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Proof nets and cliques: towards the understanding of analytical proofs

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[1] Senza di lui questa tesi sarebbe stata scritta in Word e i proof nets disegnati con Paint!

And thanks to the reader for bearing my poor English!

Introduction

The object of the thesis

What is an analytical proof?

Such a question has been for a long time a crucial one in logic. We think that modern proof theory has refreshed the question, setting it in a new and fruitful framework.

Traditionally, a proof of a theorem A is considered analytical when it proves A only developing concepts already present in A. In [Gen35], Gentzen provides a formal proof system which allows a mathematical definition of such an intuitive notion of analyticity – the *subformula property*. A proof of A meets the *subformula property* if all the formulas occurring in it are subformulas of A.

Of course, many proofs are not analytical. Actually when we prove a theorem by using a lemma, we lose the analyticity, since the lemma can exploit concepts unrelated with the theorem. In Gentzen's system, applying a lemma corresponds to use a *cut*, the unique logical rule which violates the subformula property. The remarkable result of [Gen35] is the *Hauptsatz theorem*, stating that cuts can be removed, i.e. any proof of classical logic can be reduced to a cut-free proof – a proof satisfying the subformula property.

A first naive answer to our original question is the following:

An analytical proof is a cut-free proof.

Well, such an answer is not so bad, since it provides a formal definition of the intuitive notion of analytical proof, but yet the framing is not well hit. Actually the most remarkable point of the Hauptsatz theorem is not its statement, but its proof. This last one has in fact disclosed a startling dynamics within classical logic, consisting in the rules which transform any proof with cuts in a cut-free proof. Such a dynamics has been a turning point in proof theory, revealing an unexpected correspondence between proofs and programs: the *Curry-Howard isomorphism*. This isomorphism associates proofs with programs, in such a way that the reduction of the cuts in a proof corresponds to the execution of the associated program.

The Curry-Howard isomorphism has leaded to a prolific exchange between logic and computer science, which is still alive. In particular, it provides a fresh answer to our original question:

An analytical proof is what is invariant under cut reduction.

The research of invariants under a transformation is a crucial one in mathematics. In logic such a research is called *semantics*. In particular denotational semantics describes the invariants under cut reduction by elements of special mathematical structures (like sets, topological spaces, coherent spaces etc. . .). The general object of our thesis is to understand how precise this description can be.

Actually, the cut reduction in classical logic is a very clumsy process. So clumsy that the notion itself of invariant has a meaning only for the *intuitionistic logic* - a restriction of classical logic. In whole Gentzen's classical logic the only invariant under cut reduction is the *provability*, i.e. the correctness of the proofs. Everything collapses, any denotational semantics associates the same element to all correct proofs.

At first, it has been thought that the cause of this collapse was in the nature of classical negation. Indeed classical negation is involutive, that is $\neg\neg A$ is A, while it is not the case for the intuitionistic one, for which A is stronger than $\neg\neg A$. The discovery of *linear logic* ([Gir87]) has demolished such a supposition. Linear logic in fact has a good denotational semantics, although its negation is involutive.

Linear logic points out that the collapse of the semantics of classical logic is due to the *structural rules* (weakening and contraction): classical logic makes an unrestricted use of such rules, intuitionistic and linear logic do not.

More precisely, linear logic is a refinement of classical and intuitionistic logic, characterized by the splitting of standard connectives ("and" and "or") in two classes (*additive* and *multiplicative*) and the introduction of new connectives (*exponentials*) which give a logical status to the structural rules of classical and intuitionistic logic.

This change of viewpoint has many striking consequences in proof theory, among which one of the most important is the introduction of *proof nets*, a graph-theoretic presentation that gives a more geometric account of proofs.

In the framework of proof nets, cut reductions become graph rewriting rules, transforming a proof net π in a proof net π'. Let us denote such a transformation by $\pi \rightarrow_\beta \pi'$, and by $=_\beta$ the equivalence relation $\rightarrow_\beta$ induces.

A denotational semantics $\mathfrak{S}$ for linear logic associates with a proof net π an element $[\![\pi]\!]_\mathfrak{S}$ of $\mathfrak{S}$, such that $\pi =_\beta \pi'$ implies $[\![\pi]\!]_\mathfrak{S} = [\![\pi']\!]_\mathfrak{S}$. The elements of $\mathfrak{S}$ provide a description of the $=_\beta$-equivalence classes, our question being how precise this description can be.

The relation $\rightarrow_\beta$ meets two crucial properties for a rewriting system: *confluence* and *strong normalization*. Confluence means that $\rightarrow_\beta$ is deterministic, i.e. all the cut reductions of π converge to a common result. Strong normalization instead means that such a result always exists, i.e. any sequence $\pi \rightarrow_\beta \pi' \rightarrow_\beta \pi'' \rightarrow_\beta \ldots$ will eventually lead to a cut-free proof net.

Both properties are crucial for comparing the $=_\beta$-equivalence classes with their interpretations in $\mathfrak{S}$. In fact confluence and normalization guarantee that each $=_\beta$-equivalence class contains exactly one cut-free proof net, which is thus its canonical representative. Hence we can compare $=_\beta$ with $\mathfrak{S}$ by checking the following two properties:

injectivity (or faithfulness): for each element s of $\mathfrak{S}$ there is *at most* one cut-free π such that $[\![\pi]\!]_{\mathfrak{S}} = s$;

surjectivity (or full completeness): for each element s of $\mathfrak{S}$ there is *at least* one cut-free π such that $[\![\pi]\!]_{\mathfrak{S}} = s$.

If $\mathfrak{S}$ is both injective (or faithful) and surjective (or fully complete), then its elements describe exactly the $=_\beta$-equivalence classes of the proof nets.

The injectivity and surjectivity of a semantics are traditional questions of theoretical computer science, but they are quite a novelty in the domain of proof theory. Actually they are in the spirit of Girard's program (see [Gir99]) of removing the strict distinction between syntax (proof nets) and semantics. In fact, a proof of the injectivity and surjectivity of $\mathfrak{S}$ provides a way for reconstructing a unique cut-free proof net π from each element s of $\mathfrak{S}$, i.e. it provides the inverse of the $\mathfrak{S}$ interpretation.

Of course our research should be developed without prejudice for a syntax or a semantics, i.e. it should renew both of them. On the one hand, we may change a semantics for getting closer to the proof nets. For example, in chapter 2 we move from the coherent semantics to the hypercoherent one in order to approach to additive proof nets. On the other hand we may change our notion of proof nets by following the suggestions of a semantics. Such is the spirit, for example, of the results on the exponential proof nets of chapter 3.

A last remark before going into the details. The set of proof nets is a subset of a wider set of graphs: the set of proof structures. More precisely, proof nets are those proof structures which correspond to correct proofs. The importance of proof structures is that cut reduction is defined directly on them, so it makes sense even without logical correctness.

Here is a crucial novelty of linear logic: it introduces a cut reduction, hence a denotational semantics, on incorrect objects. Thus the two questions of injectivity and surjectivity can be at first addressed in the less restricted framework of proof structures, and then adapted to proof nets.

Let us be more precise on this point. A proof of the injectivity and surjectivity of a semantics $\mathfrak{S}$ consists in a method for reconstructing a unique proof net from each element of $\mathfrak{S}$. The reconstruction of π can be divided in two steps: firstly, we recover from the $\mathfrak{S}$ interpretation the graphical structure of π, i.e. we reconstruct π as a proof structure; secondly, we recover from the $\mathfrak{S}$ interpretation the correctness of π, i.e. we recognize π as a proof net.

For an example of such a method look at the proof in chapter 1 of the correspondence between proof nets and complete cliques. In theorem 14 we deal with a method for reconstructing a multiplicative cut-free proof structure π from its interpretation $[\![\pi]\!]$. In theorems 24 and 25 we prove that $[\![\pi]\!]$ is a clique if and only if the proof structure π is a proof net.

Injectivity and surjectivity in linear logic

We give a brief overview on the previous works we know about the injectivity and the surjectivity in linear logic.

Injectivity. A semantics $\mathfrak{S}$ is injective if for any two proof nets π and π', $[\![\pi]\!]_{\mathfrak{S}} = [\![\pi']\!]_{\mathfrak{S}}$ implies $\pi =_\beta \pi'$.

The question of injectivity has been addressed in the framework of linear logic by Tortora in [TdF03b] (see also [TdF00] for a more detailed treatment). However it is a traditional problem in the denotational semantics of λ-calculus. In particular recall *Statman theorem*, stating that the relational model is injective for the simply typed λ-calculus ([Sta83]).[2]

The semantic injectivity is deeply related with the so-called *syntactical separability*. The most well-known example of syntactic separability is Böhm theorem for pure λ-calculus ([B$\ddot{6}$8]): if t, t' are two closed λ-terms, then $t \neq_{\beta\eta} t'$ implies that there are λ-terms $u_1, \ldots, u_n$ such that $tu_1 \ldots u_n \rightarrow_\beta 1$ and $t'u_1 \ldots u_n \rightarrow_\beta 0$. That is, t and t' compute two distinct functions on the λ-terms, $u_1, \ldots, u_n$ being an example of arguments on which t and t' give different values.

The syntactical separability is a form of injectivity with respect to a model internal to the syntax. More: it can be proven from a well-chosen result of semantic injectivity. For example, in [Jol00] Joly proves the syntactical separability of the simply typed λ-calculus by means of Statman theorem, i.e. by means of the injectivity of the relational model.[3]

In a more proof-theoretical framework, the syntactical separation is a key property of Girard's ludics ([Gir01]). Some works on the syntactical separation have been made also in linear logic. The first one is [MP94], in the framework of pure proof nets, while in the typed case it exists a work by Matsuoka ([Mat05]), dealing with the separation of the implicational multiplicative linear logic fragment.

Surjectivity. A semantics $\mathfrak{S}$ is surjective if for any element s in $\mathfrak{S}$, there is a proof net π such that $[\![\pi]\!]_{\mathfrak{S}} = s$.

As far as we know, the question of surjectivity has been addressed at first by Girard in [Gir91]. Abramsky and Jagadeesan have defined in [AJ94] the first surjective (in their terms *fully complete*) model for the multiplicative fragment of linear logic (**MLL**).

The pioneering [AJ94] was followed by a series of papers which established the surjectivity of a variety of models with respect to various versions of **MLL** (see for example [HO92], [BS96], [Tan97], [Ham01]).

More recently the surjectivity for the additive proof nets has been attacked in [AM99] and [BHS05]. Indeed both papers deal with the additive proof nets defined in [Gir96], which are not canonical, especially they do not allow injectivity results. The problem of additive canonicity has been overcome by the additive proof nets defined in [HvG03]. However there is not yet any surjectivity result with respect to this last syntax.

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[2] Statman theorem is often called *completeness theorem*, since it states the completeness of the relational model with respect to the equational theory induced by the β- and η-reductions.

Actually we have to be more precise in the definition of the equivalence between proof nets induced by the cut reduction. In the λ-calculus we have mainly two rewriting rules: the β-reduction, which induces the β-equivalence on the λ-terms, and the η-expansion, which instead induces the η-equivalence. It is well known that the *Curry-Howard isomorphism* relates the β-reduction of the λ-calculus to the cut reduction in the proof nets. What about the η-expansion? It corresponds to a rewriting rule of proof nets too, i.e. to the reduction of complex axioms in simpler ones. Indeed it is common to avoid such a further reduction by allowing only atomic axioms in the definition itself of proof nets, i.e. by restraining to the η-*long* proof nets. Following [TdF03b], we will adopt such a convention, thus the equivalence $\pi =_\beta \pi'$ implicitly means $\pi =_{\beta\eta} \pi'$.

[3] A similar result is in [DP01b].

Finally, we know only one paper dealing with the surjectivity of exponential proof nets: [Lau04] by Laurent. In that paper Laurent proves the surjectivity (and injectivity) of a game semantics for the polarized fragment of **MELL**. However there is no surjectivity result for the coherent semantics, thus we believe that our section 3.4 is a novelty.

Contents of the thesis

The thesis is divided in three chapters, dealing with respectively the multiplicatives ($\otimes$, $\parr$), the additives ($\oplus$, &) and the exponentials (!, ?).

In chapter 1, we study surjectivity and syntactical separability of multiplicative proof nets. The general method we use consists first in addressing the two questions in the less restrictive framework of proof structures, and then in adapting the results to proof nets.

In section 1.1 we recall the definition of proof structures and in subsection 1.1.1 the definition of relational semantics. The main result in subsection 1.1.1 is the semantical characterization of those sets which are interpretations of proof structures (theorem 14). In subsection 1.2.1 from this result and from a theorem by Retoré ([Ret97], here theorem 25) we deduce an alternative proof (with respect to [Tan97]) of the surjectivity of coherent semantics with respect to the proof nets of **MLL** with mix (corollary 26).

In subsection 1.1.2 we introduce an observational equivalence between proof structures (definition 15). The main result of this subsection is the separation theorem for **MLL** proof structures (theorem 16). As corollaries we prove that the defined observational equivalence coincides with the equivalence induced by cut-elimination (corollary 17) and that such an equivalence is a maximal congruence between proof structures (corollary 18). In subsection 1.2.2 we weaken the observational equivalence of definition 15 reducing the admissible contexts (definition 27) and we prove (proposition 29) that concerning this weaker equivalence the separation of **MLL** does not hold.

The contents of this chapter are in [Pag06b].

In chapter 2, we study the proof nets for the multiplicative additive fragment of linear logic (briefly **MALL**).

Firstly we give in section 2.1 an overview of the proof nets based on the additive boxes. In particular we remark that such proof nets have not a confluent cut reduction.

Later in sections 2.2 and 2.3, we analyze the proof nets based on additive slices.

In section 2.2 we introduce **MALL** proof structures as couples of a set of slices and of an equivalence relation defining the superposition of slices. Our approach is in between the sliced proof structures defined by Tortora and Laurent in [LTdF04] and the ones introduced by Hughes and van Glabbeeck in [HvG03], although we will follow [HvG03] in the two most crucial passages: the cut reduction and the correctness criterion.

In subsection 2.2.1 we recall the relational semantics for the additives. Our main results are theorem 45, extending the injectivity of relational semantics to **MALL**, and theorem 48, yielding a semantic characterization of those sets which are interpretations of **MALL** proof structures.

In subsection 2.2.2 we define an observational equivalence between **MALL** proof structures (definition 50), which is the natural extension of the **MLL** equivalence $\sim_{\mathbb{B}}$ defined in subsection 1.1.2. Contrary to the multiplicative case, we prove in proposition 52 that the separation theorem does not hold in the additive framework (at least with the present syntax).

In section 2.3 we deal with the additive proof nets and Hughes and van Glabbeeck's correctness criterion. In subsection 2.3.3 we present our ongoing research for a surjective denotational semantics for **MALL** proof nets. The crucial point is to characterize semantically the additive proof nets. In particular we refer to the hypercoherent semantics defined by Ehrhard in [Ehr93]. We prove that any interpretation of a proof net is a hyperclique (theorem 68). Conversely, it remains an open question if any cut-free proof structure, whose interpretation is a hyperclique, is a proof net (see proposition 69 and conjecture 70).

In chapter 3, we study the proof nets for the multiplicative exponential fragment of linear logic (briefly **MELL**).

In section 3.1 we introduce **MELL** proof nets.

In section 3.2 we recall the multiset based uniform coherent semantics ($\mathfrak{Coh}$) and the non-uniform one ($\mathfrak{nuCoh}$). $\mathfrak{Coh}$ has been introduced by Girard in [Gir91], while $\mathfrak{nuCoh}$ is a more recent semantics defined by Bucciarelli and Ehrhard in [BE01].

In section 3.3 we attack the question of the injectivity of $\mathfrak{Coh}$ for **MELL** proof nets. In subsection 3.3.1, we define a counter-example to the $\mathfrak{Coh}$ injectivity for the polarized fragment of **MELL**, which had been conjectured in [TdF03b]. In subsections 3.3.2, 3.3.3 instead we prove the injectivity of $\mathfrak{Coh}$ for the so-called $(?\parr)$-**MELL** proof nets (theorem 100). Theorem 100 has been proved in [TdF03b], the main novelty of our approach is to provide a different proof by means of lemma 98, based on Girard's notion of longtrip.

In section 3.4 we solve the open question of characterizing those proof structures whose interpretation is a clique in $\mathfrak{nuCoh}$ (theorems 103, 104). Such a characterization provides a new geometric criterion on **MELL** proof structures: the *weak correctness* (definition 102). The contents of this section are in [Pag06a].

Notations and conventions

We recall some basic notations and definitions.

- We denote the elements of sets by lower-case letters $a, b, u, v, x, y, z \ldots$, and sets by typewrite capital letters $\mathtt{A}, \mathtt{B}, \mathtt{X} \ldots$.

  The cartesian product of $\mathtt{A}, \mathtt{B}$ is denoted by $\mathtt{A} \times \mathtt{B}$ and defined by $\mathtt{A} \times \mathtt{B} = \{< a, b > \mid a \in \mathtt{A}, b \in \mathtt{B}\}$. If $\mathtt{C} \subseteq \mathtt{A} \times \mathtt{B}$, the projection of $\mathtt{C}$ are $p_1(\mathtt{C}) = \{a \mid \exists b \in \mathtt{B}, < a, b > \in \mathtt{C}\}$ and $p_2(\mathtt{C}) = \{b \mid \exists a \in \mathtt{A}, < a, b > \in \mathtt{C}\}$.

  The disjoint union of $\mathtt{A}, \mathtt{B}$ is denoted by $\mathtt{A} + \mathtt{B}$ and defined by $\mathtt{A} + \mathtt{B} = \mathtt{A} \times \{1\} \cup \mathtt{B} \times \{2\}$. If $\mathtt{C} \subseteq \mathtt{A} + \mathtt{B}$, the projection of $\mathtt{C}$ are $s_1(\mathtt{C}) = \{a \mid < a, 1 > \in \mathtt{C}\}$ and $s_2(\mathtt{C}) = \{b \mid < b, 2 > \in \mathtt{C}\}$.

- Let $\mathtt{X}$ be a set, a multiset of elements in $\mathtt{X}$ is a function $v : \mathtt{X} \to \mathbb{N}$. In other words, v is a set of elements of $\mathtt{X}$ in which repetitions can occur: for any

$x \in \mathtt{X}$, the value $v(x)$ tell us how many times x occurs in v. We denote multisets by square brackets, for example $[a, a, b, c, c, c]$ is the multiset containing twice a, once b and three times c.

The support of a multiset v, denoted by $Supp(v)$, is the $\mathcal{X}$ subset $v^{-1}(\mathbb{N}/\{0\})$. For example $Supp([a, a, b, c, c, c]) = \{a, b, c\}$.

By the plus symbol $+$ we denote the disjoint union of multisets, for example $[a, a, b] + [a, c, c] = [a, a, a, b, c, c]$. The neutral element of $+$ is the empty multiset, denoted by $\emptyset$. If n is a number and v a multiset, we denote by nv the multiset $\underbrace{v + \ldots + v}_{n \text{ times}}$.

By $\mathbf{M}(\mathtt{X})$ (resp. $\mathbf{M_{fin}}(\mathtt{X})$) we mean the set of all multisets (resp. finite multisets) of $\mathtt{X}$.

- We denote the formulas by capital letters $A, B, C \ldots$, and the multisets of formulas by Greek capital letters $\Gamma, \Delta, \Sigma \ldots$.

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Contents

Chapter 1

Multiplicatives

In this chapter we study surjectivity and syntactical separability of multiplicative proof nets. The general method we use consists first in addressing the two questions in the less restrictive framework of proof structures, and then in adapting the results to proof nets.

In section 1.1 we recall the definition of proof structures and in subsection 1.1.1 the definition of relational semantics. The main result in subsection 1.1.1 is the semantical characterization of those sets which are interpretations of proof structures (theorem 14). In subsection 1.2.1 from this result and from a theorem by Retoré ([Ret97], here theorem 25) we deduce an alternative proof (with respect to [Tan97]) of the surjectivity of coherent semantics with respect to the proof nets of **MLL** with mix (corollary 26).

In subsection 1.1.2 we introduce an observational equivalence between proof structures (definition 15). The main result of this subsection is the separation theorem for **MLL** proof structures (theorem 16). As corollaries we prove that the defined observational equivalence coincides with the equivalence induced by cut-elimination (corollary 17) and that such an equivalence is a maximal congruence between proof structures (corollary 18). In subsection 1.2.2 we weaken the observational equivalence of definition 15 reducing the admissible contexts (definition 27) and we prove (proposition 29) that concerning this weaker equivalence the separation of **MLL** does not hold.

