldots, A_n \vdash B$ denote the sequent $\vdash_0 (A_1^\perp, \ldots, A_n^\perp, B)$, and $A \dashv\vdash B$ denote the two sequents $A \vdash B$ and $B \vdash A$.

Proposition 3. *The following sequents are provable in permutative logic:*

$$\begin{array}{lll}
A \vdash \flat A & \flat A \vdash A \invamp \hbar & \\
(A \invamp B) \invamp C \dashv\vdash A \invamp (B \invamp C) & \flat\flat A \dashv\vdash \flat A & A \invamp \flat B \dashv\vdash \flat B \invamp A \\
A \invamp \bot \dashv\vdash A & \flat\bot \dashv\vdash \bot & \flat(A \invamp \flat B) \dashv\vdash \flat A \invamp \flat B \\
\bot \invamp A \dashv\vdash A & \flat\hbar \dashv\vdash \hbar & \flat(A \invamp B) \dashv\vdash \flat(B \invamp A).
\end{array}$$

Proof —

$A \vdash \flat A$	$\flat A \vdash A \invamp \hbar$	$A \invamp \flat B \dashv\vdash \flat B \invamp A$
$\dfrac{\text{divide}\;\dfrac{\dfrac{}{\vdash_0 (A^\perp, A)}}{\vdash_0 (A^\perp), (A)}}{\vdash_0 (A^\perp, \flat A)}$	$\dfrac{\dfrac{\text{merge}\;\dfrac{\dfrac{\dfrac{}{\vdash_0 (A^\perp, A)}}{\vdash_0 (\#A^\perp), (A)}}{\vdash_1 (\#A^\perp, A)}}{\vdash_0 (\#A^\perp, A, \hbar)}}{\vdash_0 (\#A^\perp, A \invamp \hbar)}$	$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_0 (B^\perp, B)}}{\vdash_0 (\#B^\perp), (B)} \qquad \dfrac{}{\vdash_0 (A^\perp, A)}}{\vdash_0 (\#B^\perp \otimes A^\perp, A), (B)}}{\vdash_0 (\#B^\perp \otimes A^\perp, \flat B, A)}}{\vdash_0 (\#B^\perp \otimes A^\perp, \flat B \invamp A)}$

$\flat\flat A \vdash \flat A$	$\flat\bot \vdash \bot$	$\flat\hbar \vdash \hbar$	$\flat(A \parr B) \dashv\vdash \flat(B \parr A)$
$\vdash_0 (A^\bot, A)$ $\vdash_0 (\#A^\bot), (A)$ $\vdash_0 (\#\#A^\bot), (), (A)$ $\cdots\cdots$ $\vdash_0 (\#\#A^\bot), (A)$ $\vdash_0 (\#\#A^\bot, \flat A)$	$\vdash_0 (1)$ $\vdash_0 (\#1), ()$ $\cdots\cdots$ $\vdash_0 (\#1)$ $\vdash_0 (\#1, \bot)$	$\vdash_1 (h)$ $\vdash_1 (\#h), ()$ $\cdots\cdots$ $\vdash_1 (\#h)$ $\vdash_0 (\#h, \hbar)$	$\vdash_0 (B, B^\bot) \qquad \vdash_0 (A, A^\bot)$ $\vdash_0 (B^\bot \otimes A^\bot, A, B)$ $\vdash_0 (\#(B^\bot \otimes A^\bot)), (A, B)$ $\vdash_0 (\#(B^\bot \otimes A^\bot)), (B \parr A)$ $\vdash_0 (\#(B^\bot \otimes A^\bot), \flat(B \parr A))$

$\flat(A \parr \flat B) \vdash \flat A \parr \flat B$	$\flat A \parr \flat B \vdash \flat(A \parr \flat B)$
$\vdash_0 (B, B^\bot)$ $\vdash_0 (B), (\#B^\bot) \qquad \vdash_0 (A, A^\bot)$ $\vdash_0 (\#B^\bot \otimes A^\bot, A), (B)$ $\vdash_0 (\#(\#B^\bot \otimes A^\bot)), (A), (B)$ $\vdash_0 (\#(\#B^\bot \otimes A^\bot), \flat A), (B)$ $\vdash_0 (\#(\#B^\bot \otimes A^\bot), \flat A, \flat B)$ $\vdash_0 (\#(\#B^\bot \otimes A^\bot), \flat A \parr \flat B)$	$\vdash_0 (B, B^\bot) \qquad \vdash_0 (A, A^\bot)$ $\vdash_0 (B), (\#B^\bot) \qquad \vdash_0 (\#A^\bot), (A)$ $\vdash_0 (\#B^\bot \otimes \#A^\bot), (A), (B)$ $\vdash_0 (\#B^\bot \otimes \#A^\bot), (A, \flat B)$ $\vdash_0 (\#B^\bot \otimes \#A^\bot), (A \parr \flat B)$ $\vdash_0 (\#B^\bot \otimes \#A^\bot, \flat(A \parr \flat B))$