The **formulas** of **MLL** are:

$$F ::= \quad X \quad | \quad X^{\perp} \quad | \quad F \parr F \quad | \quad F \otimes F$$

As always we set $(A \parr B)^{\perp} = B^{\perp} \otimes A^{\perp}$ and $(A \otimes B)^{\perp} = B^{\perp} \parr A^{\perp}$. We denote by capital Greek letters $\Sigma, \Pi, \ldots$ the sets of formulas. We write $A_1 \odot \ldots \odot A_{n-1} \odot A_n$ for $A_1 \odot (\ldots \odot (A_{n-1} \odot A_n) \odot)$, where $\odot$ is $\parr$ or $\otimes$.

The rules of the **MLL sequent calculus** are as follows ([Gir87]):

$$\frac{}{\vdash X, X^{\perp}}\, ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\, cut$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\, \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\, \otimes$$

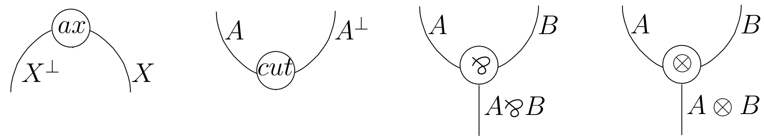


Figure 1.1: **MLL** links.

We restrict ourself to axioms introducing just atomic formulas: this is a common way to avoid the η-expansion rule (see for example [TdF03b]). **MLL** can be extended with the mix rule:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\, mix$$

Remark that every rule of **MLL** is free from conditions on the context: it deals exclusively with its active formulas. Nevertheless the structure of a sequent proof yields further inessential dependencies among the rules, by which a proof appears as a tree. If we instead consider just the logical order between such rules, what we get is a graph less restrictive than a tree: a proof net.

The set of proof nets is a subset of a wider set of graphs: the set of proof structures. More precisely, proof nets are those proof structures which correspond to correct proofs. The importance of proof structures is that cut reduction is defined directly on them, so it makes sense even without logical correctness. This is indeed one remarkable novelty of linear logic.

1.1 Proof structures

In this section we recall the **MLL** proof structures and the cut reduction rules defined on them. We introduce for proof structures a denotational semantics (the relational model) in subsection 1.1.1, and an observational equivalence in subsection 1.1.2.

Proof structures. Proof structures are oriented graphs (even empty) whose nodes are called **links** and whose edges are labeled by formulas of linear logic. When drawing a proof structure we represent edges oriented up-down so that we may speak of moving *downwardly* or *upwardly* in the graph. Links are defined together with both an arity (the number of incident edges, called the **premises of the link**) and a coarity (the number of emergent edges, called the **conclusions of the link**). **MLL** links are the following (see figure 1.1):

1. the **axiom** (*ax*-link), which has two conclusions labeled by dual atomic formulas, but no premise;

2. the **cut** (*cut*-link), which has two premises labeled by dual formulas but no conclusion;

3. the **par** (⅋-link), which has two ordered premises and one conclusion. If the left premise is labeled by the formula A and the right premise is

labeled by the formula B, then the conclusion is labeled by the formula $A⅋B$;

4. the **tensor** ($\otimes$-link), which has two ordered premises and one conclusion. If the left premise is labeled by the formula A and the right premise is labeled by the formula B, then the conclusion is labeled by the formula $A \otimes B$.

Each edge is the conclusion of a unique link and the premise of at most one link. Edges which are not the premise of any link are the **conclusions** of the proof structure. A link l of a proof structure π is **terminal** if all the conclusions of l are conclusions of π. π is **closed** if it has only one conclusion. If π is a proof structure with conclusions $C_1, \ldots, C_n$, we define the **closure of** π **with conclusion** $\mathbf{C_1}⅋\ldots⅋\mathbf{C_n}$ as the proof structure obtained from π by adding the necessary ⅋-links below $C_1, \ldots, C_n$.

Proof structures are denoted by Greek letters: $\pi, \sigma, \tau, \ldots$, the edges by initial Latin letters: $a, b, c \ldots$ and the links by middle-position Latin letters: $l, m, n, o \ldots$. We write $a : A$ if a is an edge labeled by the formula A.

We define by PS^m the set of **MLL** proof structures.

An **oriented edge** is an edge together with a direction *upward*, denoted by $\uparrow a$, or *downward*, denoted by $\downarrow a$. We write $\updownarrow a$ in case we do not want to specify if we mean either $\uparrow a$ or $\downarrow a$. An **oriented path** (or simply path) from $\updownarrow a_0$ to $\updownarrow a_n$ in a proof structure π is a sequence of π oriented edges $<\updownarrow a_0, \ldots, \updownarrow a_n>$ such that for any $i < n$:

- if $\updownarrow a_i =\uparrow a_i$, $\updownarrow a_{i+1} =\uparrow a_{i+1}$, then a_i is conclusion of the link of which a_{i+1} is premise;
- if $\updownarrow a_i =\uparrow a_i$, $\updownarrow a_{i+1} =\downarrow a_{i+1}$, then a_i and a_{i+1} are conclusions of the same link;
- if $\updownarrow a_i =\downarrow a_i$, $\updownarrow a_{i+1} =\downarrow a_{i+1}$, then a_i is the premise of the link of which a_{i+1} is conclusion;
- if $\updownarrow a_i =\downarrow a_i$, $\updownarrow a_{i+1} =\uparrow a_{i+1}$, then a_i and a_{i+1} are premises of the same link;

morally $\updownarrow a_i =\uparrow a_i$ (resp. $\updownarrow a_i =\downarrow a_i$) when the path crosses the edge a_i from the link it is conclusion (resp. premise) to the link it is premise (resp. conclusion). We say that **a path crosses a link l** if it contains a sequence of two edges having l as a vertex.

A path is **up-oriented** (resp. **down-oriented**) if all its edges are upward (resp. downward) oriented. **An edge** a **is above an edge** b ($a \geq b$) if there is a path down-oriented from $\downarrow a$ to $\downarrow b$.

We denote paths by Greek letters $\phi, \tau, \psi, \ldots$. We write $\updownarrow a \in \phi$ to mean that $\updownarrow a$ occurs in ϕ, sometimes we write simply $a \in \phi$ for meaning that $\uparrow a$ or $\downarrow a$ occurs in ϕ. We denote by $\psi \sqsubseteq \phi$ when ψ is a subpath of ϕ. We may denote a path $<\updownarrow a_0, \ldots, \updownarrow a_n>$ by a simple succession of oriented edges, i.e. $\updownarrow a_0 \ldots \updownarrow a_n$.

We recall in the proof structures framework the notion of congruent equivalence, defined by Girard in [Gir91]:

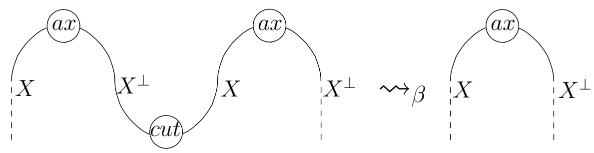


Figure 1.2: axiom cut reduction 1.

Definition 1 (from [Gir91]) *An equivalence $\equiv$ between proof structures is a **congruence** (or is **congruent**) when for all proof structures π_1, π_2, if $\pi_1 \equiv \pi_2$ then π_1 and π_2 have the same conclusions, and whenever π_1' and π_2' have been obtained from π_1 and π_2 by adding the same links, then $\pi_1' \equiv \pi_2'$.*

Cut reduction. The cut defines the composition between proof structures: if π and σ are two proof structures with conclusions respectively Π, A and $\Sigma, A^\perp$, the **composition** of π and σ on $A, A^\perp$, denoted by $[\pi, \sigma]_{\mathbf{A},\mathbf{A}^\perp}$, is the proof structure with conclusions Π, Σ obtained by joining π and σ with a new cut with premises A and $A^\perp$. We omit the indexes $_{A,A^\perp}$ in case it is clear which are the premises of the cut.

A proof structure without cuts is called **cut-free**. The **MLL cut reduction rules** are graph rewriting rules which modify a proof structure π, obtaining a proof structure π' with same conclusions as π. We denote the cut reduction relation between π and π' as $\pi \rightsquigarrow_\beta \pi'$, recalling the β-reduction of λ-calculus.

Let l be a cut in a proof structure. l can be of two types:

- an axiom cut, whose premises are labeled by dual atomic formulas X and $X^\perp$;
- a $⅋/\otimes$ cut, whose premises are labeled by dual multiplicative formulas $A⅋B$ and $A^\perp \otimes B^\perp$.

The reduction rule for l is defined as follows:

- if l is an axiom cut, let m be the axiom of which a conclusion is the premise of l labeled by X and let n be the axiom of which a conclusion is the premise of l labeled by $X^\perp$. If $m \neq n$, then l is reduced erasing l, m, n and the l premises, and later on linking the remained m, n conclusions through a new axiom link (see figure 1.2). If $m = n$, then l is reduced simply erasing l, m and the l premises (see figure 1.3);
- if l is a $⅋/\otimes$ cut, let m be the par whose conclusion is the premise of l labeled by $A⅋B$ and let n be the tensor whose conclusion is the premise of l labeled by $A^\perp \otimes B^\perp$ (remember that compound formulas do not label conclusions of axioms). Let a, b (resp. a', b') be the left and right premises of m (resp. n). Then l is reduced simply erasing l, m, n and l premises, and later on linking respectively a, a' and b, b' by two new cuts (see figure 1.4).

The reduction in figure 1.3 is maybe unusual, indeed it has a dubious logical meaning. Yet we are not at logic level: we study the reduction rules just as

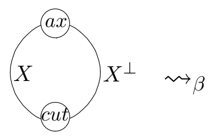


Figure 1.3: axiom cut reduction 2.

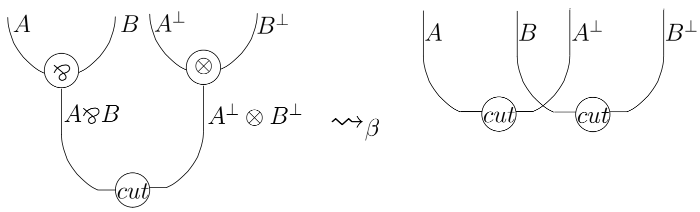


Figure 1.4: ⅋/⊗ cut reduction.

rewriting rules for proof structures. In section 1.2 we will upgrade to proof nets, the links will acquire a logical meaning as well as the reduction rules. In particular proof nets do not allow *"vicious cuts"* as the cut between the two conclusions of an axiom.

The reflexive and transitive closure of $\rightsquigarrow_\beta$ is denoted by $\rightarrow_\beta$. The symmetric closure of $\rightarrow_\beta$ is denoted by $=_\beta$ and called **β-equivalence**.

As well-known, $\rightarrow_\beta$ enjoys confluence and strong normalization:

Theorem 2 (Confluence) *For every proof structure π_1, π_2 and π_3, s.t. $\pi_1 \rightarrow_\beta \pi_2$ and $\pi_1 \rightarrow_\beta \pi_3$, there is a proof structure π_4, s.t. $\pi_2 \rightarrow_\beta \pi_4$ and $\pi_3 \rightarrow_\beta \pi_4$.*

Theorem 3 (Strong normalization) *For every proof structure π, there is no infinite sequence of proof structures π_0, π_1, π_2, ... s.t. $\pi_0 = \pi$ and $\pi_i \rightsquigarrow_\beta \pi_{i+1}$.*

Confluence and strong normalization assure that in each equivalence class of $=_\beta$ there is one and only one cut-free proof structure. We remark that the only cut-free proof structure without conclusions is the empty graph, hence all the proof structures without conclusions are reduced to the empty graph.

It is well-known that the conclusions of a cut-free proof structure determine it up to the axioms: a cut-free proof structure with conclusions $C_1, \ldots, C_n$ is the forest of the n syntax trees of the formulas $C_1, \ldots, C_n$ and a set of axioms linking in pairs such forest leaves.

1.1.1 Relational semantics

A denotational semantics defines an invariant under cut reduction. In this subsection we recall the relational semantics for **MLL**, which associates with formulas sets and with proof structures relations. The main result is the semantic characterization of those relations which are interpretations of proof structures (theorem 14).

Let $\mathtt{X}$ be a set, a **relational model on** $\mathtt{X}$ ($\mathfrak{Rel}^{\mathtt{X}}$) associates with formulas sets, in the following way:

- $\mathtt{X}$ is associated with the atomic formulas $X, X^{\perp}$;
- if $\mathtt{A}$ and $\mathtt{B}$ are associated respectively with A and B, then $\mathtt{A}\times\mathtt{B}$ is associated with $A⅋B$ and $A \otimes B$.

We recall that we denote the elements of sets by lower-case letters $a, b, u, v, x, y, z \ldots$, and sets by typewrite capital letters $\mathtt{A}, \mathtt{B}, \mathtt{X} \ldots$. If $\mathtt{C} \subset \mathtt{A} \times \mathtt{B}$ we define the projections $p_1(\mathtt{C}) = \{a \mid \exists b \in \mathtt{B}, < a, b > \in \mathtt{C}\}$ and $p_2(\mathtt{C}) = \{b \mid \exists a \in \mathtt{A}, < a, b > \in \mathtt{C}\}$.

For each proof structure π, we define the **interpretation of π in $\mathfrak{Rel}^{\mathtt{X}}$**, denoted by $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$, where the index $_{\mathfrak{Rel}^{\mathtt{X}}}$ is omitted if it is clear which model we refer to.

In case π has no conclusion, let $[\![\pi]\!]$ set as undefined. Otherwise, let $c_1 : C_1, \ldots, c_n : C_n$ be the conclusions of π, $[\![\pi]\!]$ is a subset of $\mathtt{C}_1 \times \ldots \times \mathtt{C}_n$, which we define using the notion of experiment. The experiments have been introduced by Girard in [Gir87], and extensively studied in [TdF00] by Tortora de Falco.

Definition 4 (Experiment [Gir87]) *A $\mathfrak{Rel}^{\mathtt{X}}$* ***experiment*** *$e$* ***on a proof structure*** *π, denoted by $e : \pi$, is a function associating with every edge $a : A$ of π an element of $\mathtt{A}$, so that the following conditions are respected:*

axiom: *if a, b are the conclusions of an axiom, then $e(a) = e(b)$;*

cut: *if a, b are the premises of a cut, then $e(a) = e(b)$;*

multiplicative: *if c is the conclusion of a ⅋ or $\otimes$ with premises a and b, then $e(c) =< e(a), e(b) >$.*

The experiments can be viewed as π edges decorations either from axioms to conclusions or vice-versa from conclusions to axioms: multiplicative condition determines an experiment either assigning values to the axioms, if cut-condition is satisfied, or assigning values to the conclusions and to the cuts of π, if axiom-condition is satisfied.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$ and $e : \pi$ be an experiment, then the **result of e** is the element $< e(c_1), \ldots, e(c_n) >$ of $\mathtt{C}_1 \times \ldots \times \mathtt{C}_n$. The interpretation of π in $\mathfrak{Rel}^{\mathtt{X}}$ is the set of the results of all the $\mathfrak{Rel}^{\mathtt{X}}$ experiments on π:

$$[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = \{< e(c_1), \ldots, e(c_n) > \mid e \text{ is a } \mathfrak{Rel}^{\mathtt{X}} \text{ experiment on } \pi\}$$

For each formula C we have on the one hand the proof structures with conclusion C, on the other hand the subsets of $\mathtt{C}$, being $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ a function from the proof structures to the subsets of $\mathtt{C}$. It is well known that[1]:

Theorem 5 (Soundness of $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$, [Gir87]) *For every proof structures π, π', $\pi =_\beta \pi'$ implies $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = [\![\pi']\!]_{\mathfrak{Rel}^{\mathtt{X}}}$.*

Theorem 6 (Injectivity of $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$, from [TdF03b]) *If $\mathtt{X}$ is infinite, then for every proof structures π, π', $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = [\![\pi']\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ implies $\pi =_\beta \pi'$.*

[1] Actually in [Gir87] (resp. [TdF03b]) the author proves the semantical soundness (resp. injectivity) in the more restricted case of proof nets. We remark that those proofs can be extended straightforwardly to the general case of proof structures.

The rest of the subsection is devoted to characterizing those subsets of $\mathtt{C}$, called complete subsets, which are the interpretations of proof structures with conclusion C (theorem 14). In this way, $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ becomes a bijection between the cut-free proof structures with conclusion C and the complete subsets of $\mathtt{C}$.

To achieve theorem 14 let us start from the proof of the injectivity of $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$. Let π be a cut-free proof structure with conclusion C, we have already noticed that π can be presented as a set of axioms linking the leaves of the syntax tree of C. The proof of the injectivity of $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ mainly uses the fact that there exists $u \in [\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ which codes all the pairs of dual leaves linked by an axiom of π. Indeed such an element u is the result of an injective experiment:

Definition 7 ([TdF00]) *Let π be a cut-free proof structure and $e : \pi$ be an experiment. e is* ***injective*** *when for any two different edges a, a' labeled by X, $e(a) \neq e(a')$.*

We remark that, if $\mathtt{X}$ is infinite, any cut-free proof structure has injective experiments: simply take an injective assignment of values to the axioms of the proof structure. By an easy induction we can prove that injective experiments are actually injective on edges of any type, not only atomic:

Fact 8 *Let π be a cut-free proof structure and $e : \pi$ be an injective experiment, for any two different edges a, a' labeled by the same formula A, $e(a) \neq e(a')$.*

The results of injective experiments are the most informative points of $\mathtt{C}$: we define a pre-order $\succeq$ on the elements of $\mathtt{C}$ (definition 9), measuring how much information on proof structures is coded by an element; as expected, the results of injective experiments are maximal among the (balanced, see definition 10) elements of $\mathtt{C}$. Conversely, in lemma 11 we prove that all the maximals among the (balanced) elements of $\mathtt{A}$ are results of injective experiments.

In lemma 12, we prove that for every proof structure π, the set $[\![\pi]\!]$ has the shape $\{v | u \succeq v\}$, where u is the result of an injective experiment on π. Therefore we define the complete subsets of $\mathtt{C}$ as those subsets of the form $\{v | u \succeq v\}$, for a maximal u among the (balanced) elements of $\mathtt{C}$. In this way we get a characterization for those subsets of $\mathtt{C}$ which are interpretations of proof structures (theorem 14).

An element u of a set $\mathtt{C}$ is a sequence of elements of the basic set $\mathtt{X}$ and the symbols $<, >$. We call the elements of $\mathtt{X}$ which are in u the **atoms** of u. We remark that any element u in $\mathtt{C}$ defines a labeling of the syntax tree of C: the atoms of u will label the leaves of such a tree. An occurrence of an atom x in u is a **positive occurrence** if it labels a subformula X of C, it is a **negative occurrence** if it labels a subformula $X^\perp$ of C.

Having given two elements $x, y \in \mathtt{X}$, we define $u[y/x]$ as the element of $\mathtt{C}$ obtained from u by substituting y for each occurrence of x. As always, we extend the definition to simultaneous substitutions $u[y_1/x_1, \ldots, y_n/x_n]$.

Definition 9 *Let C be an* **MLL** *formula, $\mathtt{C}$ its associated set and $u, u' \in \mathtt{C}$. We write $\mathbf{u} \succeq \mathbf{u'}$ if there is a substitution $[y_1/x_1, \ldots, y_n/x_n]$ so that $u[y_1/x_1, \ldots, y_n/x_n] = u'$. We set $\mathbf{u} \approx \mathbf{u'}$ if $u \succeq u'$ and $u' \succeq u$.*

In general $\approx$ identifies the results of the experiments, different just for a renaming of the values appointed to the conclusions of the axioms.

The following definition allows to take out from $\mathtt{C}$ those elements which cannot be in the interpretation of a proof structure:

Definition 10 *An element $u \in \mathtt{C}$ is **balanced**, if for every atom the number of its positive occurrences in u is equal to the number of its negative occurrences.*

The property of being balanced is stable by substitution: if u is a balanced element, then $u[y_1/x_1, \ldots, y_n/x_n]$ is balanced for every substitution $[y_1/x_1, \ldots, y_n/x_n]$.