□

Corollary 1. *The following sequents are provable in permutative logic:*

$$\bot \vdash \hbar \qquad A \parr \hbar \dashv\vdash \hbar \parr A \qquad \flat(\hbar \parr A) \dashv\vdash \hbar \parr \flat A$$

Proof — Easy from the previous proposition. □

As a consequence, we have the following corollary.

Corollary 2. *Any negative formula is equivalent to a formula of the form:*

$$N = \flat(P^1_\parr) \parr \cdots \parr \flat(P^k_\parr) \parr Q_1 \parr \cdots \parr Q_\ell \parr \hbar \parr \cdots \parr \hbar$$

with d occurrences of $\hbar$ and $k, \ell, d \geq 0$. For each $i = 1, \ldots, k$ the formula $P^i_\parr$ is of the form $P^i_1 \parr \cdots \parr P^i_{n_i}$ for some $n_i \geq 1$, and the P^i_j and Q_i are positive formulas. Explicit parentheses for associativity have been omitted.

As a special case, $\bot$ corresponds to the case where $k, \ell, d = 0$. Note that each of P^i_j and Q_i being positive, they can in turn be decomposed as above (by duality).

The following defined connectives $\dot\parr$ and $\dot\otimes$ are useful for the embedding of LL into PL (Theorem 3): $A \dot\parr B = A \parr \flat B$. Its dual is: $A \dot\otimes B = \#A \otimes B$. The following properties are straightforward.

$$A \dot\parr (B \dot\parr C) \dashv\vdash A \dot\parr (C \dot\parr B) \qquad A \parr B \vdash A \dot\parr B$$
$$A \parr (B \dot\parr C) \dashv\vdash (A \parr B) \dot\parr C \dashv\vdash (A \dot\parr C) \parr B \qquad A \dot\parr \bot \dashv\vdash A$$

3.2 Subsystems

The surface associated by Métayer [14] to a proof net in multiplicative linear logic can be explicitly computed by the sequent calculus introduced by Gaubert [6]. In this calculus, there are no structural rules, and the logical rules only deal with the connectives $\otimes$ and $\parr$. The cycles correspond to the conclusions on the same border and the number attached to the sequent is the number of handles of the surface. Our $\otimes$ rule is the same, and the two $\parr$ rules in [6] are recovered as follows:

$$
\begin{array}{c}
\vdash_d \Sigma,(\Gamma,A,\Delta,B) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\text{divide}\ \dfrac{\vdash_d \Sigma,(B,\Gamma,A,\Delta)}{\vdash_d \Sigma,(B,\Gamma,A),(\Delta)} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\text{par}\ \dfrac{\vdash_d \Sigma,(\Gamma,A,B),(\Delta)}{\vdash_d \Sigma,(\Gamma,A\parr B),(\Delta)}
\end{array}
\qquad\qquad
\text{par}\ \dfrac{\text{merge}\ \dfrac{\vdash_d \Sigma,(\Gamma,A),(B,\Delta)}{\vdash_{d+1} \Sigma,(\Gamma,A,B,\Delta)}}{\vdash_{d+1} \Sigma,(\Gamma,A\parr B,\Delta)}
$$

Observe that the two $\parr$ rules in [6] are not reversible, hence calculus in [6] cannot have the focussing property. Melliès' planar logic [13] exactly corresponds to the $(\otimes,\parr)$ fragment of PL restricted to proofs with 0 handle. The following theorem shows that PL is a conservative extension of cyclic linear logic [19] and linear logic [7].