The pre-order $\succeq$ evaluates how much informative the elements of $\mathtt{C}$ are. The results of the injective experiments are balanced and maximal among the balanced elements of $\mathtt{C}$. We prove the vice-versa in the next lemma:

Lemma 11 *Let $\mathtt{X}$ be an infinite set, $\mathcal{C}$ be a set associated with a formula C in $\mathfrak{Rel}^{\mathtt{X}}$. Let $u \in \mathtt{C}$ be a balanced element which is maximal among the balanced elements of $\mathtt{C}$. There is a cut-free closed proof structure π^u with conclusion C and an injective experiment $e^u : \pi^u$ so that the result of e^u is u.*

Proof. From the C tree we get π^u up to the axioms. Since u is balanced and maximal among the balanced elements and $\mathtt{X}$ is infinite, each atom x of u has exactly one positive and one negative occurrence in u, hence each atom x defines a pair of leaves $X, X^\perp$ of the C tree. We get π^u by linking with axioms such pairs.

Clearly u is the result of the injective experiment on π^u which takes the value x on the pair of edges of type X, $X^\perp$ associated with x in u. $\square$

Lemma 11 defines a function from the balanced elements maximal among the balanced elements of $\mathtt{C}$ to the closed cut-free proof structures with conclusion C:

$$u \Longrightarrow \pi^u$$

such a function is a bijection between the $\approx$-equivalence classes of the balanced maximal elements of $\mathtt{A}$ and the closed cut-free proof structures with conclusion A.

Lemma 12 *Let $\mathtt{X}$ be an infinite set and π be a closed cut-free proof structure with conclusion C. There is a balanced element u in $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ maximal among the balanced elements of $\mathtt{C}$. Moreover for any such balanced and maximal u, $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = \{v | u \succeq v\}$.*

Proof. Since $\mathtt{X}$ is infinite, there are injective experiments on π. Let $e : \pi$ be an injective experiment, and u its result. Clearly u is balanced and maximal among the balanced elements of $\mathtt{C}$. Now, take any such u.

Let $a_1, \ldots, a_n$ be the conclusions of type X of the axioms of π. Let $e' : \pi$ be an experiment and v its result. Clearly $v = u[e'(a_1)/e(a_1), \ldots, e'(a_n)/e(a_n)]$, therefore $u \succeq v$.

Conversely, let $v \in \mathtt{C}$ be so that $u \succeq v$, then there is a substitution $[y_1/e(a_1), \ldots, y_n/e(a_n)]$, so that $v = u[y_1/e(a_1), \ldots, y_n/e(a_n)]$. Let e' be the experiment so that $e'(a_1) = y_1, \ldots, e'(a_n) = y_n$, clearly e' has v as result. $\square$

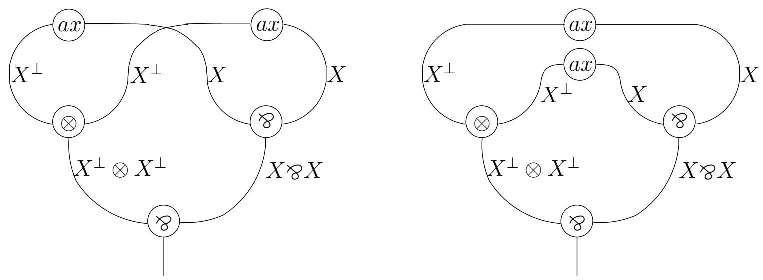


Figure 1.5: observational values $\mho$ and Ω

Definition 13 *A subset* $\mathtt{P} \subset \mathtt{C}$ *is* ***complete*** *if there is a balanced element* $u \in \mathtt{P}$ *which is maximal among the balanced elements of* $\mathtt{C}$ *and*

$$\mathtt{P} = \{v | u \succeq v\}$$

Theorem 14 *Let* $\mathtt{X}$ *be an infinite set. Let* C *be an* **MLL** *formula and* $\mathtt{C}$ *its interpretation in* $\mathfrak{Rel}^{\mathtt{X}}$*. A subset* $\mathtt{P}$ *of* $\mathtt{C}$ *is the interpretation of a closed proof structure with conclusion* C *if and only if* $\mathtt{P}$ *is complete.*

Proof. Let $\mathtt{P}$ be a complete set of $\mathtt{C}$. By its definition there is a balanced element $u \in \mathtt{P}$ which is maximal among the balanced elements of $\mathtt{C}$ and $\mathtt{P} = \{v | u \succeq v\}$. By lemma 11 there is a proof structure π^u and an injective experiment $e^u : \pi^u$ so that the result of e^u is u. By lemma 12, $[\![\pi^u]\!] = \mathtt{P}$.

Conversely, let π be a closed proof structure with conclusion C. By lemma 12, $[\![\pi]\!]$ is complete. □

1.1.2 Observational equivalence

In definition 15 we introduce an observational equivalence $\sim_{\mathbb{B}}$ between **MLL** proof structures. The main result of this subsection is theorem 16 by which follows that $=_\beta$ and $\sim_{\mathbb{B}}$ are the same equivalence (corollary 17) and that such an equivalence is a maximal congruence (corollary 18).

We choose as observational values the only two cut-free proof structures with conclusion $(X^\perp \otimes X^\perp) ⅋ (X ⅋ X)$ (see figure 1.5). We denote the formula $(X^\perp \otimes X^\perp) ⅋ (X ⅋ X)$ by $\mathbb{B}$, and the two cut-free proof structures with conclusion $\mathbb{B}$ resp. by $\mho$ and Ω.

A **proper axiom with conclusions** $\mathbf{C_1}, \ldots, \mathbf{C_n}$ is a link without premises but with n conclusions labeled respectively by $C_1, \ldots, C_n$. A **context of type** $\mathbf{C_1}, \ldots, \mathbf{C_n}$ is a proof structure with conclusion $\mathbb{B}$ where proper axioms with conclusions $C_1, \ldots, C_n$ can occur. We denote a context by $C[\,]$.

Let π be a proof structure with conclusions $C_1, \ldots, C_n$ and let $C[\,]$ be a context of the same type. By $C[\pi]$ we denote the proof structure with conclusion $\mathbb{B}$ obtained from $C[\,]$ substituting π for each occurrence of the proper axiom.

Definition 15 *Let* π_1, π_2 *be proof structures with conclusions* $C_1, \ldots, C_n$*. We say that* π_1 *and* π_2 *are* ***observationally equal*** *(denoted by* $\pi_1 \sim_{\mathbb{B}} \pi_2$*) if for all contexts* $C[\,]$ *of type* $C_1, \ldots, C_n$*,* $C[\pi_1] =_\beta C[\pi_2]$*.*

Clearly $\sim_{\mathbb{B}}$ is a congruence. By theorem 16 we prove that $\sim_{\mathbb{B}}$ and $=_\beta$ are indeed the same equivalence (corollary 17):

Theorem 16 (Separation of MLL) *Let π_1 and π_2 be two closed proof structures with conclusion C. If $\pi_1 \neq_\beta \pi_2$, then there is a proof structure σ with conclusion $C^\perp, \mathbb{B}$, such that $[\sigma, \pi_1] \to_\beta \mho$ and $[\sigma, \pi_2] \to_\beta \Omega$.*

Proof. Let π_1, π_2 be two different cut-free proof structures with conclusion C. Let $1, \ldots, 2n$ be an enumeration of the leaves of the syntax tree of C, so that the odd numbers enumerate the leaves labeled by X and the even numbers those labeled by $X^\perp$.

We have already noticed that π_1, π_2 can be presented as bijections from the odd to the even numbers of $\{1, \ldots, 2n\}$. Since $\pi_1 \neq \pi_2$, there is an odd number $o \leq 2n$ such that $\pi_1(o) = e$ and $\pi_2(o) = e'$ for $e \neq e'$.

We define the proof structure σ with conclusions $C^\perp, \mathbb{B}$, such that $[\pi_1, \sigma] \to_\beta \mho$ and $[\pi_1, \sigma] \to_\beta \Omega$. The forest of the syntax trees of $C^\perp, \mathbb{B}$ has $2n+4$ leaves. The enumeration given above of the leaves of the syntax tree of C induces an enumeration $1, \ldots, 2n, 2n+1, \ldots, 2n+4$ of the leaves of the forest, so that:

- the odd (resp. even) numbers in $\{1, \ldots, 2n\}$ enumerate the leaves labeled by $X^\perp$ (resp. X) above $C^\perp$;
- the odd (resp. even) numbers in $\{2n+1, \ldots, 2n+4\}$ enumerate the leaves labeled by X (resp. $X^\perp$) above $\mathbb{B}$.

In particular we remark that e, e' are now associated with leaves labeled by X above $C^\perp$ and o with a leaf labeled by $X^\perp$ above $C^\perp$, finally $2n+1$ and $2n+3$ (resp. $2n+2$ and $2n+4$) are the two leaves labeled by X (resp. $X^\perp$) above $\mathbb{B}$.

σ is any bijection between the leaves labeled by X and those labeled by $X^\perp$, so that $\sigma(o) = 2n+1$, $\sigma(e) = 2n+2$ and $\sigma(e') = 2n+4$. Clearly we have that $[\sigma, \pi_1] \to_\beta \mho$ and $[\sigma, \pi_2] \to_\beta \Omega$. □

Corollary 17 (Equality of $\sim_{\mathbb{B}}$ and $=_\beta$) *Let π_1 and π_2 be two proof structures with same conclusions, $\pi_1 \sim_{\mathbb{B}} \pi_2$ iff $\pi_1 =_\beta \pi_2$.*

Proof. Let π_1 and π_2 be two proof structures with same conclusions, we may suppose π_1, π_2 closed, since both $\sim_{\mathbb{B}}$ and $=_\beta$ are congruences. By the confluence of $=_\beta$, if $\pi_1 =_\beta \pi_2$ then $\pi_1 \sim_{\mathbb{B}} \pi_2$, the converse holds by theorem 16. □

Corollary 18 (Maximality of $=_\beta$) *Let $\equiv$ be a congruence which contains $=_\beta$, then either $\equiv$ is equal to $=_\beta$ or $\equiv$ collapses.*

Proof. Let $\equiv$ be a congruence containing $=_\beta$ and let us suppose that there are two distinct proof structures π_1, π_2 such that $\pi_1 \equiv \pi_2$ but $\pi_1 \neq_\beta \pi_2$. We prove $\tau_1 \equiv \tau_2$, for every proof structure τ_1, τ_2 with same conclusions.

Since $\equiv$ is a congruence we can suppose π_1 and π_2 being closed with same conclusion C. Since $\pi_1 \neq_\beta \pi_2$, by theorem 16 there is a proof structure σ with conclusions $C^\perp, \mathbb{B}$, such that $[\pi_1, \sigma] \to_\beta \mho$ and $[\pi_2, \sigma] \to_\beta \Omega$. By the congruence of $\equiv$, we deduce $[\pi_1, \sigma] \equiv [\pi_2, \sigma]$, hence $\mho \equiv \Omega$.

Let τ_1, τ_2 be two distinct proof structures with same conclusions, we prove that $\tau_1 \equiv \tau_2$. Since $\equiv$ is a congruent extension of $=_\beta$, we can suppose τ_1, τ_2 to be cut-free and with only one conclusion D. Let $1, \ldots, 2n$ be an enumeration of the leaves of the syntax tree of D, so that the odd numbers enumerate the leaves labeled by X and the even numbers the ones labeled by $X^\perp$. Since D has at least two distinct cut-free proof structures (i.e. τ_1, τ_2), D has at least two occurrences of X and two of $X^\perp$, i.e. $n \geq 2$.

We have already seen that τ_1, τ_2 can be presented as bijections from the odd to the even numbers of $\{1, \ldots, 2n\}$. Since $\tau_1 \neq \tau_2$, there is an odd number $o \leq 2n$ such that $\tau_1(o) \neq \tau_2(o)$, let us choose o minimal and let $\tau_1(o) = e$, $\tau_2(o) = e'$ and $\tau_2^{-1}(e) = o'$. By minimality of o, $o < o'$.

Now, we define a proof structure σ with conclusions $D^\perp, D, \mathbb{B}^\perp$. The forest of the syntax trees of such conclusions has $2n + 2n + 4$ leaves. The above enumeration of the leaves of the syntax tree of D induces an enumeration $1, \ldots, 2n, 2n + 1, \ldots, 4n, 4n + 1, \ldots, 4n + 4$ of the forest leaves, so that:

- the odd (resp. even) numbers in $\{1, \ldots, 2n\}$ enumerate the leaves labeled by $X^\perp$ (resp. X) above $D^\perp$;
- the odd (resp. even) numbers in $\{2n + 1, \ldots, 4n\}$ enumerate the leaves labeled by X (resp. $X^\perp$) above D;
- the odd (resp. even) numbers in $\{4n+1, \ldots, 4n+4\}$ enumerate the leaves labeled by $X^\perp$ (resp. X) of the tree of $\mathbb{B}^\perp$.

In particular we remark that e and e' are associated with leaves labeled by X above $D^\perp$, while $2n+e$ and $2n+e'$ are associated with leaves labeled by $X^\perp$ above D, and finally $4n + 1$ and $4n + 3$ (resp. $4n + 2$ and $4n + 4$) are the two leaves labeled by X (resp. $X^\perp$) above $\mathbb{B}$.

We set $\sigma(e) = 4n+2$, $\sigma(e') = 4n+4$, $\sigma(2n+e) = 4n+1$, $\sigma(2n+e') = 4n+3$, and for all the others $i \leq 2n$, $\sigma(i) = 2n + i$.

The peculiarity of σ is that the action of $[\sigma, \mho]$ is the identity, while the action of $[\sigma, \Omega]$ is the flip of e and e'. More precisely, for any proof structure π with conclusion D, $[[\sigma, \mho], \pi] \to_\beta \pi$, while $[[\sigma, \Omega], \pi] \to_\beta \pi'$, where π' is obtained from π by flipping e and e'. Moreover, by the congruence of $\equiv$ and the fact that $\mho \equiv \Omega$, we have $[\sigma, \mho] \equiv [\sigma, \Omega]$.

Now, by induction on $2n - o$ we prove that $\tau_1 \equiv \tau_2$:

- if $2n - o = 1$, then $o = 2n - 1$ and $o' = 2n$ and $\tau_1(o') = e'$. As we have remarked, $[[\sigma, \mho], \tau_1] \to_\beta \tau_1$ and $[[\sigma, \Omega], \tau_1] \to_\beta \tau_2$. Since $[\sigma, \mho] \equiv [\sigma, \Omega]$, we get $\tau_1 \equiv \tau_2$;
- if $2n - o > 1$. As we have remarked, $[[\sigma, \mho], \tau_1] \to_\beta \tau_1$ and $[[\sigma, \Omega], \tau_1] \to_\beta \tau_3$, where τ_3 is obtained from τ_1 by flipping e and e'. In particular $\tau_3(e') = o$, so that τ_2 and τ_3 at most differ on an $o'' > o$, thus, by induction hypothesis $\tau_3 \equiv \tau_2$. Therefore, $\tau_1 \equiv [[\sigma, \mho], \tau_1] \equiv [[\sigma, \Omega], \tau_1] \equiv \tau_3 \equiv \tau_2$.

□

Relational semantics defines a congruence $\equiv_{\mathfrak{Rel}}$ between proof structures, what means $\pi_1 \equiv_{\mathfrak{Rel}} \pi_2$ if for all $\mathtt{X}$, $[\![\pi_1]\!]_{\mathtt{X}} = [\![\pi_2]\!]_{\mathtt{X}}$. By the soundness of the relational semantics we know that $=_\beta \subseteq \equiv_{\mathfrak{Rel}}$. Now, by corollary 18 we get the converse $\equiv_{\mathfrak{Rel}} \subseteq =_\beta$, i.e. a proof of the injectivity of the relational semantics, alternative to that in [TdF03b].

1.2 Proof nets

In this section we recall **MLL** proof nets, which are those proof structures which correspond to correct proofs. We introduce coherent semantics in subsection 1.2.1 and an observational equivalence for proof nets in subsection 1.2.2.

The proofs of the **MLL** sequent calculus can be translated into proof structures by a function called **desequentialization**. This translation associates with a sequent proof σ a proof structure $(\sigma)^\bullet$, defined by induction on σ (see [Gir87]):

- if σ is an axiom with conclusions $X, X^\perp$, then $(\sigma)^\bullet$ is an axiom link with conclusions $X, X^\perp$;
- if σ ends in a $\parr$-rule, having as premise the subproof σ', then $(\sigma)^\bullet$ is obtained by adding to $(\sigma')^\bullet$ the link $\parr$ corresponding to the $\parr$-rule;
- if σ ends in a $\otimes$-rule (resp. *cut*-rule), with premises the subproofs σ' and σ'', then $(\sigma)^\bullet$ is obtained by connecting $(\sigma')^\bullet$ and $(\sigma'')^\bullet$ by means of the link $\otimes$ (resp. cut) corresponding to the $\otimes$-rule (resp. *cut*-rule);
- if σ ends in a *mix*-rule, with premises the subproofs σ' and σ'', then $(\sigma)^\bullet$ is obtained by taking the disjoint union of $(\sigma')^\bullet$ and $(\sigma'')^\bullet$.

A **proof net** π is a proof structure associated with a sequent proof, moreover π is said **without mix** if it is associated with a sequent proof without the mix rule. A unique proof net can be associated with several calculus proofs: it yields a canonical representation of sequent proofs modulo inessential commutation of rules (see [BdW95]). We highlight that both semantic injectivity and syntactical separability can be studied in linear logic thanks to this canonical representation of proofs.

Many criteria have been proposed for characterizing **MLL** proof nets independently from $(\ )^\bullet$. We recall here the criterion by Danos and Regnier in [DR89].

A **correctness graph** of a proof structure π is a π subgraph which is obtained by erasing one premise for each $\parr$.

Definition 19 *A proof structure is* ***correct*** *(resp.* ***strongly correct****) if all its correctness graphs are acyclic (resp. acyclic and connected).*

Theorem 20 ([DR89]) *Let $\pi \in PS^m$. π is a proof net (resp. a proof net without mix) iff π is correct (resp. strongly correct).*

In the sequel we will largely use paths which are feasible in the correctness graphs of a proof structure. Let π be a proof structure, a path ϕ in π **comes back** if there is an edge a s.t. $\uparrow a, \downarrow a \in \phi$; a **switching edge** of π is a $\parr$ premise; a path ϕ is **switching** if it never comes back and it does not contain two switching edges of a same link. Of course a switching path in π is a path in at least one correctness graph of π. A **switching cycle** is a switching path from $\updownarrow a$ to $\updownarrow a$. Thus π is correct iff π does not contain any switching cycle.

We denote by PN^{mx} the set of **MLL** proof nets and by PN^m that of **MLL** proof nets without mix. Clearly:

$$PN^m \subset PN^{mx} \subset PS^m$$

In this chapter we study both PN^m and PN^{mx}. We are interested in proof nets with mix mainly for two reasons. Firstly, the mix rule holds in coherent spaces, so when investigating the correspondence between proof nets and coherent spaces (subsection 1.2.1), it is convenient to refer to PN^{mx}. Secondly, in chapter 3 we introduce the weakening link: in presence of weakening it is not very clear what is the connectivity of a correctness graph, so sometimes it is simpler asking just for the acyclicity.

1.2.1 Coherent semantics

In this subsection we upgrade to coherent semantics by enriching relational semantics with a coherence relation on the sets associated with formulas. We recall the semantical characterization of proof structure correctness, proved in [Ret97] by Retoré. The novelty of our approach is corollary 26, stating the correspondence between proof nets and complete cliques.

Definition 21 ([Gir87]) *A* ***coherent space*** $\mathcal{X}$ *is a couple* $(|\mathcal{X}|, \mathrel{\substack{\frown\\\smile}})$*, where* $|\mathcal{X}|$ *is a set, called the* ***web*** *of* $\mathcal{X}$*, and* $\mathrel{\substack{\frown\\\smile}}$ *is a binary relation in* $|\mathcal{X}|$ *which is reflexive and symmetric, called the* ***coherence*** *of* $\mathcal{X}$*.*

A ***clique*** *of* $\mathcal{X}$ *is a subset* $\mathtt{C}$ *of* $|\mathcal{X}|$ *such that for every* $x, y \in \mathtt{C}$, $x \mathrel{\substack{\frown\\\smile}} y$.

We will write $x \mathrel{\substack{\frown\\\smile}} y\,[\mathcal{X}]$ if we want to explicit the coherent space $\mathrel{\substack{\frown\\\smile}}$ refers to. We introduce the following notation, well-known in the framework of coherent spaces:

- $x \frown y$, if $x \mathrel{\substack{\frown\\\smile}} y$ and $x \neq y$;
- $x \mathrel{\substack{\smile\\\frown}} y$, if not $x \frown y$;
- $x \smile y$,if not $x \mathrel{\substack{\frown\\\smile}} y$.

Remark that we may define a coherent space specifying its web and one among its relations $\mathrel{\substack{\frown\\\smile}}$, $\mathrel{\substack{\smile\\\frown}}$, $\frown$, $\smile$.

A coherent space is identified with a graph whose vertex set is $|\mathcal{X}|$ and whose edges set is the extension of $\mathrel{\substack{\frown\\\smile}}$.

Let $\mathcal{X}$ be a coherent space, a **coherent model on** $\mathcal{X}$ ($\mathfrak{Coh}^{\mathcal{X}}$) associates with **MLL** formulas coherent spaces, defined by induction on the formulas, as follows:

- with X it is associated $\mathcal{X}$;
- with $A^\perp$ it is associated $\mathcal{A}^\perp$ defined as follows: $|\mathcal{A}^\perp| = |\mathcal{A}|$, the coherence of $\mathcal{A}^\perp$ is the incoherence of $\mathcal{A}$, i.e. $x \mathrel{\substack{\frown\\\smile}} y \left[\mathcal{A}^\perp\right]$ iff $x \mathrel{\substack{\smile\\\frown}} y\,[\mathcal{A}]$;
- with $A \otimes B$ it is associated $\mathcal{A} \otimes \mathcal{B}$ defined as follows: $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ and $< a, b > \mathrel{\substack{\frown\\\smile}} < a', b' > [\mathcal{A} \otimes \mathcal{B}]$ iff $a \mathrel{\substack{\frown\\\smile}} a'\,[\mathcal{A}]$ and $b \mathrel{\substack{\frown\\\smile}} b'\,[\mathcal{B}]$.

Of course, the space $\mathcal{A}⅋\mathcal{B}$ is defined by $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$.

Remark that the web associated with a formula A by $\mathfrak{Coh}^{\mathcal{X}}$ is precisely the interpretation of A in $\mathfrak{Rel}^{|\mathcal{X}|}$.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$ the **interpretation** of π in $\mathfrak{Coh}^{\mathcal{X}}$ is a subset of $|\mathsf{C}_1⅋\ldots⅋\mathsf{C}_n|$, denoted by $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$, where the index $_{\mathfrak{Coh}^{\mathcal{X}}}$ is omitted in the case it is clear which model we refer to.

$[\![\pi]\!]$ is defined exactly in the same way as in relational semantics (see section 1.1.1). We have the same definitions concerning the **experiment** e on a proof structure π, its **result**, and the interpretation $[\![\pi]\!]$. The relational interpretation of π differs from the coherent one only in presence of exponentials (see section 3.2): if π is an **MLL** proof structure, then $[\![\pi]\!]_{\mathfrak{Rel}^{|\mathcal{X}|}} = [\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

What we achieve introducing coherence is that the set $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ can be or not a clique. Girard proves in [Gir87] that if π is a proof net then $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is a clique. Retoré proves the converse for the cut-free proof nets, hence the correctness of a cut-free proof structure corresponds to the pairwise coherence of the results of its experiments.

In this thesis we will study several extensions of Girard's and Retoré's theorems. Here we give the proofs of both theorems in a slightly different way from the original proofs, our aim is to underline their symmetry. In particular the implication π *proof net* $\Rightarrow [\![\pi]\!]$ *clique* is an immediate consequence of lemma 22, while the one $[\![\pi]\!]$ *clique* $\Rightarrow \pi$ *proof net* is a consequence of lemma 23. The lemmas 22 and 23 show the correspondence between the switching paths of a proof structure π and the way the coherence spreads over the edges of π.

Lemma 22 (from [Gir87]) *Let π be a proof net with conclusions $c_1 : C_1, \ldots, c_n : C_n$. If e_1, e_2 are two experiments on π such that $e_1(c_1) \smile e_2(c_1) \left[\mathcal{C}_1^\perp\right]$, then there is a switching path ϕ from c_1 to a conclusion c_i such that $e_1(c_i) \frown e_2(c_i) [\mathcal{C}_i]$.*

Proof. Let $e_1(c_1) \smile e_2(c_1)$. We define a sequence of paths $\phi_1 \subset \phi_2 \subset \ldots \subset \phi_k$, such that ϕ_1 is exactly $\uparrow c_1$, ϕ_k starts from $\uparrow c_1$ and ends in $\downarrow c_i$, and for each ϕ_j among $\phi_1, \ldots, \phi_k$:

1. for each edge $a : A$, if $\uparrow a \in \phi_j$, then $e_1(a) \smile e_2(a) [\mathcal{A}]$, if $\downarrow a \in \phi_j$, then $e_1(a) \frown e_2(a) [\mathcal{A}]$;
2. ϕ_j is a switching path.

Let us define ϕ_{j+1} from ϕ_j, which we suppose satisfies conditions 1 and 2. Let $a : A$ be the last edge of ϕ_j. Then:

- if $\downarrow a \in \phi_j$, by hypothesis $e_1(a) \frown e_2(a) [\mathcal{A}]$:
  - if a is premise of a ⅋ with conclusion $c : C$, then $e_1(c) \frown e_2(c) [\mathcal{C}]$. We define $\phi_{j+1} = \phi_j * \downarrow c$;
  - if a is premise of a $\otimes$ with conclusion $c : C$ and premises $a : A, b : B$. In case $e_1(c) \frown e_2(c) [\mathcal{C}]$, we define $\phi_{j+1} = \phi_j * \downarrow c$; otherwise $e_1(b) \smile e_2(b) [\mathcal{B}]$, in this case we define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is premise of a cut with premises $a : A, b : A^\perp$, than $e_1(b) \smile e_2(b) \left[\mathcal{A}^\perp\right]$, so let $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is conclusion of π, then we define ϕ_j as ϕ_n.

- if $\uparrow a \in \phi_j$, by hypothesis $e_1(a) \smile e_2(a)\,[\mathcal{A}]$:
  - if a is conclusion of a $\parr$ or a $\otimes$, then exists a premise $b : B$ s.t. $e_1(b) \smile e_2(b)\,[\mathcal{B}]$. We define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is conclusion of an axiom l with conclusions $a : A, b : A^\perp$, then $e_1(b) \frown e_2(b)\,\left[A^\perp\right]$, thus we define $\phi_{j+1} = \phi_j * \downarrow b$.

Clearly ϕ_{j+1} satisfies condition 1. Let us prove that it is a switching path. Since ϕ_j is a switching path, we have to prove that the new edge added to ϕ_{j+1} is not a premise of a $\parr$ of which the other premise is already in ϕ_j.

Let b be the edge added to ϕ_{j+1}. We split in two cases, depending if $\uparrow b$ or $\downarrow b$ is added to ϕ_{j+1}.

In case $\phi_{j+1} = \phi_j * \downarrow b$, where b is a premise of a $\parr$ with premises a, b and conclusion c, we suppose ϕ_j contains a and we prove a contradiction. Of course if $a \in \phi_j$, then $c \in \phi_j$. By condition 1 and the hypothesis $\downarrow b \in \phi_{j+1}$, $e_1(b) \frown e_2(b)$, hence $e_1(c) \frown e_2(c)$, from which we deduce $\downarrow c \in \phi_j$. So ϕ_j has the following shape:

$$\phi_j = \phi'_j * \downarrow a \downarrow c * \phi''_j$$

but then $\downarrow c * \phi''_j * \downarrow b \downarrow c$ is a switching cycle, violating the correctness of π.

In case $\phi_{j+1} = \phi_j * \uparrow b$, where b is premise of a $\parr$ with premises a, b and conclusion c, then $\phi_j = \phi'_j * \uparrow c$. In this case it is immediate that $a \notin \phi_j$, otherwise ϕ_j should contains a switching cycle from $\uparrow c$ to $\uparrow c$.

So we have proved that all the paths $\phi_1, \phi_2, \phi_3, \dots$ are switching. Since π is correct, none of them can be a cycle, thus the sequence $\phi_1, \phi_2, \phi_3, \dots$ will eventually meet a conclusion c_i of π, so terminating in a path ϕ_k satisfying the lemma. □

Lemma 23 (from [Ret97]) *Let $\mathfrak{Coh}^{\mathcal{X}}$ be defined from a coherent space $\mathcal{X}$ with at least $x, y, z \in |\mathcal{X}|$, such that $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$.*

Let π be a cut-free proof net, ϕ be a switching path from a conclusion of π $c : C$ to a conclusion of π $c' : C'$.

There are two experiments e_1, e_2 on π such that $e_1(c) \smile e_2(c)\,[\mathcal{C}]$, $e_1(c') \frown e_2(c')\,[\mathcal{C}']$ and for any further conclusion $d : D$, $e_1(d) \asymp e_2(d)\,[\mathcal{D}]$.

Proof. We recall that an experiment on a proof structure π is completely determined by its values on the axioms' conclusions. Moreover, since we suppose π cut-free, every choice of values on the axioms' conclusions respecting the axiom-condition (see definition 4) determines an experiment on π.

Thus we define e_1, e_2 by declaring their values on the axioms of π. Let us fix $x, y, z \in |\mathcal{X}|$, such that $x \frown y$ and $x \smile z$. For every edge a of type X let us set:

- if $\uparrow a \in \phi$, then $e_1(a) = x$, $e_2(a) = y$;
- if $\downarrow a \in \phi$, then $e_1(a) = x$, $e_2(a) = z$;
- otherwise, $e_1(a) = x = e_2(a)$.

Remark that e_1 is a x-costant function on the axioms. In chapter 3 we will define such a kind of experiment *simple*, by following [TdF00].

For every edge $d : D$, we prove that:

1. if $\exists d' \geq d, d' \in \phi$ (where recall $d' \geq d$ means that d' is above d in π) then $e_1(d) \neq e_2(d)$;

2. if $\uparrow d \notin \phi$, then $e_1(d) \underset{\frown}{\smile} e_2(d)\,[\mathcal{D}]$;

3. if $\forall d' \geq d, \downarrow d' \notin \phi$, then $e_1(d) \underset{\smile}{\frown} e_2(d)\,[\mathcal{D}]$.

Condition 1 is immediate. Simply remark that for every edge d only the atom x occurs in $e_1(d)$, being e_1 x-constant on the atomic edges. On the other hand, if $\exists d' \geq d$, then there is an axiom edge $a \geq d$, $a \in \phi$. So that the either atom y or z occurs in $e_2(d)$.

Instead we prove $2 - 3$ by induction on the type of d:

Atom: in case d is atomic, then the assertion is immediate by definition of e_1, e_2.

Tensor: in case $d : A \otimes B$, then let $a : A$, $b : B$ be the premises of the $\otimes$ with conclusion d:

2. if $\uparrow d \notin \phi$, we split in three cases.
   In case $\uparrow a \in \phi$ then $\downarrow b \in \phi$, which implies $e_1(b) \smile e_2(b)$, by induction hypothesis and condition 1. Thus we deduce $e_1(d) \smile e_2(d)$. The same if $\uparrow b \in \phi$.
   In case both $\uparrow a, \uparrow b \notin \phi$, then by induction hypothesis $e_1(a) \underset{\frown}{\smile} e_2(a)$ and $e_1(b) \underset{\frown}{\smile} e_2(b)$, which implies $e_1(d) \underset{\frown}{\smile} e_2(d)$;
3. if $\forall d' \geq d, \downarrow d' \notin \phi$, then of course $\forall d' \geq a$ and $\forall d' \geq b$, $\downarrow d' \notin \phi$, which by induction implies $e_1(a) \underset{\smile}{\frown} e_2(a)$ and $e_1(b) \underset{\smile}{\frown} e_2(b)$, so $e_1(d) \underset{\smile}{\frown} e_2(d)$.

Par: in case $d : A \parr B$, then let $a : A$, $b : B$ be the premises of the $\parr$ with conclusion d:

2. if $\uparrow d \notin \phi$, then both $\uparrow a, \uparrow b \notin \phi$, being ϕ a switching path. By induction hypothesis, $e_1(a) \underset{\frown}{\smile} e_2(a)$ and $e_1(b) \underset{\frown}{\smile} e_2(b)$, which implies $e_1(d) \underset{\frown}{\smile} e_2(d)$;
3. if $\forall d' \geq d, \downarrow d' \notin \phi$, then of course $\forall d' \geq a$ and $\forall d' \geq b$, $\downarrow d' \notin \phi$, which implies $e_1(a) \underset{\smile}{\frown} e_2(a)$ and $e_1(b) \underset{\smile}{\frown} e_2(b)$, that is $e_1(d) \underset{\smile}{\frown} e_2(d)$.

Recall that ϕ starts with $\uparrow c$ and ends with $\downarrow c'$. By the properties 1-3 we know that $e_1(c) \smile e_2(c)$, $e_1(c') \frown e_2(c')$ and for any further π conclusion $d : D$, $e_1(d) \underset{\frown}{\smile} e_2(d)$. □

Theorem 24 ([Gir87]) *Let π be a proof structure. If π is correct then $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is a clique, for every coherent space $\mathcal{X}$.*

Proof. It is a direct consequence of lemma 22 and the definition of the coherence on the $\parr$ coherent spaces. □

Theorem 25 ([Ret97]) *Let π be a cut-free proof structure and $\mathcal{X}$ be a coherent space with $x, y, z \in |\mathcal{X}|$, such that $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$.*

If $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is a clique then π is correct.

Proof. Let π be a cut-free not correct proof structure and $\mathcal{X}$ be a coherent space with $x, y, z \in |\mathcal{X}|$, such that $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$. By induction on the number of links in π we prove that $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is not a clique.

If π has a terminal $\parr$ l, then let π' be obtained from π erasing l and its conclusion. Clearly π' is not correct, so $[\![\pi']\!] = [\![\pi]\!]$ is not a clique.

If π has a terminal $\otimes$ l, then let π' be obtained from π by erasing l and its conclusion. If π' is still not correct, we get the assertion by induction hypothesis. If instead π' is correct, then all the switching cycles in π cross l, in particular there is a switching path in the cut-free proof net π' from the left premise $a : A$ of l to the right premise $b : B$ (which are conclusions in π'). By means of lemma 23 we define two experiments e_1, e_2 such that $e_1(a) \smile e_2(a)\,[\mathcal{A}]$, $e_1(b) \frown e_2(b)\,[\mathcal{B}]$ and on any further conclusion $c : C$, $e_1(c) \asymp e_2(c)\,[\mathcal{C}]$. By extending e_1, e_2 to π, we get two experiments whose results are strictly incoherent, hence $[\![\pi]\!]$ is not a clique.

Finally, if π has no terminal $\parr$ or $\otimes$, then π is correct. □

A nice corollary of theorem 14 in subsection 1.1.1 and the above theorems 24, 25 is the semantical characterization of those sets which are interpretations of proof nets (corollary 26).

Since the web of a coherent space is a set, we can introduce the pre-order $\succeq$ (definition 9) and the notion of **complete subset** (definition 13) on webs exactly in the same way as we did with relational semantics.

If $\mathcal{A}$ is a coherent space, a **complete clique** of $\mathcal{A}$ is a complete subset of $|\mathcal{A}|$ which is a clique.

Corollary 26 *Let $\mathcal{X}$ be a coherent space whose web is infinite and with x, y, z, such that $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$. Let C be an* **MLL** *formula and $\mathcal{C}$ its interpretation in $\mathfrak{Coh}^{\mathcal{X}}$.*

A subset $\mathtt{P}$ of $\mathcal{C}$ is the interpretation of a closed proof net with conclusion C if and only if $\mathtt{P}$ is a complete clique.

Proof. Let $\mathtt{P}$ be a complete clique of $\mathcal{C}$. Since $\mathtt{P}$ is complete, by theorem 14, there is a closed cut-free proof structure π with conclusion C such that $[\![\pi]\!] = \mathtt{P}$. Since $\mathtt{P}$ is a clique, by theorem 25, π is a proof net.

Conversely, if π is a proof net, by theorem 24 $[\![\pi]\!]$ is a clique, and by theorem 14 $[\![\pi]\!]$ is complete. □

1.2.2 Observational equivalence

The observational equivalence $\sim_{\mathbb{B}}$ (definition 15) depends on the proof structures behaviors within all possible contexts. In this subsection we would like to restrict the observations just to the correct contexts, defining a weak observational equivalence $\sim^{w}_{\mathbb{B}}$ (definition 27). The main result of this subsection is proposition 29, stating that the $\sim^{w}_{\mathbb{B}}$ is strictly larger than $=_{\beta}$.

At first, we remark that the only two proof structures ℧ and Ω with conclusion $\mathbb{B}$ (figure 1.5) are correct, therefore we can keep them as observational values. At second, we extend the correctness criterion to contexts. A **correctness graph** of a context is a subgraph obtained by erasing one premise for each ⅋-link. A context **is correct** if all its correctness graphs are acyclic.

Definition 27 *Let π_1, π_2 be two proof nets with conclusions $C_1, \ldots, C_n$. We say that π_1 and π_2 are **observationally weak equal** ($\pi_1 \sim^w_{\mathbb{B}} \pi_2$) if for all the correct contexts $C[\,]$ of type $C_1, \ldots, C_n$, $C[\pi_1] =_\beta C[\pi_2]$.*

Clearly $=_\beta \subseteq \sim^w_{\mathbb{B}}$. The main result of this section is proposition 29, which states that $=_\beta \subsetneq \sim^w_{\mathbb{B}}$: there are proof nets which are observationally weak equal but not β-equivalent (hence, neither observationally equal).

Such a result does not clash with corollary 18, stating that $=_\beta$ is a maximal congruence. It means that $\sim^w_{\mathbb{B}}$ is not a congruence when extended to proof structures. Indeed $\sim^w_{\mathbb{B}}$ is defined only between proof nets but not between proof structures in general, therefore if π_1 and π_2 are two observationally weak equal proof nets, and if π_1' and π_2' have been obtained from π_1 and π_2 by adding the same links, $\pi_1' \not\sim^w_{\mathbb{B}} \pi_2'$ could happen simply because π_1' and π_2' are not correct.

Remark that in general a context can be quite complex, namely the proper axioms might be whenever and wherever we want them. Before attacking proposition 29, it is thus convenient to restrain our observations to the simplest contexts, which are the proof nets themselves:

Lemma 28 (Context lemma) *Let π_1 and π_2 be two proof nets with conclusions $C_1, \ldots, C_n$. Let π_1^* and π_2^* be the two closures of π_1, π_2 with conclusion $C_1 ⅋ \ldots ⅋ C_n$. Then $\pi_1 \not\sim^w_{\mathbb{B}} \pi_2$ iff there is a proof net σ with conclusions $C_1^\perp \otimes \ldots \otimes C_n^\perp, \mathbb{B}$, such that $[\pi_1^*, \sigma] \neq_\beta [\pi_2^*, \sigma]$.*

Proof. The "if" part is immediate. Conversely, let π_1 and π_2 be two proof nets with same conclusions $C_1, \ldots, C_n$ such that $\pi_1 \not\sim_{\mathbb{B}} \pi_2$. We prove that there is a proof net σ with conclusions $C_1^\perp \otimes \ldots \otimes C_n^\perp, \mathbb{B}$, such that $[\pi_1^*, \sigma] \neq_\beta [\pi_2^*, \sigma]$.

By definition 27, there is a correct context $C[\,]$ such that $C[\pi_1] \neq_\beta C[\pi_2]$. We enumerate by $1, \ldots, k$ the occurrences of the proper axiom in $C[\,]$. For each $i \leq k$, let σ_i be the proof net obtained from $C[\,]$ substituting π_1 to the occurrences $1, \ldots, i$ of the proper axiom and π_2 to the occurrences $i+1, \ldots, k$. Clearly, $\sigma_0 = C[\pi_2] \neq_\beta C[\pi_1] = \sigma_k$, hence there is an i such that $\sigma_i \neq_\beta \sigma_{i+1}$. σ is obtained from $C[\,]$ in two steps. At first, we substitute π_1 to the occurrences $1, \ldots, i$ of the proper axiom in $C[\,]$ and π_2 to the occurrences $i+2, \ldots, k$. At second, we substitute the $i+1$-th occurrence of the proper axiom with the set of the n axioms with conclusions respectively $C_1^\perp, C_1, \ldots, C_n^\perp, C_n$ and we link the conclusions $C_1^\perp, \ldots, C_n^\perp$ with tensors, so as to get a unique conclusion $C_1^\perp \otimes \ldots \otimes C_n^\perp$.