Theorem 3. *Any formula A of CyLL (resp. LL) is turned into a formula A^{cy} (resp. A^{li}) of PL by $a^{\mathrm{cy}} = a^{\mathrm{li}} = a$ for an atom a and by:*

$$
\begin{array}{ll}
(A\parr B)^{\mathrm{cy}} = A^{\mathrm{cy}} \parr B^{\mathrm{cy}} \qquad & (A\parr B)^{\mathrm{li}} = A^{\mathrm{li}} \dot{\parr} B^{\mathrm{li}} \\
(A\otimes B)^{\mathrm{cy}} = A^{\mathrm{cy}} \otimes B^{\mathrm{cy}} \qquad & (A\otimes B)^{\mathrm{li}} = A^{\mathrm{li}} \dot{\otimes} B^{\mathrm{li}}
\end{array}
$$

A formula A of CyLL (resp. LL) is provable in CyLL (resp. LL) if, and only if, A^{cy} (resp. A^{li}) is provable in PL.

Proof — The case of CyLL is an obvious induction: essentially, CyLL is the $(\otimes,\parr)$ fragment of permutative logic with 1 cycle and 0 handle (i.e., rank 0).

For LL, we extend the translation to sequents, and we first show that if $\vdash A_1,\ldots,A_n$ is provable in LL then $\vdash_0 (A_1^{\mathrm{li}}),\ldots,(A_n^{\mathrm{li}})$ is provable in PL, by induction of a proof of $\vdash A_1,\ldots,A_n$ in LL. Since clearly $A^{\perp\mathrm{li}} = A^{\mathrm{li}\perp}$, axiom and the $\parr$ and $\otimes$ rules of LL are translated as follows in PL:

$$
\text{divide}\ \dfrac{\dfrac{}{\vdash_0 (A^{\mathrm{li}},A^{\perp\mathrm{li}})}}{\vdash_0 (A^{\mathrm{li}}),(A^{\perp\mathrm{li}})}
\quad
\dfrac{\dfrac{\vdash_0 \Sigma^{\mathrm{li}},(A^{\mathrm{li}}),(B^{\mathrm{li}})}{\vdash_0 \Sigma^{\mathrm{li}},(A^{\mathrm{li}},\flat B^{\mathrm{li}})}}{\vdash_0 \Sigma^{\mathrm{li}},(A^{\mathrm{li}}\parr \flat B^{\mathrm{li}})}
\quad
\dfrac{\dfrac{\vdash_0 \Sigma^{\mathrm{li}},(A^{\mathrm{li}})}{\vdash_0 \Sigma^{\mathrm{li}},(\# A^{\mathrm{li}})} \qquad \vdash_0 (B^{\mathrm{li}}),\Theta^{\mathrm{li}}}{\vdash_0 \Sigma^{\mathrm{li}},(\# A^{\mathrm{li}}\otimes B^{\mathrm{li}}),\Theta^{\mathrm{li}}}
$$

To show the converse, we associate to any formula, sequent, proof of PL its *linear skeleton* in the evident way, by forgetting the information specific to PL, i.e., by forgetting $\flat,\#$ and by mapping $\hbar, h$ to $\perp, 1$, by mapping the variety underlying a sequent to its support set, and by forgetting the $\flat,\#$ rules and the structural rules. It is straightforward to check that a proof in PL is thus mapped to a proof in LL, and this is enough to conclude. $\square$

It is not obvious however that pomset calculus [17], non-commutative logic [2] or ordered calculus [16] are subsystems of PL.

By using the linear skeleton just defined, it is possible to show that PL behaves as a topological decoration of (essentially the multiplicative fragment of) LL. This observation should be a basis for a theory of proof nets for PL.

Theorem 4. *If $\vdash_d \Sigma$ is a sequent of PL and π is a proof of its skeleton in LL, then for some $e \geq 0$, there is a proof of $\vdash_{d+e} \Sigma$ in PL whose skeleton is π. In particular, if the linear skeleton of a sequent $\vdash_d \Sigma$ of PL is provable in LL, then for some $e \geq 0$, $\vdash_{d+e} \Sigma$ is provable in PL.*

Proof — We only need to show the first assertion. It is obtained by induction on π, together with the observation that a connective of PL in Σ which is forgotten by the skeleton operation can always be decomposed in PL, possibly at a certain cost (in terms of structural rules, hence in terms of handles). We omit the details here. □

4 Cut elimination and focussing

4.1 Cut elimination

Theorem 5. *Any proof in PL can be transformed into a proof without cut.*

Proof — The proof follows the usual pattern [3], where cuts are eliminated by repetitive application of reduction rules to the proofs. There are three kinds of reductions: axiom case (when one of the premisses of the cut is an axiom), commutative conversion (when the principal formula in one of the premisses of the cut is not the cut-formula) and symmetric reductions (when the principal formula in both premisses of the cut is the cut formula). Some cases are detailed below, the other configurations being treated similarly. □