Clearly σ is correct, moreover $[\pi_1^*, \sigma] =_\beta \sigma_i \neq_\beta \sigma_{i+1} =_\beta [\pi_2^*, \sigma]$. □

Now, let us prove that $\sim^w_{\mathbb{B}}$ is a strict extension of $=_\beta$:

Proposition 29 *There are proof nets π_1, π_2 such that $\pi_1 \neq_\beta \pi_2$ and $\pi_1 \sim^w_{\mathbb{B}} \pi_2$.*

Proof. Let C be the formula $((X \otimes X) ⅋ X) ⅋ (X^\perp \otimes X^\perp) ⅋ X^\perp$, and π_1, π_2 be any two different cut-free proof nets with conclusion C (take for example those in figure 1.6).

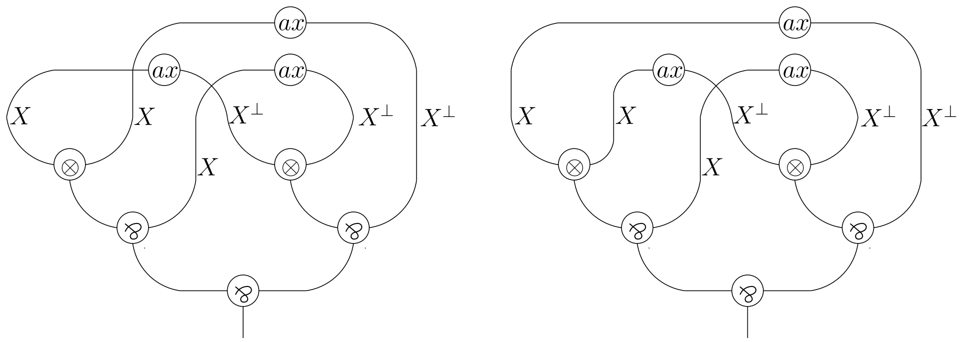


Figure 1.6: example of proof nets π_1, π_2 with conclusion $((X \otimes X) \parr X) \parr (X^\perp \otimes X^\perp) \parr X^\perp$.

Let us suppose $\pi_1 \not\sim_{\mathbb{B}}^{w} \pi_2$ and let us prove the absurdity. By lemma 28 there is a proof net σ with conclusions $C^\perp, \mathbb{B}$, such that $[\pi_1, \sigma] \neq_\beta [\pi_2, \sigma]$. Since $\mho$ and Ω are the only two cut-free proof nets with conclusion $\mathbb{B}$, we may suppose $[\pi_1, \sigma] \rightarrow_\beta \mho$ and $[\pi_2, \sigma] \rightarrow_\beta \Omega$.

Let $\mathtt{X}$ be a coherent space with $x, y, z \in |\mathtt{X}|$, such that $x \frown y$, $x \smile z$ and $y \smile z$: we will prove that $[\![\sigma]\!]_{\mathtt{X}}$ is not a clique, hence contraddicting theorem 24.

We remark that $\langle \langle x, z\rangle, \langle x, z\rangle\rangle \in [\![\mho]\!]$ and $\langle \langle x, z\rangle, \langle z, x\rangle\rangle \in [\![\Omega]\!]$, therefore there are $u \in [\![\pi_1]\!]$ and $v \in [\![\pi_2]\!]$ such that $\langle u, \langle \langle x, z\rangle, \langle x, z\rangle\rangle\rangle$, $\langle v, \langle \langle z, x\rangle, \langle z, x\rangle\rangle\rangle \in [\![\sigma]\!]$.

By theorem 26, $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ are complete cliques, thus for all $u', v' \in |\mathcal{C}|$, s.t. $u' \preceq u$ (resp. $v' \preceq v$), $u' \in [\![\pi_1]\!]$ (resp. $v' \in [\![\pi_2]\!]$). In particular, let $w_1, \ldots, w_n$ be the atoms different from z and x in u and v. We define $u' = u[x/w_1, \ldots, x/w_n]$ (resp. $v' = v[x/w_1, \ldots, x/w_n]$). Since $u' \preceq u$ (resp. $v' \preceq v$) , $u' \in [\![\pi_1]\!]$ (resp. $v' \in [\![\pi_2]\!]$); moreover, since $[\![\sigma]\!]$ is a complete clique too and $\langle u', \langle \langle x, z\rangle, \langle x, z\rangle\rangle\rangle \preceq \langle u, \langle \langle x, z\rangle, \langle x, z\rangle\rangle\rangle$ (resp. $\langle v', \langle \langle x, z\rangle, \langle z, x\rangle\rangle\rangle \preceq \langle v, \langle \langle x, z\rangle, \langle z, x\rangle\rangle\rangle$), we have that $\langle u', \langle \langle x, z\rangle, \langle x, z\rangle\rangle\rangle$, $\langle v', \langle \langle x, z\rangle, \langle z, x\rangle\rangle\rangle \in [\![\sigma]\!]$.

Now, let us look at the atom a (resp. b) of u' (resp. v') corresponding to the bold occurrence of X in $C^\perp = ((X^\perp \parr X^\perp) \otimes X^\perp) \otimes (X \parr X) \otimes \mathbf{X}$.

If $a = x$ and $b = z$ (or vice-versa, $a = z$, $b = x$), then $a \smile b\,[X]$, which implies $u' \smile v' \left[C^\perp\right]$ by the definition of the coherent spaces associated with $C^\perp$. Moreover, $\langle \langle x, z\rangle, \langle x, z\rangle\rangle \smile \langle \langle x, z\rangle, \langle z, x\rangle\rangle\ [\mathbb{B}]$, by the definition of the coherent spaces associated with $\mathbb{B} = (X^\perp \otimes X^\perp) \parr (X \parr X)$. Thus, $\langle u', \langle \langle x, z\rangle, \langle x, z\rangle\rangle\rangle \smile \langle v', \langle \langle x, z\rangle, \langle z, x\rangle\rangle\rangle \left[C^\perp \parr \mathbb{B}\right]$, i.e. $[\![\sigma]\!]$ is not a clique.

If $a = b$, let us suppose $a, b = x$ (the case $a, b = z$ being similar). In this case we consider $u'' = u'[y/z]$ and $v'' = v'[z/x, x/z]$. Since $\langle u', \langle \langle x, z\rangle, \langle x, z\rangle\rangle\rangle \approx \langle u'', \langle \langle x, y\rangle, \langle x, y\rangle\rangle\rangle$ (resp. $\langle v', \langle \langle x, z\rangle, \langle z, x\rangle\rangle\rangle \approx \langle v'', \langle \langle z, x\rangle, \langle x, z\rangle\rangle\rangle$), we deduce that $\langle u'', \langle \langle x, y\rangle, \langle x, y\rangle\rangle\rangle$, $\langle v'', \langle \langle z, x\rangle, \langle x, z\rangle\rangle\rangle \in [\![\sigma]\!]$. Since $x \smile z\ [X]$, we infer $u'' \smile v'' \left[C^\perp\right]$ by the definition of the coherent spaces associated with $C^\perp$. Moreover, $\langle \langle x, y\rangle, \langle x, y\rangle\rangle \smile \langle \langle z, x\rangle, \langle x, z\rangle\rangle\ [\mathbb{B}]$, by the definition of the coherent spaces associated with $\mathbb{B}$. Thus, $\langle u'', \langle \langle x, y\rangle, \langle x, y\rangle\rangle\rangle \smile \langle v'', \langle \langle z, x\rangle, \langle x, z\rangle\rangle\rangle \left[C^\perp \parr \mathbb{B}\right]$, i.e. $[\![\sigma]\!]$ is not a clique. □

We end this section with some remarks on the above proposition.

The failure of the equality between $=_\beta$ and $\sim_{\mathbb{B}}^{w}$ does not depend on the

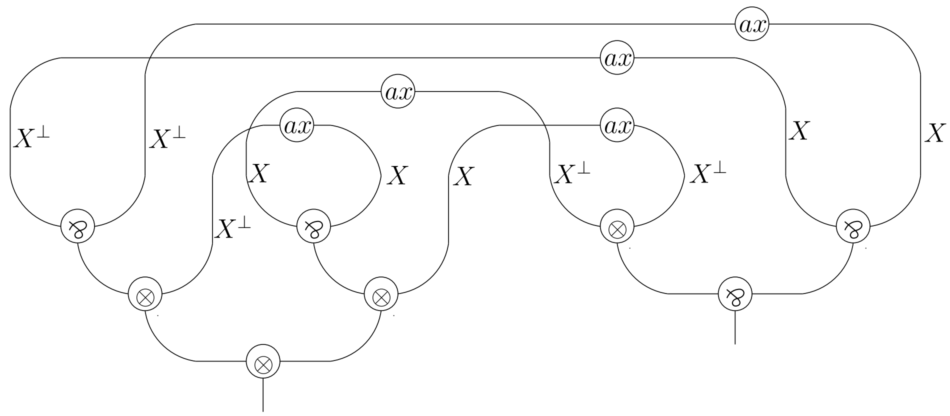


Figure 1.7: non-correct proof structure σ with conclusion $C^{\perp}, \mathbb{B}$.

formula $\mathbb{B}$ chosen as the type for the observational values. Indeed for any formula A we may denote by $\sim^w_A$ the observational weak equivalence defined by looking at the correct contexts with conclusion A instead of $\mathbb{B}$, getting all the same $=_\beta \subsetneq \sim^w_A$.

In simple typed λ-calculus we can prove a separation theorem (analogous to theorem 16) only if we substitute the atom X with more complex formulas (see [Sta83] and [Jol00]). One might thus think that proposition 29 is due to the fact that we have not allowed the substitution of the atom X in definition 27. It is not so. Actually the atom substitution is usefull in presence of exponentials (like in λ-calculus), but it is useless in a linear framework (like **MLL**). Indeed the proof of proposition 29 can be easily extended to the case we allow the substitution of X with more complex **MLL** formulas.

The failure of the equality between $=_\beta$ and $\sim^w_{\mathbb{B}}$ is actually due to the lack of garbage collectors among the correct contexts. Proof structures have garbage collectors (the cyclic cuts, erased by β-reduction), hence we can prove theorem 16, but the proof nets (which have to be correct) have not. For example recall the proof nets π_1, π_2 in figure 1.6: π_1 and π_2 are separable by the non-correct proof structure in figure 1.7, in fact $[\pi_1, \sigma] \rightarrow_\beta \mho$ and $[\pi_2, \sigma] \rightarrow_\beta \Omega$. Remark that during the reductions of $[\pi_1, \sigma]$ and $[\pi_2, \sigma]$ we meet cyclic cuts.

In this framework there is an interesting result by Matsuoka in [Mat05], dealing with the intuitionistic multiplicative linear logic fragment (which corresponds to the linear λ-calculus with pairing). The author notices that such a fragment has correct garbage collectors; from that, he proves a separation theorem.

Chapter 2

Additives

In this chapter we study the proof nets for the multiplicative additive fragment of linear logic (briefly **MALL**).

Firstly we give in section 2.1 an overview of the proof nets based on the additive boxes. In particular we remark that such proof nets have not a confluent cut reduction.

Later in sections 2.2 and 2.3, we analyze the proof nets based on additive slices.

In section 2.2 we introduce **MALL** proof structures as couples of a set of slices and of an equivalence relation defining the superposition of slices. Our approach is in between the sliced proof structures defined by Tortora and Laurent in [LTdF04] and the ones introduced by Hughes and van Glabbeeck in [HvG03], although we will follow [HvG03] in the two most crucial passages: the cut reduction and the correctness criterion.

In subsection 2.2.1 we recall the relational semantics for the additives. Our main results are theorem 45, extending the injectivity of relational semantics to **MALL**, and theorem 48, yielding a semantic characterization of those sets which are interpretations of **MALL** proof structures.

In subsection 2.2.2 we define an observational equivalence between **MALL** proof structures (definition 50), which is the natural extension of the **MLL** equivalence $\sim_{\mathbb{B}}$ defined in subsection 1.1.2. Contrary to the multiplicative case, we prove in proposition 52 that the separation theorem does not hold in the additive framework (at least with the present syntax).

In section 2.3 we deal with the additive proof nets and Hughes and van Glabbeeck's correctness criterion. In subsection 2.3.3 we present our ongoing research for a surjective denotational semantics for **MALL** proof nets. The crucial point is to characterize semantically the additive proof nets. In particular we refer to the hypercoherent semantics defined by Ehrhard in [Ehr93]. We prove that any interpretation of a proof net is a hyperclique (theorem 68). Conversely, it remains an open question if any cut-free proof structure, whose interpretation is a hyperclique, is a proof net (see proposition 69 and conjecture 70).

The **formulas** of **MALL** are defined by the following grammar:

$$F ::= \quad X \quad | \quad X^{\perp} \quad | \quad F \parr F \quad | \quad F \& F \quad | \quad F \otimes F \quad | \quad F \oplus F$$

As always we set $(A\&B)^\perp = A^\perp \oplus B^\perp$ and $(A \oplus B)^\perp = A^\perp \& B^\perp$.

The rules of the **sequent calculus** of **MALL** are those of **MLL** extended by the rules for the additives:

$$\frac{\vdash \Sigma, A \qquad \vdash \Sigma, B}{\vdash \Sigma, A\&B}\,\&$$

$$\frac{\vdash \Sigma, A}{\vdash \Sigma, A \oplus B}\,\oplus_1 \qquad \frac{\vdash \Sigma, B}{\vdash \Sigma, A \oplus B}\,\oplus_2$$

Remark that the &-rule requires the same context in both its premises, and that such a context is superimposed in the conclusion.

The context rôle in the &-rule is hard to be represented in a proof net. First of all because the notion of context itself is quite unnatural for proof nets, requiring to link non-active formulas. But worse is the context superposition, which brings in linear logic old problems dating back to the disjunction elimination rule in natural deduction. In particular the context superposition has a kind of ubiquity, in the sense that it can be placed before or after Σ without really changing the rule.

A naive way to represent the &-rule in the proof nets is by using the *additive box*, which is a literal translation of the sequent &-rule. In particular the additive box has auxiliary conclusions, having the same state of ubiquity as the &-rule context. Such auxiliary conclusions can be placed before or after most other links, thus yielding a wide range of commutation equivalences, which are opposite to the spirit itself of proof nets.

On the contrary, the *slices* are a subtler approach to the &-rule. They avoid to associate explicitly with a & link its context, deferring such a problem as a step of sequentialization. In this way the proof nets became the canonical representatives for the commutation equivalences induced by the additives. As far as we know the sliced proof nets are the unique **MALL** syntax overcoming such commutation equivalences. A proof of this canonicity is the relational semantics injectivity, which holds in the sliced proof nets (theorem 45) but not in the proof nets based on the additive boxes (see the counter-example in figures 2.5 and 2.6).

But using the slices has a price. Since they do not explicit the &-rule context, the problem of sequentializing a proof net as well as that of defining a correctness criterion become very hard. Indeed they have been open problems for fifteen years, since the inception of linear logic in 1987. Recently in [LTdF04], Laurent and Tortora de Falco have solved such problems for the cut-free proof structures of the polarized fragment of linear logic (with exponentials); while in [HvG03], Hughes and van Glabbeek have given a definitive solution for **MALL**.

2.1 Additive boxes

In this subsection we give an overview of the proof nets based on additive boxes, introduced in [Gir87] (see also [TdF00] for an extensively study of the subject). Our aim is to present their main weakness - a cut reduction which is not confluent.

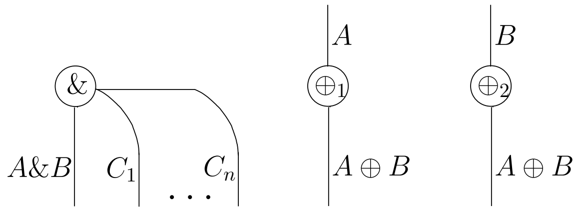


Figure 2.1: **MALL** links for the proof nets based on additive boxes.

The section is divided in three paragraphs. The first one, called *proof structures*, introduces the notions of additive box and of proof structure. The following paragraph, called *proof nets*, defines the correspondence between sequent proofs and correct proof structures. In this paragraph appears clearly that an additive box is nothing more than a step of sequentialization placed in the framework of proof nets. Finally in the third paragraph, called *cut reduction*, we clash with the non-confluent cut reduction.

Proof structures. **MALL** proof structures are defined as a straightforward extension of the **MLL** ones. We add to the **MLL** links (figure 1.1) the following **additive links** (figure 2.1):

1. the **with** link (&), which has no premise and $n+1$ ordered conclusions ($n \geq 0$). Its first conclusion is the **principal conclusion** of the link and it is labeled by a formula $A \& B$. The others (if exist) are the **auxiliary conclusions** of the link, labeled by formulas $C_1, \ldots, C_n$;

2. the **plus**$_1$ and **plus**$_2$ links ($\oplus_1$, $\oplus_2$), which have one premise and one conclusion. If the conclusion of $\oplus_1$ (resp. $\oplus_2$) is labeled by the formula $A \oplus B$, then its premise is labeled by the formula A (resp. B).

To sum up, the **MALL** links are divided in three groups: the structural links (axiom and cut), the multiplicative links ($\parr$ and $\otimes$) and the additive ones (&, $\oplus_1$, $\oplus_2$).

A set of links σ is a **surface** if each edge of σ is premise of at most one link and conclusion of exactly one link of σ. The edges which are not any link premise are called **conclusions** of the surface.

A **proof structure of additive depth** 0 (or simply depth 0) is a surface without &. A **proof structure of additive depth at most** $n+1$ is a surface such that each & w with conclusions $A\&B, C_1, \ldots, C_n$ is associated with two proof structures π_A, π_B of additive depth at most n and conclusions respectively $A, C_1, \ldots, C_n$ and $B, C_1, \ldots, C_n$. $\{\pi_A, \pi_B\}$ is called the **additive box** (or simply box) of w, π_A (resp. π_B) being its **left** (resp. **right**) **component**.

Proof nets. The proofs of the sequent calculus of **MALL** can be easily translated into proof structures: the **desequentialization** for **MALL** proofs is the (straightforward) extension of the desequentialization $(\ )^\bullet$ for **MLL** (see section 1.2) to the additive rules.

If σ is a proof in the sequent calculus, then $(\sigma)^\bullet$ is defined by induction on σ. In case σ ends with a **MLL** rule then $(\sigma)^\bullet$ is defined as in section 1.2. In case σ ends in an additive rule then $(\sigma)^\bullet$ is defined as follows:

- if σ ends in a &-rule with premises the subproofs σ' and σ'' with conclusions respectively $\vdash \Sigma, A$ and $\vdash \Sigma, B$, $(\sigma)^\bullet$ is the link & with conclusions $A\&B, \Sigma$ and additive box the set $\{(\sigma_1)^\bullet, (\sigma_2)^\bullet\}$;
- If σ ends in a $\oplus_i$-rule (for $i = 1, 2$), having as premise the subproof σ', then $(\sigma^\bullet)$ is obtained by adding to $(\sigma')^\bullet$ the link $\oplus_i$ corresponding the $\oplus_i$-rule.

Remark that the translation of the sequent &-rule is a *false* desequentialization, in the sense that the auxiliary doors of the link & impose a strict distinction between the links before and those after the &.

The **proof nets** are those proof structures which are in the range of $(\ )^\bullet$. The correctness criteria, i.e criteria characterizing the proof nets independently from $(\ )^\bullet$, are straightforward extensions of the ones in **MLL**. For example we recall here the extension of the Danos and Regnier's criterion.

A **correctness graph of a surface** σ is a subgraph of σ obtained by erasing one premise for each $\invamp$. A **correctness graph of a proof structure** π is a correctness graph of one of the surfaces of π.

Definition 30 *A proof structure is* ***correct*** *(resp.* ***strongly correct****) if all its correctness graphs are acyclic (resp. acyclic and connected).*

Theorem 31 *An additive proof structure π is correct (resp. strongly correct) iff π is a proof net (resp. a proof net without mix).*

PROOF [SKETCH]. The difficult part is the *only if* part. The proof defines a (non-deterministic) procedure of sequentialization of a correct proof structure π into a sequent calculus proof. Such a sequentialization is defined by induction on the additive depth of π and on the number of links in π at depth 0.

In case π has links at additive depth 0 different from & or it is not connected, we use the **MLL** sequentialization procedure, straightforward extended to the plus.

Otherwise π has only one link at depth 0 which is a &. By induction on the additive depth we have the sequentialization of the right and left components of the box associated with the &. By composing these two sequent proofs with a &-rule we get the sequentialization of π. □

In the proof of theorem 31 appears clearly the sequential nature of the link &, which translates literally the &-rule of sequent calculus.

Cut reduction. The right judge for a syntax is the cut reduction: such a judge shows the weakness of the additive boxes.

The reduction of a cut between the principal conclusion $A_1\&A_2$ of a & and the conclusion $A_1^\perp \oplus A_2^\perp$ of a $\oplus_i$ ($i = 1, 2$) is easily definable as in figure 2.2: morally the $\oplus_i$ chooses one component of the & box. How to reduce instead a cut of which one premise is the auxiliary conclusion of a &?

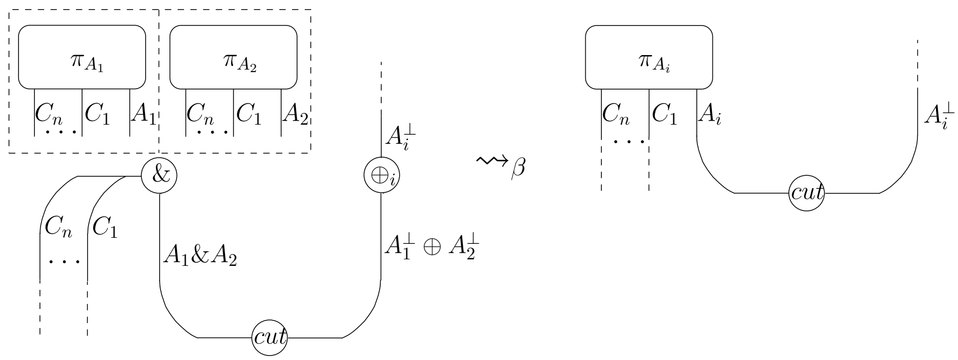


Figure 2.2: & / $\oplus_i$ cut reduction for the proof nets based on additive boxes ($i = 1, 2$).

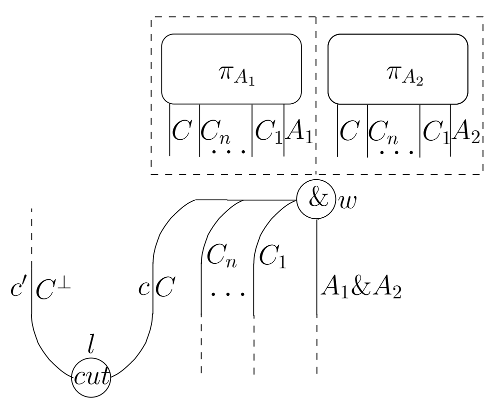


Figure 2.3: commutative additive cut.

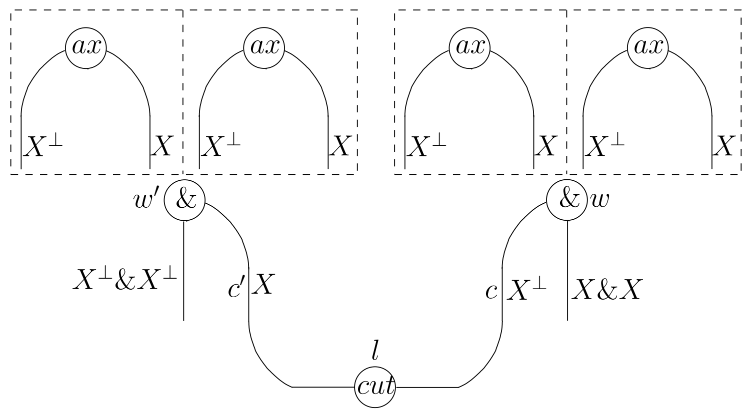


Figure 2.4: example of a commutative additive cut.

Let π be a proof net, l be a cut in π of premises $c : C$ and $c' : C^\perp$, such that c is an auxiliary conclusion of a & link w (see figure 2.3): such a cut l is called **commutative additive**.

For reducing l we have to put c' inside both the components of the box associated with w. Putting c' inside the box associated with w means putting inside a sub-net with c' as conclusion. Which sub-net? In [Gir87] it is suggested to put the maximal sub-net containing c', defined by the notion of *empire of* c'. In [TdF03a] it is shown that almost[1] any choice of a sub-net with c' as conclusion defines a reduction respecting the same denotational semantics as the one defined in [Gir87].

Worse, even if we have decided which sub-net putting inside the box associated with w, what happens if also the premise c' of l is an auxiliary conclusion of another & w'? Let us take for example the proof net π in figure 2.4.

If we want to reduce the cut l, do we have to put c' inside the box associated with w, or c inside the box associated with w'? By putting c' inside the w box we get the proof net in figure 2.5, while by putting c in the w' box we get the proof net in figure 2.6.

Such a choice generates from π two cut-free proof nets, so showing that the cut reduction is not confluent.

2.2 Proof structures

In this section we introduce the additive proof structures starting from the notion of slice. We proceed in this way: in the first paragraph, called *slices*, we define the slices - morally multiplicative proof structures with possibly unary additive links. In the second paragraph, called *proof structures*, we introduce additive proof structures as couples of a set of slices and an equivalence defining the slices superposition. Finally in the third paragraph, called *cut reduction*, we describe the reduction of a cut in a proof structure as a parallel reduction of superposed slices cuts.

[1]Some restrictions are needed in case of proof nets with mix.

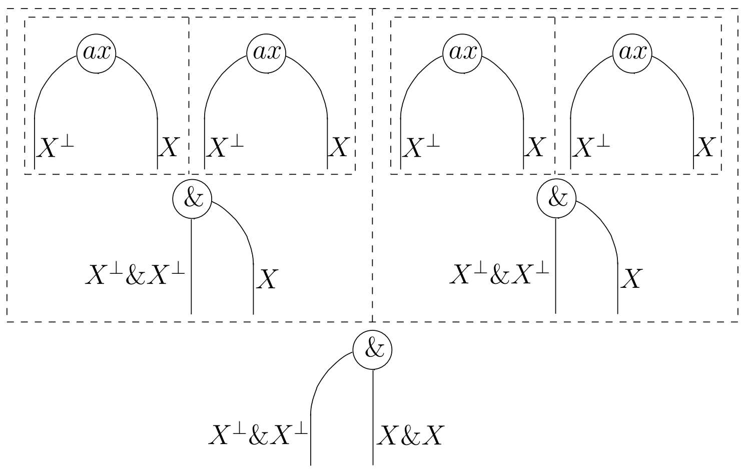


Figure 2.5: example 1 of the cut reduction of the proof net in figure 2.4.

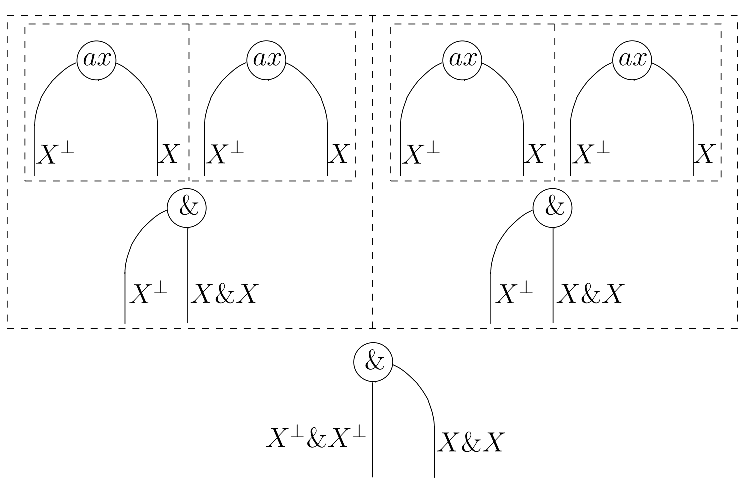


Figure 2.6: example 2 of the cut reduction of the proof net in figure 2.4.

The syntax we present here is on the one hand in the spirit of [HvG03], particularly in the definition of a cut reduction as parallel reduction of different slices cuts. On the other hand we define more freely the proof structures, without taking in account Hughes and van Glabbeek's resolution condition, in this following the spirit of [LTdF04].

Slices. The starting point we propose for understanding the additives is the ingenious idea of *slice* defined by Girard already in [Gir87]: an additive proof is a superposition of slices of multiplicative proofs.

Let us look for example at the following sequent proof π of $(X \otimes X) \oplus (X \otimes X), (X^{\perp} \& X^{\perp}) \parr X^{\perp}$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash X \otimes X, X^{\perp}, X^{\perp}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp}, X^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash X \otimes X, X^{\perp}, X^{\perp}$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp}, X^{\perp}$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp} \& X^{\perp}, X^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), (X^{\perp} \& X^{\perp}) \parr X^{\perp}$}
\end{prooftree}
$$

for recovering a slice of π we erase a branch of each &-rule, in this case just one. For example, by erasing the right branch we get the following slice α_1:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash X \otimes X, X^{\perp}, X^{\perp}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp}, X^{\perp}$}
\RightLabel{$\&$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp} \& X^{\perp}, X^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), (X^{\perp} \& X^{\perp}) \parr X^{\perp}$}
\end{prooftree}
$$

and by erasing the left branch, we get the following slice α_2:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash X, X^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash X \otimes X, X^{\perp}, X^{\perp}$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp}, X^{\perp}$}
\RightLabel{$\&$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), X^{\perp} \& X^{\perp}, X^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (X \otimes X) \oplus (X \otimes X), (X^{\perp} \& X^{\perp}) \parr X^{\perp}$}
\end{prooftree}
$$

Both α_1 and α_2 are multiplicative proofs with some unary additive rules. They can be represented as multiplicative proof structures with some unary links $\oplus$ and & as in figure 2.7.

It is simple to represent the slices by proof structures. What we only need is to extend the set of **MLL** links with the following **additive links** (see figure 2.8):

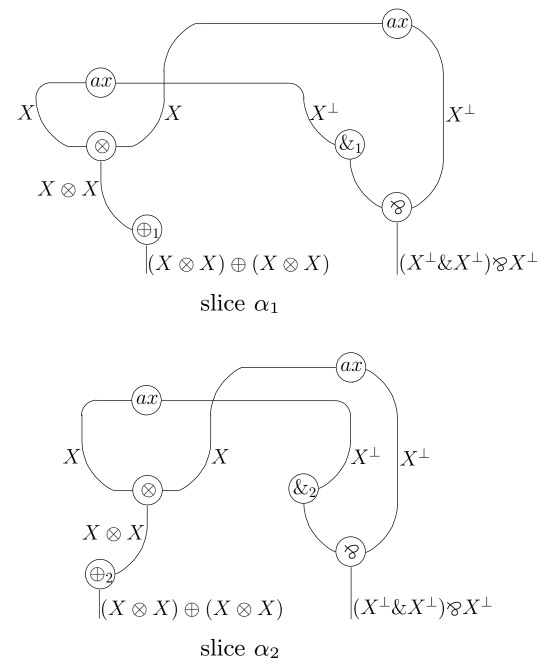


Figure 2.7: example of additive slices α_1 and α_2.

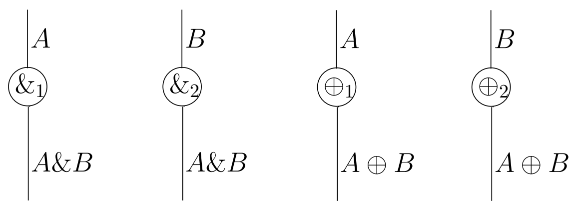


Figure 2.8: **MALL** links for slices.

1. the **with**$_1$ and **with**$_2$ links ($\&_1$, $\&_2$), which have one premise and one conclusion. If the premise of $\&_1$ (resp. $\&_2$) is labeled by a formula A (resp. B) then the conclusion of $\&_1$ (resp. $\&_2$) is labeled by a formula $A\&B$;

2. the **plus**$_1$ and **plus**$_2$ links ($\oplus_1$, $\oplus_2$), which have one premise and one conclusion. If the premise of $\oplus_1$ (resp. $\oplus_2$) is labeled by a formula A (resp. B), then the conclusion of $\oplus_1$ (resp. $\oplus_2$) is labeled by a formula $A \oplus B$.

To sum up, the **MALL** links are of three types: the structural links (axiom and cut), the multiplicative links ($\parr$ and $\otimes$) and the additive links ($\&_{1,2}$ and $\oplus_{1,2}$). A **slice** is a graph (even empty) whose nodes are the **MALL** links and such that each edge is premise of at most one link and conclusion of exactly one link. The edges which are not any link premise are the **conclusions of the slice**.

We denote the slices by the initial Greek letter $\alpha, \beta, \dots$ and the sets of slices by capital Latin letters $S, Q, \dots$.

Let l, m be two links of a slice α. We say that **l is a predecessor of m**, denoted by $\mathbf{l} \rightarrow \mathbf{m}$, if a conclusion of l is premise of m. Let m, m' be two links and l (resp. l') be a predecessor of m (resp. of m'), we say that **l and l′ are similar predecessors** if one of the following conditions holds:

- m, m' are multiplicative links and the conclusions of l, l' are both right or both left premises;
- m, m' are the same kind of additive link, i.e. both $\&_i$ or both $\oplus_i$ for $i = 1, 2$;
- m, m' are cuts and the conclusions of l, l' have the same type.

As always, a slice is **cut-free** if it has no cut. Remark that, contrary to **MLL**, the conclusions of a cut-free slice does not define the slice up to the axioms, since we do not know the premise of an additive link from its conclusion.

Proof structures. An additive proof is a superposition of slices, so once we have defined what is a slice, we have to understand what is a superposition of slices.

Let us come back to our example: the sequent proof π of $(X \otimes X) \oplus (X \otimes X)$, $(X^\perp \& X^\perp) ⅋ X^\perp$. The question is defining the links of α_1 and α_2 which are superposed in π, or, otherwise stated, the links of π which are shared by α_1 and α_2.

Clearly the terminal links of π are shared by both α_1 and α_2, so are the ⅋ link with conclusion $(X^\perp \& X^\perp) ⅋ X^\perp$ and the $\oplus$ link with conclusion $(X \otimes X) \oplus (X \otimes X)$:[2] the slices un-thread once we go up the $\oplus$ premise.

By looking at the example we remark that: if l is a multiplicative link shared by α_1 and α_2, for example the ⅋ link, then the l predecessors are still shared by both the slices; if l is an additive link shared by α_1 and α_2, then its predecessors can be no more shared by both the slices; conversely if a link is shared by α_1 and α_2, then so are all the links below it.

These simple remarks help us to fix the idea of shared link and to introduce an equivalence relation between the links of the slices:

Definition 32 *Let S be a set (even empty) of slices with same conclusions $C_1, \dots, C_n$. A **sharing equivalence** on S is an equivalence relation $\equiv$ on the links of the slices in S such that for any links l, l', m:*

identity: *if l, l' belong to the same slice then $l \equiv l'$ iff $l = l'$;*

bottom: *if l, l' are terminal, then $l \equiv l'$ iff l, l' have the same conclusion among $C_1, \dots, C_n$;*

cut: *if $l \equiv l'$ and l is a cut with premises of type $A, A^\perp$, then l' is a cut with premises of same types;*

[2] We allow to share an additive link by two slices α_1, α_2 even if it occurs in α_1 as $\oplus_1$ (resp. $\&_1$) and in α_2 as $\oplus_2$ (resp. $\&_2$), like in [HvG03].

bottom-up: *if $m \to l$, and $l \equiv l'$ then for any m', $m' \to l'$, $m \equiv m'$ iff m and m' are similar predecessors;*

up-bottom: *if $l \to m$ and $l \equiv l'$, then there is m', $l' \to m'$ and $m \equiv m'$.*

If $l \equiv l'$, we say that $\mathbf{l}, \mathbf{l'}$ ***are superimposed in* S *by*** $\equiv$. *Conversely, let $[l]$ be the equivalence class of a link l and $\alpha_1, \dots, \alpha_n$ be the slices of S which have links in $[l]$, we say that* $[\mathbf{l}]$ ***is a link of* S *shared by*** $\alpha_\mathbf{1}, \dots, \alpha_\mathbf{n}$.

Here are two propositions which will be used later and which we hope will help the reader to take confidence with the sharing equivalences:

Proposition 33 *Let S be a set of slices with same conclusions, $\equiv$ be a sharing equivalence on S, and l, l' be two links of slices in S. If $l \equiv l'$ then both l, l' are cuts with same type premises, or l, l' have the conclusions of same type.*

Proof. We prove the proposition by induction on the number of links below l. If l is terminal or a cut then the proposition is a consequence of condition **bottom** or **cut**. If l is predecessor of m then by condition **up-bottom** there is an m' of which l' is predecessor and such that $m \equiv m'$; by induction m and m' have conclusions of same type. Hence, by condition **bottom-up** and by definition of similar predecessor the conclusions of l, l', which are premises respectively of m, m', have same type. Moreover, if l, l' are axioms it is clear that the other conclusions than those premises of m, m' are of same type too.
□

Proposition 34 *A sharing equivalence $\equiv$ is completely determined once we define $\equiv$ on the cuts. In particular, the sharing equivalence on a set of cut-free slices is unique.*

Proof. Let us suppose $\equiv_1$ and $\equiv_2$ are two sharing equivalences on a set of slices with same conclusions, such that:

(*) for any two cuts n, n', $n \equiv_1 n'$ iff $n \equiv_2 n'$.

For any two links l, l' we prove by induction on the number of links below l that $l \equiv_1 l'$ iff $l \equiv_2 l'$.

If l is terminal or a cut then the statement is a consequence of condition **bottom** or of (*).

If l is predecessor of m. If $l \equiv_1 l'$ then by condition **up-bottom** there is an m' of which l' is predecessor such that $m \equiv_1 m'$, moreover, by condition **bottom-up** we have that l, l' are similar predecessors. By induction $m \equiv_2 m'$, so by condition **bottom-up** and the fact that l, l' are similar predecessors, we have $l \equiv_2 l'$. Similarly, we get that $l \equiv_2 l'$ implies $l \equiv_1 l'$. □

We remark that in case of a set S of cut-free slices, the unique sharing equivalence on S is exactly the one defined by Tortora de Falco and Laurent in [LTdF04]. We have extended that equivalence to the case with cuts for comparing Hughes and van Glabbeek's syntax with the syntax used in [LTdF04].

A sharing equivalence $\equiv$ on S can be easily extended to an equivalence on the edges of the slices of S. Let a, a' be two edges of the slices of S, we set $\mathbf{a} \equiv \mathbf{a'}$ if one of the following cases holds:

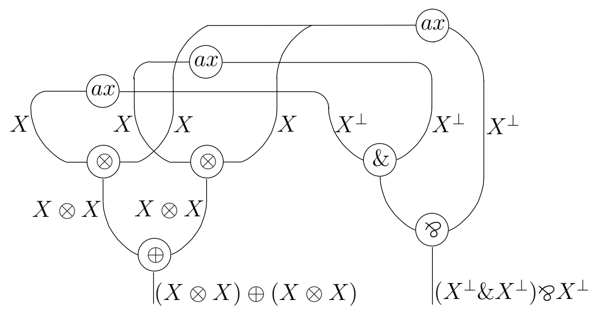


Figure 2.9: sharing quotient of the slices in figure 2.7

- a and a' are the same conclusion of S;
- a (resp. a') is an edge conclusion of l (resp. l') and premise of m (resp. m'), and $l \equiv l'$, $m \equiv m'$.

Remark that by proposition 33 if $a \equiv a'$ then a and a' are labeled by the same formula. Actually the following fact holds:

Fact 35 *Let S be a set of slices with same conclusions, let $\equiv$ denote a sharing equivalence on S extended to the edges of the slices in S. If a is an edge of a slice of S, s.t. a is conclusion (resp. premise) of a link m, then all the edges in $[a]$ are conclusion (resp. premise) of links in $[m]$.*

Fact 35 allows to define from $\equiv$ and S the graph ${}^{S}/_{\equiv}$, whose nodes are the equivalence classes of the links of the slices in S, and whose edges are the equivalence classes of the edges of the slices in S (see definition 36).