Symmetric reductions:

$$\text{cut}\ \dfrac{\dfrac{\vdash_d \Sigma, (\Gamma), (A)}{\vdash_d \Sigma, (\Gamma, \flat A)} \qquad \dfrac{\vdash_e \Theta, (\Delta, A^\perp)}{\vdash_e \Theta, (\Delta), (\# A^\perp)}}{\vdash_{d+e} \Sigma, \Theta, (\Gamma), (\Delta)} \quad \rightsquigarrow \quad \text{cut}\ \dfrac{\vdash_d \Sigma, (\Gamma), (A) \qquad \vdash_e \Theta, (\Delta, A^\perp)}{\vdash_{d+e} \Sigma, \Theta, (\Gamma), (\Delta)}$$

$$\text{cut}\ \dfrac{\dfrac{\vdash_d \Sigma, (\Gamma, A, B)}{\vdash_d \Sigma, (\Gamma, A \parr B)} \qquad \dfrac{\vdash_f \Xi, (\Lambda, B^\perp) \qquad \vdash_e \Theta, (\Delta, A^\perp)}{\vdash_{e+f} \Theta, \Xi, (\Lambda, B^\perp \otimes A^\perp, \Delta)}}{\vdash_{d+e+f} \Sigma, \Theta, \Xi, (\Gamma, \Delta, \Lambda)} \quad \rightsquigarrow$$

$$\text{cut}\ \dfrac{\text{cut}\ \dfrac{\vdash_d \Sigma, (\Gamma, A, B) \qquad \vdash_f \Xi, (\Lambda, B^\perp)}{\vdash_{d+f} \Sigma, \Xi, (\Gamma, A, \Lambda)} \qquad \vdash_e \Theta, (\Delta, A^\perp)}{\vdash_{d+e+f} \Sigma, \Theta, \Xi, (\Gamma, \Delta, \Lambda)}$$

Commutative conversions:

$$\text{cut}\ \dfrac{\dfrac{\vdash_d \Sigma, (\Gamma, C), (A)}{\vdash_d \Sigma, (\Gamma, C, \flat A)} \qquad \vdash_e \Theta, (\Delta, C^\perp)}{\vdash_{d+e} \Sigma, \Theta, (\Gamma, \Delta, \flat A)} \quad \rightsquigarrow \quad \dfrac{\text{cut}\ \dfrac{\vdash_d \Sigma, (\Gamma, C), (A) \qquad \vdash_e \Theta, (\Delta, C^\perp)}{\vdash_{d+e} \Sigma, \Theta, (\Gamma, \Delta), (A)}}{\vdash_{d+e} \Sigma, \Theta, (\Gamma, \Delta, \flat A)}$$

$$\text{cut}\ \dfrac{\text{merge}\ \dfrac{\vdash_d \Sigma, (\Gamma), (\Delta, C)}{\vdash_{d+1} \Sigma, (\Gamma, \Delta, C)} \qquad \vdash_e \Theta, (\Lambda, C^\perp)}{\vdash_{d+e+1} \Sigma, \Theta, (\Gamma, \Delta, \Lambda)} \quad \rightsquigarrow \quad \text{merge}\ \dfrac{\text{cut}\ \dfrac{\vdash_d \Sigma, (\Gamma), (\Delta, C) \qquad \vdash_e \Theta, (\Lambda, C^\perp)}{\vdash_{d+e} \Sigma, \Theta, (\Gamma), (\Delta, \Lambda)}}{\vdash_{d+e+1} \Sigma, \Theta, (\Gamma, \Delta, \Lambda)}$$

Identity

$$\text{Ax}\ \frac{}{\vdash_0 p \blacktriangleright p^{\perp}}\ \text{if } p \text{ is a negative atom}$$

Structural rules

$$\text{unfocus}\ \frac{\vdash_d \Sigma, (\Gamma, A)}{\vdash_d \Sigma \blacktriangleright \Gamma, A}\ \text{if } A \text{ is negative} \qquad \text{focus1}\ \frac{\vdash_d \Sigma \blacktriangleright}{\vdash_d \Sigma}\ \text{if } \Sigma \text{ is reduced} \qquad \text{focus2}\ \frac{\vdash_d \Sigma \blacktriangleright \Gamma, A}{\vdash_d \Sigma, (\Gamma, A) \blacktriangleright}\ \text{if } A \text{ is positive}$$

$$\text{divide}\ \frac{\vdash_d \Sigma, (\Gamma, \Delta) \blacktriangleright}{\vdash_d \Sigma, (\Gamma), (\Delta) \blacktriangleright} \qquad \text{merge}\ \frac{\vdash_d \Sigma, (\Gamma), (\Delta) \blacktriangleright}{\vdash_{d+1} \Sigma, (\Gamma, \Delta) \blacktriangleright}$$

Logical rules

$$\text{par}\ \frac{\vdash_d \Sigma, (\Gamma, A, B)}{\vdash_d \Sigma, (\Gamma, A \parr B)} \qquad \text{tensor}\ \frac{\vdash_d \Sigma \blacktriangleright \Gamma, A \qquad \vdash_e \Theta \blacktriangleright \Delta, B}{\vdash_{d+e} \Sigma, \Theta \blacktriangleright \Delta, \Gamma, A \otimes B}$$

$$\text{flat}\ \frac{\vdash_d \Sigma, (\Gamma), (A)}{\vdash_d \Sigma, (\Gamma, \flat A)} \qquad \text{sharp}\ \frac{\vdash_d \Sigma \blacktriangleright \Gamma, A}{\vdash_d \Sigma, (\Gamma) \blacktriangleright \# A}$$

$$\text{hbar}\ \frac{\vdash_{d+1} \Sigma, (\Gamma)}{\vdash_d \Sigma, (\Gamma, \hbar)} \qquad \text{h}\ \frac{}{\vdash_1 \blacktriangleright h}$$

$$\text{bottom}\ \frac{\vdash_d \Sigma, (\Gamma)}{\vdash_d \Sigma, (\Gamma, \perp)} \qquad \text{one}\ \frac{}{\vdash_0 \blacktriangleright 1}$$

Table 2. The focussed sequent calculus of permutative logic

4.2 Focussing

As with any coloured logic derived from a variety-presentation framework, the sequent calculus has a remarkable property called focussing, which eliminates irrelevant non-determinism in proof construction. It reflects general permutability properties of inferences: any positive (resp. negative) inference can be permuted upward (resp. downward) if the active formulas of the lower inference are not principal in the upper inference. Thus, inferences of the same polarity can be grouped together. This can be captured in a variant of the sequent calculus called the focussing sequent calculus.

Definition 8. *The sequents of the focussing sequent calculus are of two types:*

- *standard sequents of the form* $\vdash_d \Sigma$*;*
- *focussed sequents of the form* $\vdash_d \Sigma \blacktriangleright \Gamma$ *where the list of formulas* Γ *has been singled out. Note that, here,* Γ *is not taken modulo cyclic exchange.*

A structure on formulas (eg. sequent) is said to be reduced *if it does not contain any negative compound formula.*

The focussing sequent calculus is given in Table 2. Its negative logical inferences are identical to those of the standard sequent calculus. Its positive logical inferences are also those of the standard calculus, except that the principal formula is syntactically distinguished as *focus*, and, when read bottom-up, the focus is passed to its active formulas as long as they remain positive. In the generic variety-presentation framework, the positive rule for an n-ary connective is:

$$\frac{\omega_1 \blacktriangleright A_1 \qquad \cdots \qquad \omega_n \blacktriangleright A_n}{\tau(\omega_n, \ldots, \omega_1) \blacktriangleright \tau^+(A_1, \ldots, A_n)}$$

The unfocus rule models the loss of focus due to a change of polarity of the focus (from positive to negative). The rules divide, merge, focus1 and focus2 are in fact a single rule in the generic variety-presentation framework:

$$\text{focus}\ \frac{\omega \blacktriangleright A}{\alpha} \qquad \xrightarrow{\text{in PL}} \qquad \text{focus}\ \frac{\vdash_e \Theta \blacktriangleright \Gamma, A}{\vdash_d \Sigma}$$

if α is reduced, A is positive and $\alpha \preccurlyeq \omega * A$ $\xrightarrow{\text{in PL}}$ if Σ is reduced, A is positive and $\vdash_d \Sigma \ \preccurlyeq \vdash_e \Theta, (\Gamma, A)$

It is easy to show that the calculus with the focus rule is equivalent to that with the divide, merge, focus1 and focus2 rules. The latter has been adopted only to make explicit the use of the divide and merge rules, which are implicit in the side relaxation condition of the focus rule.