For example, let us come back to the proof π of $(X \otimes X) \oplus (X \otimes X)$, $(X^\perp \& X^\perp) \parr X^\perp$. Its slices α_1 and α_2 (figure 2.7) are cut-free, hence the set $\{\alpha_1, \alpha_2\}$ has a unique sharing equivalence $\equiv$. The graph ${}^{\{\alpha_1,\alpha_2\}}/_{\equiv}$ induced by such an equivalence is in figure 2.9.

Remark that in the graph of figure 2.9 there are binary additive links and axioms with more than two conclusions. To be pedantic we have to extend the set of links with the following shared links (figure 2.10):

1. the **shared axiom** link, which has no premise, $n > 0$ conclusions of type X and $m > 0$ conclusions of type $X^\perp$;

2. the **shared with** link (&), which has two ordered premises and one conclusion. If a shared with has the left premise labeled by the formula A and the right premise labeled by the formula B, then the conclusion is labeled by the formula $A\&B$;

3. the **shared plus** link ($\oplus$), which has one or two ordered premises and one conclusion. If a binary shared plus has the left premise labeled by the formula A and the right premise labeled by the formula B, then the conclusion is labeled by the formula $A \oplus B$.

The shared links do not occur in the slices but only in their superposition. Anyway the most of times we will write just link, omitting if we refer to a shared or slice link, being it clear from the context.

We define in general our graph representation of a superposition of slices:

Definition 36 *Let S be a set of slices with same conclusions, and let $\equiv$ denote a sharing equivalence on S extended to the edges of slices in S. The* ***$\equiv$-sharing quotient of* S**, *denoted by $^S/_\equiv$, is the graph whose links (resp. edges) are the equivalence classes w.r.t. $\equiv$ of the links (resp. edges) of the slices in S.*

In particular if m is a link of a slice in S, then:

1. *in case m is an axiom with conclusions a, b, then $[m]$ is a shared axiom of $^S/_\equiv$ with among its conclusions $[a], [b]$;*
2. *in case m is a cut with premises a, b, then $[m]$ is a cut of $^S/_\equiv$ with premises $[a]$ and $[b]$;*
3. *in case m is a $⅋$ (resp. $\otimes$) with premises a, b and conclusion c, then $[m]$ is a $⅋$ (resp. $\otimes$) of $^S/_\equiv$ with premises $[a], [b]$ and conclusion $[c]$;*
4. *in case m is a $\&_1$ (resp. $\oplus_1$) with premise a and conclusion c, then $[m]$ is a sharing $\&$ (resp. sharing $\oplus$) in $^S/_\equiv$ with left premise $[a]$ and conclusion $[c]$;*
5. *if m is a $\&_2$ (resp. plus$_2$) in S with premise a and conclusion c, then $[m]$ is a sharing $\&$ (resp. sharing $\oplus$) in $^S/_\equiv$ with right premise $[a]$ and conclusion $[c]$.*

For another example of sharing quotient recall the multiplicative proof nets $\mho$ and Ω of figure 1.5. Clearly $\mho$ and Ω can be considered cut-free slices with conclusion $\mathbb{B}$, hence by proposition 34 the sharing equivalence $\equiv$ on $\{\mho, \Omega\}$ is unique. In figure 2.11 there is the sharing quotient $^{\{\mho,\Omega\}}/_\equiv$.

Definition 37 *A* **MALL** ***proof structure*** *π with conclusions $C_1, \ldots, C_n$ is a couple $(|\pi|, \equiv_\pi)$, where $|\pi|$ is a set (even empty) of slices with conclusions $C_1, \ldots, C_n$ and $\equiv_\pi$ is a sharing equivalence on $|\pi|$.*

We call ***links*** *(resp.* ***edges****)* ***of*** *π the links (resp. edges) of $^{|\pi|}/_{\equiv_\pi}$.*

We stress the fact that the above definition authorize to speak of an empty set of slices as a proof structure with conclusions $C_1, \ldots, C_n$. We need such improper proof structures to define a cut elimination on **MALL** proof structures preserving the type of the conclusions (see the following paragraph *cut reduction*).

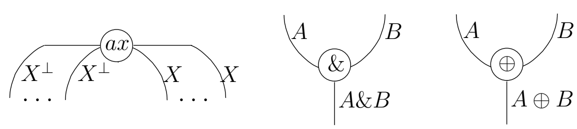


Figure 2.10: shared **MALL** links.

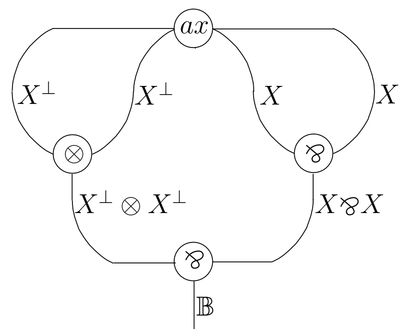


Figure 2.11: sharing quotient of $\{\mho, \Omega\}$.

The proof structures are denoted by final Greek letters $\pi, \sigma, \rho, \ldots$. We define PS^{ma} as the set of the **MALL** proof structures.

A proof structure is **closed** if it has only one conclusion. If π is a proof structure with conclusions $C_1, \ldots, C_n$, we define **the closure of** π with conclusion $C_1 ⅋ \ldots ⅋ C_n$ as the proof structure π^* obtained by adding to each slice of π the necessary $⅋$ links below $C_1, \ldots, C_n$.

Remark that if $|\pi|$ is a set of cut-free slices or it has at most one element then the sharing equivalence on $|\pi|$ is unique: in such cases we may speak of a proof structure π without making explicit $\equiv_\pi$. In particular a **MLL** proof structure can be considered as a single slice **MALL** proof structure.

A proof structure is **cut-free** if so are all its slices. Like in **MLL**, the cut defines a composition between proof structures. Let $\pi = (\{\alpha_1, \ldots, \alpha_n\}, \equiv_\pi)$ and $\sigma = (\{\beta_1, \ldots, \beta_m\}, \equiv_\sigma)$ be two proof structures with conclusions respectively A, Π and $A^\perp, \Sigma$, the **composition of** π **and** σ **on** $\mathbf{A}, \mathbf{A}^\perp$, denoted by $[\pi, \sigma]_{A,A^\perp}$, is the proof structure defined as follows:

- $|[\pi, \sigma]_{A,A^\perp}|$ is obtained by connecting every slice of π and every slice of σ by means of a cut with premises the conclusions A and $A^\perp$ of respectively π and σ;
- $\equiv_{[\pi,\sigma]_{A,A^\perp}}$ is the smallest sharing equivalence on $|[\pi, \sigma]_{A,A^\perp}|$ containing $\equiv_\pi \cup \equiv_\sigma$ and equaling all the cuts with premises the conclusions $A, A^\perp$ of respectively π and σ.

We omit the index ${}_{A,A^\perp}$ in $[\pi, \sigma]_{A,A^\perp}$ when it is clear which are the conclusions on which compose. Remark that if π is the empty proof structure, then $[\pi, \sigma]$ is empty for any proof structure σ.

Cut reduction. A cut l of a **MALL** proof structure π is now an equivalence class of cuts in the slices of π. We define the reduction of l as the simultaneous reduction of all the cuts superposed in l.

We proceed in this way: firstly we define the reduction of a cut in a single slice as an easy extension of **MLL** cut reduction; secondly, we define the reduction of a cut in a proof structure as a simultaneous reduction of the corresponding cuts in the slices of the proof structure.

Let l be a cut in a slice α. l can be of three types:

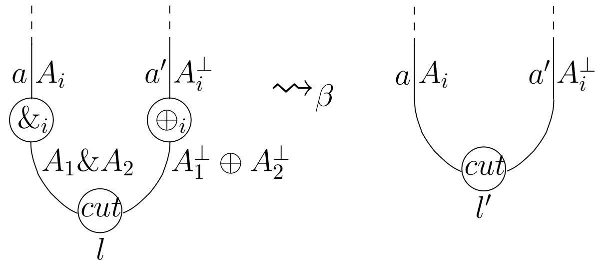


Figure 2.12: $\&_i/\oplus_i$ slice cut reduction, for $i = 1, 2$.

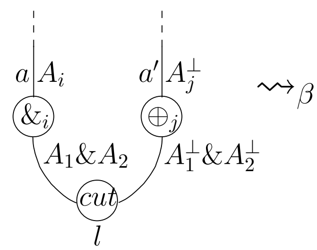


Figure 2.13: $\&_i/\oplus_j$ slice cut reduction, for $i \neq j$.

- an axiom cut, whose premises are labeled by dual atomic formulas X and $X^\perp$;
- a $⅋/\otimes$ cut, whose premises are labeled by dual multiplicative formulas $A⅋B$ and $A^\perp \otimes B^\perp$;
- a $\&/\oplus$ cut, whose premises are labeled by dual additive formulas $A\&B$ and $A^\perp \oplus B^\perp$.

In the first two cases, we reduce l as in **MLL** (see section 1.1). In case l is an additive cut, let $A_1\&A_2$ and $A_1^\perp \oplus A_2^\perp$ be the types of the l premises, $\&_i$ and $\oplus_j$ be the l predecessors. Let us call a (resp. a') the premise of $\&_i$ (resp. of $\oplus_j$). If $i = j$ we reduce l erasing l itself, its premises and predecessors and by adding a cut l' with premises a and a' (figure 2.12). If $i \neq j$ we reduce l erasing completely the slice (figure 2.13).

We write $\alpha \rightsquigarrow_\beta \alpha'$ if α' is the result of the reduction of a cut in the slice α.

Now, let $\pi = (|\pi|, \equiv_\pi)$ be a proof structure with conclusions $C_1, \ldots, C_n$. As written above, a cut l of π is an equivalence class of cuts in the slices of π. We define the reduction of l as the simultaneous reduction of all the cuts superposed in l. That is, if $|\pi| = \{\alpha_1, \ldots, \alpha_n\}$, for each α_i we define α_i' as the reduction of the α_i cut superposed in l, if it exists, or, in case α_i does not share l, we set $\alpha_i' = \alpha_i$. So, the result of the reduction of l is the proof structure π' with conclusions $C_1, \ldots, C_n$ defines as follows:

- $|\pi'| = \{\alpha_1', \ldots, \alpha_n'\}$;

- $\equiv_{\pi'}$ is the smallest sharing equivalence which contains the restriction of $\equiv_\pi$ to the links persisting in π' (morally we extend the restriction of $\equiv_\pi$ in order to meet the cut condition of definition 32 for the new cuts in π').

Remark that a non-empty proof structure can reduce to an empty one, since the cut reduction may erase slices, as in figure 2.13. Thus, for preserving the conclusions of a proof structure under cut reduction we allow the empty proof structure with conclusions $C_1, \ldots, C_n$, for any formulas $C_1, \ldots, C_n$.

We write $\pi \rightsquigarrow_\beta \pi'$ if π' is the result of a cut reduction of π. As always, $\rightarrow_\beta$ is the reflexive and transitive closure of $\rightsquigarrow_\beta$ and $=_\beta$ is the symmetrical closure of $\rightarrow_\beta$.

Contrary to the proof structures based on additive boxes, sliced proof structures enjoy confluence:

Theorem 38 (Confluence) *For every $\pi, \pi', \pi'' \in PS^{ma}$ s.t. $\pi \rightarrow_\beta \pi'$ and $\pi \rightarrow_\beta \pi''$, there is $\pi''' \in PS^{ma}$, s.t. $\pi' \rightarrow_\beta \pi'''$ and $\pi'' \rightarrow_\beta \pi'''$.*

PROOF [SKETCH]. Simply notice that the cut reduction on a single slice is confluent, being a straightforward extension of the **MLL** cut reduction.

The cut reduction on a proof structure is confluent, since it is a parallel reduction of slices cuts, and the reduction of a cut in a slice does not interfer with the one of a cut in another slice. □

Of course $\rightarrow_\beta$ enjoys strong normalization:

Theorem 39 (Strong normalization) *For every $\pi \in PS^{ma}$, there is no infinite sequence of proof structures π_0, π_1, π_2, $\ldots$ s.t. $\pi_0 = \pi$ and $\pi_i \rightsquigarrow_\beta \pi_{i+1}$.*

PROOF [SKETCH]. As in **MLL** remark that any cut reduction either reduces the number of cuts or the complexity of the formulas labelling the premises of the cuts. Hence by an easy induction we get the assertion. □

2.2.1 Relational semantics

In this subsection we extend **MLL** relational semantics to the additives. The main results of this subsection is theorem 45, stating the injectivity of the relational semantics for **MALL** proof structures, and theorem 48, extending to the additives the characterization of those subsets which are interpretations of proof structures.

Let $\mathtt{X}$ be a set, the **relational model on** $\mathtt{X}$, denoted by $\mathfrak{Rel}^{\mathtt{X}}$, associates with **MALL** formulas sets, in the following way:

- $\mathtt{X}$ is associated with the atomic formulas $X, X^\perp$;
- if $\mathtt{A}$ and $\mathtt{B}$ are associated resp. with A and B, then $\mathtt{A} \times \mathtt{B}$ is associated with $A \parr B$ and $A \otimes B$;
- if $\mathtt{A}$ and $\mathtt{B}$ are associated resp. with A and B, then the disjoint union $\mathtt{A} + \mathtt{B}$ of $\mathtt{A}$ and $\mathtt{B}$ is associated with $A \& B$ and $A \oplus B$.

We recall that the disjoint union of two sets $\mathtt{A}$ and $\mathtt{B}$ is defined as: $\mathtt{A}+\mathtt{B} = (\{1\}\times\mathtt{A})\cup(\{2\}\times\mathtt{B})$. If $\mathtt{C}\subseteq\mathtt{A}+\mathtt{B}$ we denote by $s_1(\mathtt{C})$ (resp. by $s_2(\mathtt{C})$) the subset of $\mathtt{A}$ (resp. of $\mathtt{B}$) defined as $\{a \mid\ <1, a>\in\mathtt{C}\}$ (resp. $\{b \mid\ <2, b>\in\mathtt{C}\}$).

We extend the definition of experiment of subsection 1.1.1 to the additives in the following way:

Definition 40 *A $\mathfrak{Rel}^{\mathtt{X}}$* ***experiment*** **e** ***on a slice*** *α, denoted by $e:\alpha$, is a function associating with every edge $a:A$ of α an element of $\mathtt{A}$, such that the following conditions are respected:*

axiom: *if a, b are the conclusions of an axiom, then $e(a)=e(b)$;*

cut: *if a, b are the premises of a cut, then $e(a)=e(b)$;*

multiplicative: *if c is the conclusion of a $\parr$ or $\otimes$ with premises a and b, then $e(c)=<e(a), e(b)>$;*

additive: *if c is the conclusion of a $\&_i$ or $\oplus_i$ $(i=1,2)$ with premise a, then $e(c)=\langle i, e(a)\rangle$.*

If α has conclusions $c_1:C_1,\ldots,c_n:C_n$, the ***result of an experiment*** *$e:\alpha$ is $<e(c_1),\ldots,e(c_n)>$. An* ***experiment on a proof structure*** *is an experiment of one among its slices.*

The **interpretation of a slice** α in $\mathfrak{Rel}^{\mathtt{X}}$, denoted by $[\![\alpha]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$, is undefined in case α is empty, otherwise it is the set of its experiments results, i.e. if $c_1:C_1,\ldots,c_n:C_n$ are the conclusions of α:

$$[\![\alpha]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = \{<e(c_1),\ldots,e(c_n)> \ \mid e \text{ is a } \mathfrak{Rel}^{\mathtt{X}} \text{ experiment on } \alpha\}$$

Finally, the **interpretation of a proof structure** π is the union of the interpretations of its slices:

$$[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = \bigcup_{\alpha\in|\pi|}[\![\alpha]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$$

where in case π is empty, $\bigcup_{\alpha\in|\pi|}[\![\alpha]\!]_{\mathfrak{Rel}^{\mathtt{X}}}=\emptyset$. We omit the index $_{\mathfrak{Rel}^{\mathtt{X}}}$ if it is clear which model we refer to.

It is well known that relational semantics is sound for additive proof structures:

Theorem 41 (Soundness.) *Let $\pi,\pi'\in PS^{ma}$, if $\pi\to_\beta\pi'$ then $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = [\![\pi']\!]_{\mathfrak{Rel}^{\mathtt{X}}}$.*

Conversely, we extend theorem 6 to the additives, proving (easily) the injectivity of the relational semantics for **MALL** proof structures (theorem 45).

Injectivity. We recall the definition of injective experiment in the slice framework:

Definition 42 *Let α be a cut-free slice and $e:\alpha$ be an experiment. e is* ***injective*** *when for any two edges a, a' labeled by X, $e(a)\neq e(a')$.*

Like in **MLL** we remark that:

Fact 43 *An injective experiment is actually injective on any two edges of same type, i.e. for any two edges $a, a' : A$, $e(a) \neq e(a')$.*

Fact 44 *If* $\mathtt{X}$ *is infinite then any cut-free slice* α *has injective experiments.*

Theorem 45 (Injectivity.) *Let* $\mathtt{X}$ *be an infinite set and* $\pi, \pi' \in PS^{ma}$*, if* $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = [\![\pi']\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ *then* $\pi =_\beta \pi'$*.*

Proof. Let $\mathtt{X}$ be an infinite set and π, π' be two proof structures with same conclusions $c_1 : C_1, \ldots, c_n : C_n$, such that $[\![\pi]\!]_{\mathfrak{Rel}^{\mathtt{X}}} = [\![\pi']\!]_{\mathfrak{Rel}^{\mathtt{X}}}$. We prove that $\pi =_\beta \pi'$.

Since $\rightarrow_\beta$ is confluent and normalizing, we can suppose π and π' are cut-free. Hence we have to prove that $\pi = \pi'$.

Since both π and π' are cut-free, their sharing equivalences are unique, so it will be enough to show $|\pi| = |\pi'|$.

Let $\alpha \in |\pi|$, we will prove that $\alpha \in |\pi|'$. Let e be an injective experiment on α, which it exists by fact 44. Since the result of e is in $[\![\pi]\!] = [\![\pi']\!]$, then there is an experiment e' on a slice $\alpha' \in |\pi'|$, such that e and e' have the same result. Now, let c be a conclusion of α, and c' be the correspondent of α'. Since c and c' have same type and $e(c) = e(c')$, it is simple to note that c and c' are conclusions of links of same type and that the values of e and e' on the correspondent premises of such links are equals. Hence by going from the conclusions $c_1, \ldots, c_n$ to the atomic edges, we can prove that α and α' are the same graph up to the axioms. Now since e' has the same values as e, e' is injective too, therefore the two slices have the same axioms, that is $\alpha = \alpha'$. By symmetry we have that if $\alpha' \in |\pi'|$ then $\alpha \in |\pi|$, so $|\pi| = |\pi'|$. □

Surjectivity. For each formula C we have on the one hand the proof structures with conclusion C, on the other hand the subsets of $\mathtt{C}$. Theorems 41 and 45 prove that $[\![\ ]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ is an injective function from the β-equivalence classes of proof structures with conclusion C to the subsets of $\mathtt{C}$. The rest of the subsection is devoted to extend theorem 14 to the additives, i.e. to characterize those subsets of $\mathtt{C}$ which are interpretations of proof structures with conclusion C (theorem 48).

Let C be a **MALL** formula, an element $u \in \mathcal{C}$ is a sequence of elements of the basic set $\mathtt{X}$ and of the symbols $<, >, 1, 2$. Hence we may define all the notions introduced in subsection 1.1.1. In particular we obtain the following lemmas:

Lemma 46 *Let* $\mathtt{X}$ *be an infinite set,* $\mathtt{C}$ *be a set associated with a formula* C *in* $\mathfrak{Rel}_{\mathtt{X}}$*. Let* $u \in \mathtt{C}$ *be a balanced element which is maximal among the balanced elements of* $\mathtt{C}$*. There is a closed cut-free slice* α^u *with conclusion* C *and an injective experiment* $e^u : \alpha^u$ *such that the result of* e *is* u*.*

Proof. Actually we prove that if u is a balanced maximal element of $\mathtt{C}_1 \parr \ldots \parr \mathtt{C}_n$, then there are a slice α^u with conclusions $c_1 : C_1, \ldots, c_n : C_n$ and an injective experiment $e^u : \alpha^u$ with result u. This is achieved by an easy induction on the formulas $C_1, \ldots, C_n$. If $C_1, \ldots, C_n$ are all atomic formulas, being u balanced we have an equal number of occurrences of X and of $X^\perp$. Moreover, being u maximal among the balanced elements, u links in pairs of types $X, X^\perp$ the conclusions $c_1, \ldots, c_n$. In this way we get both the axioms defining α^u and the values of e^u on the conclusions of such axioms.