Theorem 6. *A standard sequent is provable in PL if and only if it is provable in the focussing sequent calculus of PL.*

Proof — It is straightforward to map any inference of the focussing calculus into an inference of the standard calculus (or a dummy inference): just drop the $\blacktriangleright$ sign when it appears. Hence, this ensures the soundness of the focussing calculus. Its completeness is much more involved. It relies exclusively on the axioms of variety-presentation frameworks. It is shown in three steps. First, the negative rules are shown to be invertible in the focussing calculus. Second, the "focus" rule is shown to hold even when Σ is not reduced (using the previous result). And third, the positive rules of the standard calculus are shown to hold in the focussing calculus (using the previous results). In fact, all these properties result from generic permutability properties between inferences, depending on their polarities. The interested reader is referred to [3] for details. □

5 Future work

Permutative logic opens new perspectives in the design of non-commutative logical systems. It not only quantifies the use of the structural rule of exchange but also allows to put constraints on that use. Many aspects of the logic have not been studied in this paper. For example, proof-nets in PL deserve a study of their own. It can be expected that a correctness criterion for PL should be found which extends that for CyLL: a cut free proof structure is CyLL-correct if it is LL-correct and planar. In PL, the planarity condition should be replaced by some condition involving more complex surfaces. Another interesting aspect is proof construction. It is very easy to show that, unlike the entropy of non-commutative logic [2], relaxation in PL cannot be optimised so that, during proof construction, the positive inferences perform only the “minimal” amount of relaxation that is strictly needed. However, it is conjectured that if proof-construction is viewed as a constraint propagation problem, this optimality can be recovered. Finally, we have not explored semantics issues. A trivial but uninformative phase semantics can be derived, as in any coloured logic (i.e., based on a variety-presentation framework). More work is needed to achieve interesting semantics interpretation of formulas and proofs.

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A Axioms of variety-presentation frameworks

The axioms of variety-presentation frameworks are the following (see [3]):

- **Composition**:
  For any presentations ω_1, ω_2,

$$|\omega_1| \cap |\omega_2| = \emptyset \;\Rightarrow\; \begin{cases} |\omega_1 * \omega_2| = |\omega_1| \cup |\omega_2| \\ \omega_1 * \omega_2 = \omega_2 * \omega_1 \end{cases}$$

- **Decomposition**:
  For any variety α, place x and presentation ω:

$$\begin{array}{l} x \in |\alpha| \;\Rightarrow\; x \notin |(\alpha)_x| \;\wedge\; x * (\alpha)_x = \alpha \\ x \notin |\omega| \;\wedge\; \omega * x = \alpha \;\Rightarrow\; \omega = (\alpha)_x \end{array}$$

  This implies, by composition, that if $x \in |\alpha|$ then $|(\alpha)_x| = |\alpha| \setminus \{x\}$. Hence, for a given x, the mappings $\alpha \mapsto (\alpha)_x$ (for any variety α having occurrence x) and $\omega \mapsto \omega * x$ (for any presentation ω not having occurrence x) are inverse of each other.
- **Commutation**:
  For any variety α, presentations ω_1, ω_2 and places x_1, x_2,

$$\left.\begin{array}{l} |\alpha| \cap (\{x_1\} \cup |\omega_1|) = \{x_1\} \\ |\alpha| \cap (\{x_2\} \cup |\omega_2|) = \{x_2\} \\ (\{x_1\} \cup |\omega_1|) \cap (\{x_2\} \cup |\omega_2|) = \emptyset \end{array}\right\} \Rightarrow ((\alpha)_{x_1} * \omega_1)_{x_2} * \omega_2 = ((\alpha)_{x_2} * \omega_2)_{x_1} * \omega_1$$

  From the previous axioms, it is easy to show that, under the stated condition, the two sides of the equality have the same occurrence set. This axiom asserts that they are equal.
- **Relaxation**:
  For any varieties α_1, α_2, presentation ω and place x,

$$\alpha_1 \preccurlyeq \alpha_2 \;\Rightarrow\; \begin{cases} |\alpha_1| = |\alpha_2| = D \\ (|\omega| \cup \{x\}) \cap D = \{x\} \;\Rightarrow\; (\alpha_1)_x * \omega \preccurlyeq (\alpha_2)_x * \omega \end{cases}$$

  Hence, relaxation applies only to varieties with the same occurrence set and is compatible with decomposition/composition.