The induction step splits in four cases, depending on the type of a non atomic formulas among $C_1, \ldots, C_n$. We prove just one case: say there is C_i of type $A_1 \& A_2$. In this case we remark that the point of u corresponding with C_i has the shape $u_i =< j, p >$ for a $j \in \{1,2\}$ and a $p \in A_j$. From j we are able to define the link $\&_j$ of which c_i is conclusion and to get an element $u' \in \mathtt{C}_1 ⅋ \ldots ⅋ \mathtt{A}_j ⅋ \ldots ⅋ \mathtt{C}_n$ which is balanced and maximal among the balanced elements. By induction we get $\alpha^{u'}$ and $e^{u'}$, hence α^u by adjoining c_i below A_j and e^u by extending $e^{u'}$ to the adjoined edge. □

Lemma 47 *Let* $\mathtt{X}$ *be an infinite set and* α *be a cut-free closed slice with conclusion* C*, then* $[\![\alpha]\!]_{\mathfrak{Rel}^{\mathtt{X}}}$ *is a complete subset of* $\mathtt{C}$*.*

Proof. Similar to the proof of lemma 12. □

We thus have the following theorem:

Theorem 48 *Let* $\mathtt{X}$ *be an infinite set. Let* C *be an* **MALL** *formula and* $\mathtt{C}$ *its interpretation in* $\mathfrak{Rel}^{\mathtt{X}}$*. A subset of* $\mathtt{C}$ *is the interpretation of a closed proof structure with conclusion* C *if and only if it is a (finite) union of complete sets.*

Proof. Let U be a (finite) union of complete subsets of $\mathtt{C}$, for example $U = U_1 \cup \ldots \cup U_n$ (actually, $U_1, \ldots, U_n$ are uniquely determined by U). For each U_i, let $u_i \in U_i$ be a maximal among the balanced elements of $\mathcal{C}$. By lemma 46 there is a cut-free slice α^{u_i} such that, by lemma 47, $[\![\alpha^{u_i}]\!] = U_i$. Let $|\pi| = \{\alpha^{u_1}, \ldots, \alpha^{u_n}\}$, since all slices of $|\pi|$ are cut-free, it is unequivocally determined a sharing equivalence on $|\pi|$. Clearly, $[\![\pi]\!] = U$.

Conversely, let π be a cut-free proof structure with conclusion C. By lemma 47, $[\![\pi]\!]$ is a union of complete sets. □

2.2.2 Observational equivalence

In definition 50 we extend the **MLL** observational equivalence $\sim_{\mathbb{B}}$ to the additives. The main result of this subsection is proposition 52, stating that $\sim_{\mathbb{B}}$ is strict larger than $=_{\mathbb{B}}$.

Recall the formula $\mathbb{B} = (X^\perp \otimes X^\perp) ⅋ (X ⅋ X)$, defined in subsection 1.1.2. Actually the **MALL** inhabitants of $\mathbb{B}$ are quite different from the **MLL** ones: there are four **MALL** cut-free proof structures with conclusion $\mathbb{B}$.

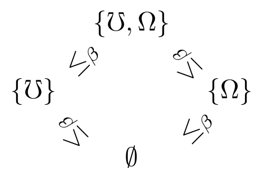
Recall $\mho$ and Ω of figure 1.5. $\mho$ and Ω are the only two cut-free slices with conclusion $\mathbb{B}$, but the cut-free proof structures of $\mathbb{B}$ are four, i.e. $\emptyset$, $\{\mho\}$, $\{\Omega\}$ and $\{\mho, \Omega\}$.

This remark shows that on **MALL** proof structures we can speak not only of β-equivalence, view as the identity between cut-free proof structures, but more finely of a pre-order $\leq_\beta$, which is the set inclusion between cut-free proof structures:

Definition 49 *Let* $\pi_1, \pi_2 \in PS^{ma}$*,* π_1^* *(resp.* π_2^**) be the cut-free proof structure* β*-equivalent with* π_1 *(resp. with* π_2*), then we set* $\boldsymbol{\pi_1} \leq_\beta \boldsymbol{\pi_2}$ *iff* $|\pi_1^*| \subseteq |\pi_2^*|$*.*

Notice that the definition 49 is meaningful since $\to_\beta$ enjoys confluence and (strong) normalization. Moreover, the pre-order $\leq_\beta$ is actually an order on the cut-free proof structures; for any formula C the empty proof structure is the minimum and the *total* proof structure, which is the set of all cut-free slices with conclusion C, is the maximum among the cut-free proof structures with conclusion C.

In particular the cut-free proof structures with conclusion $\mathbb{B}$ are ordered as follows:



A **proper axiom with conclusions $\mathbf{C_1, \ldots, C_n}$** is a link with no premises and n conclusions labeled respectively by $C_1, \ldots, C_n$. A **context of type $\mathbf{C_1, \ldots, C_n}$** is a proof structure with conclusion $\mathbb{B}$, the slices of which can have proper axioms with conclusions $C_1, \ldots, C_n$. We denote a context by $C\,[\,]$.

Let π be a proof structure with conclusions $C_1, \ldots, C_n$ and $C\,[\,]$ be a context of same type. Let us suppose that $\alpha_1, \ldots, \alpha_k$ are the slices of π and $\beta_1, \ldots, \beta_l$ those of $C\,[\,]$, we denote by $C\,[\pi]$ the following proof structure:

- let $i \leq k$ and $j \leq l$, we denote by $\gamma_{i,j}$ the slice obtained from β_j by substituting the slice α_i to each occurrence of the proper axiom in β_j. We define $|C\,[\pi]\,| = \{\gamma_{i,j} \mid i \leq k \text{ and } j \leq l\}$;
- $\equiv_{C[\pi]}$ is the smaller sharing equivalence containing both $\equiv_{C[\,]}$ and $\equiv_\pi$.

As for the β-equivalence, the observational equivalence on **MALL** proof structures is naturally refined in a pre-order, induced by our new set of values:

Definition 50 *Let $\pi_1, \pi_2 \in PS^{ma}$ be with conclusions $C_1, \ldots, C_n$. We say that π_1* ***is observationally less defined than*** *π_2 ($\pi_1 \preceq_{\mathbb{B}} \pi_2$) if for all contexts $C\,[\,]$, $C\,[\pi_1] \leq_\beta C\,[\pi_2]$. We say that π_1* ***is observationally equal to*** *π_2 ($\pi_1 \sim_{\mathbb{B}} \pi_2$) if $\pi_1 \preceq_{\mathbb{B}} \pi_2$ and $\pi_2 \preceq_{\mathbb{B}} \pi_1$.*

The context lemma still holds for such pre-order:

Lemma 51 (Context lemma) *Let $\pi_1, \pi_2 \in PS^{ma}$ be with conclusions $C_1, \ldots, C_n$. Let π_1^* and π_2^* be the two closures of π_1, π_2 with conclusion $C_1 \parr \ldots \parr C_n$. $\pi_1 \not\preceq_{\mathbb{B}} \pi_2$ iff there is a proof structure σ with conclusions $C_1^\perp \otimes \ldots \otimes C_n^\perp, \mathbb{B}$, such that $[\pi_1^*, \sigma] \not\leq_\beta [\pi_2^*, \sigma]$.*

Proof. The proof is similar to that of lemma 28, if we read $\leq_\beta$ (resp. $\preceq_{\mathbb{B}}$) instead of $=_\beta$ (resp. $\sim_{\mathbb{B}}$). □

Unfortunately our definition of additive proof structures does not meet the separation, i.e.:

Proposition 52 *There are $\pi_1, \pi_2 \in PS^{ma}$ such that $\pi_1 \sim_{\mathbb{B}} \pi_2$ but $\pi_1 \not\leq_\beta \pi_2$ and $\pi_2 \not\leq_\beta \pi_1$.*

PROOF. Let A be the formula $X⅋X⅋X⅋X$, $C = A⅋A^{\perp}$. Let α_1 and α_2 be two different cut-free slices with conclusion C, π be the *total* proof structure with conclusion C, which is the set of all cut-free slices with conclusion C. We define $|\pi_1| = |\pi| - \alpha_2$ and $|\pi_2| = |\pi| - \alpha_1$. Clearly $\pi_1 \not\leq_\beta \pi_2$ and $\pi_2 \not\leq_\beta \pi_1$, we prove that $\pi_1 \sim_{\mathbb{B}} \pi_2$. By lemma 51, it is enough to prove that for all proof structure σ with conclusions $C^{\perp}, \mathbb{B}$, $[\sigma, \pi_1] =_\beta [\sigma, \pi_2]$.

We prove that for any slice β with conclusions $C^{\perp}, \mathbb{B}$, $[\{\beta\}, \pi_1] =_\beta [\{\beta\}, \pi_2]$, which clearly implies the statement above.

Since both $C^{\perp}$ and $\mathbb{B}$ are actually formulas of **MLL**, a cut-free slice with conclusions $C^{\perp}, \mathbb{B}$ corresponds up to the axioms with the syntax trees of $C^{\perp}$ and $\mathbb{B}$. Let $\{1, \ldots, 12\}$ be an enumeration of the leaves of such a forest, such that:

- the odd (resp. even) numbers in $\{1, \ldots, 8\}$ enumerate the leaves labeled by X (resp. $X^{\perp}$) above $C^{\perp}$;
- the odd (resp. even) numbers in $\{9, \ldots, 12\}$ enumerate the leaves labeled by X (resp. $X^{\perp}$) above $\mathbb{B}$;

Let β be a cut-free slice with conclusions $C^{\perp}, \mathbb{B}$. If $\beta(9) = 10$ or $\beta(11) = 12$ then for all slices α with conclusion C, $[\beta, \alpha] \rightarrow_\beta \mho$. If $\beta(9) = 12$, or $\beta(11) = 10$ then for all slices α with conclusion C, $[\beta, \alpha] \rightarrow_\beta \Omega$. In both cases, $[\{\beta\}, \pi_1] =_\beta [\{\beta\}, \pi_2]$. Otherwise let $\beta(9) = e$ for an even number $e \leq 8$ and $\beta(10) = o', \beta(12) = o''$ for odds numbers $o', o'' \leq 8$. We remark that there is an $\alpha' \neq \alpha_1, \alpha_2$ such that $\alpha'(e) = o'$ and $\alpha'' \neq \alpha_1, \alpha_2$ such that $\alpha''(e) = o''$. Since $[\beta, \alpha'] \rightarrow_\beta \mho$ and $[\beta, \alpha''] \rightarrow_\beta \Omega$, we get $[\{\beta\}, \pi_1] =_\beta \{\mho, \Omega\} =_\beta [\{\beta\}, \pi_2]$. □

The failure of **MALL** separability is due to the large freedom we let in the proof structure definition. Actually any set of cut-free slices with same conclusions is a proof structure. We do not associate with a slice a special &-rule branch: we allow as many slices we want for any &-rule branch. For example, the proof structure $\{\mho, \Omega\}$ has two slices but no link & justifying the co-presence of both of them.

We let such a freedom since we think that the proof structures have to be the simplest structures on which cut reduction is definable. We avoid in their definition any condition not dealing with the cut reduction, such as for example Hughes and van Glabbeek's resolution condition. The problem of sequentializing a proof structure comes later, at the level of proof nets. Moreover, the problem of sequentializing a proof net in a particular logical system (for example with or without mix, with or without a zero-ary &-rule) is maybe another one coming even later.

2.3 Proof nets

We arrive to one of the most crucial points of **MALL**: the correspondence between **MALL** sequent calculus and sliced proof structures.

In subsection 2.3.1 we define the desequentialization of **MALL** sequent proofs, by following [HvG03]. Contrary to the multiplicative case, the **MALL** desequentialization is not a function, i.e. a unique sequent proof may be associated with several proof structures.

Actually such an ambiguity deals only with the desequentialization of proofs with cuts. More precisely, we will notice that the indeterminateness of a **MALL** desequentialization deals with the sharing equivalence, and not with the way we associate a set of slices with a sequent proof.

In subsection 2.3.2 we recall the **MALL** correctness criterion by Hughes and van Glabbeek. We do not give the proof of the sequentialization theorem (here theorem 58), for which we refer to [HvG03] (see also [HvG05], for a more detailed survey).

2.3.1 Desequentialization of MALL sequent proofs

MALL sequent proofs can be translated into the proof structures by means of a desequentialization procedure. Remark that such a desequentialization is not an immediate extension of the **MLL** one: a **MALL** desequentialization associates with a sequent proof a proof structure, which is now a couple of a set of slices and a sharing equivalence; it turns out that the **MALL** desequentialization is not deterministic in the definition of the sharing equivalence, i.e. with a unique sequent proof (with cuts) we can associate proof structures having the same set of slices but different sharing equivalences.

Let σ be a sequent proof, the **desequentialization** of σ is a procedure (not a function) associating with σ a proof structure $(\sigma)^\bullet$ by induction on σ as follows (recall that, by means of proposition 34, $\equiv_{(\sigma)^\bullet}$ is unequivocally determined once it is defined on the slices' cuts):

- if σ is an axiom with conclusions $X, X^\perp$, then the unique slice of $|(\sigma)^\bullet|$ is an axiom link with conclusions $X, X^\perp$. Of course in this case the sharing equivalence is straightforward;

- if σ ends in a $\parr$-rule (resp. $\oplus_i$-rule), having as premise the subproof σ_1, then $|(\sigma)^\bullet|$ is obtained by adding to every slice in $|(\sigma_1)^\bullet|$ the corresponding link $\parr$ (resp. link $\oplus_i$). Let l, l' be two cuts in the slices of $(\sigma)^\bullet$ with premises of same type. Remark that l and l' are already in the slices of $(\sigma_1)^\bullet$, we define $l \equiv_{(\sigma)^\bullet} l'$ iff $l \equiv_{(\sigma_1)^\bullet} l'$;

- if σ ends in a *mix*-rule, with premises the subproofs σ_1 and σ_2, then $|(\sigma)^\bullet|$ is obtained by taking for every slice in $|(\sigma_1)^\bullet|$ and every slice in $|(\sigma_2)^\bullet|$ their disjoint union. Notice that if $|(\sigma_1)^\bullet|$ (resp. $|(\sigma_2)^\bullet|$) contains k_1 (resp. k_2) slices, then $|(\sigma)^\bullet|$ contains $k_1 \times k_2$ slices. Let l, l' be two cuts in the slices of $(\sigma)^\bullet$ with premises of same type. Remark that l and l' are already in the slices of $(\sigma_i)^\bullet$ (resp. $(\sigma_j)^\bullet$) for $i, j \in \{1, 2\}$. We define $l \equiv_{(\sigma)^\bullet} l'$ iff $i = j$ and $l \equiv_{(\sigma_i)^\bullet} l'$;

- if σ ends in a $\otimes$-rule, with premises the subproofs σ_1 and σ_2, then $|(\sigma)^\bullet|$ is obtained by connecting every slice of $|(\sigma_1)^\bullet|$ and every slice of $|(\sigma_2)^\bullet|$ by means of the $\otimes$-link corresponding to the $\otimes$-rule. Notice that if $|(\sigma_1)^\bullet|$ (resp. $|(\sigma_2)^\bullet|$) contains k_1 (resp. k_2) slices, then $|(\sigma)^\bullet|$ contains $k_1 \times k_2$ slices. Let l, l' be two cuts in the slices of $(\sigma)^\bullet$ with premises of same type. Remark that l and l' are already in the slices of $(\sigma_i)^\bullet$ (resp. $(\sigma_j)^\bullet$) for $i, j \in \{1, 2\}$. We define $l \equiv_{(\sigma)^\bullet} l'$ iff $i = j$ and $l \equiv_{(\sigma_i)^\bullet} l'$;

- if σ ends in a *cut*-rule, with premises the subproofs σ_1 and σ_2, then $|(\sigma)^\bullet|$ is obtained by connecting every slice of $|(\sigma_1)^\bullet|$ and every slice of $|(\sigma_2)^\bullet|$ by means of the *cut*-link corresponding to the *cut*-rule. Notice that if $|(\sigma_1)^\bullet|$ (resp. $|(\sigma_2)^\bullet|$) contains k_1 (resp. k_2) slices, then $|(\sigma)^\bullet|$ contains $k_1 \times k_2$ slices. Let l, l' be two cuts in the slices of $(\sigma)^\bullet$ with premises of same type. Remark that l (resp. l') is either a new *cut*-link or it is already in the slices of $(\sigma_i)^\bullet$ (resp. $(\sigma_j)^\bullet$) for $i, j \in \{1,2\}$. We define $l \equiv_{(\sigma)^\bullet} l'$ iff l and l' are both new *cut*-links or $i = j$ and $l \equiv_{(\sigma_i)^\bullet} l'$;

- if σ ends in a &-rule with premises the subproofs σ_1 and σ_2, then $|(\sigma)^\bullet|$ is obtained by adding a $\&_1$-link (resp. $\&_2$-link) to every slice of $|(\sigma_1)^\bullet|$ (resp. $|(\sigma_2)^\bullet|$) and by taking the union of these sets of slices. Notice that if $|(\sigma_1)^\bullet|$ (resp. $|(\sigma_2)^\bullet|$) contains k_1 (resp. k_2) slices, then $|(\sigma)^\bullet|$ contains $k_1 + k_2$ slices. Let l, l' be two cuts in the slices of $(\sigma)^\bullet$ with premises of the same type. Remark that l (resp. l') is already in the slices of $(\sigma_i)^\bullet$ (resp. $(\sigma_j)^\bullet$) for $i, j \in \{1,2\}$. If $i = j$ then $l \equiv_{(\sigma)^\bullet} l'$ iff $l \equiv_{(\sigma_i)^\bullet} l'$, otherwise, if $i \neq j$, we are free to define $\equiv_{(\sigma)^\bullet}$ whatever we want, provided we respect symmetry and transitivity.

An **additive proof net** is a proof structure which is a desequentialization of an **MALL** sequent proof. Moreover a proof net is said **without mix** if it is the desequentialization of a sequent proof without mix. We denote by $\mathbf{PN^{max}}$ (resp. $\mathbf{PN^{ma}}$) the set of additive proof nets (resp. additive proof nets without mix). Clearly:

$$PN^{ma} \subset PN^{max} \subset PS^{ma}$$

We remark that $(\ )^\bullet$ is not a function because of the &-rule desequentialization. In such a case we may choose to superpose or not two cuts l, l' coming from different branchings of the &-rule. Such a freedom allows to keep a correspondence between the cut reduction in a sequent proof and the one in the associated proof net, as we show in the following example.

Let us consider the sequent proofs σ, σ' and σ'', defined as follows:

σ:

$$\dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax} \qquad \dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax} \quad \dfrac{}{\vdash X, X^\perp}{\scriptstyle ax}}{\vdash X, X^\perp \& X^\perp}{\scriptstyle \&}}{\vdash X, X^\perp \& X^\perp}{\scriptstyle cut}$$

σ':

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax} \quad \dfrac{}{\vdash X, X^\perp}{\scriptstyle ax}}{\vdash X, X^\perp \& X^\perp}{\scriptstyle \&} \qquad \dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax}}{\vdash X \oplus X, X^\perp}{\scriptstyle \oplus_1}}{\vdash X, X^\perp}{\scriptstyle cut} \qquad \dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax} \quad \dfrac{}{\vdash X, X^\perp}{\scriptstyle ax}}{\vdash X, X^\perp}{\scriptstyle cut}}{\vdash X, X^\perp \& X^\perp}{\scriptstyle \&}$$

σ'':

$$\dfrac{\dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax} \quad \dfrac{}{\vdash X, X^\perp}{\scriptstyle ax}}{\vdash X, X^\perp}{\scriptstyle cut} \qquad \dfrac{\dfrac{}{\vdash X, X^\perp}{\scriptstyle ax} \quad \dfrac{}{\vdash X, X^\perp}{\scriptstyle ax}}{\vdash X, X^\perp}{\scriptstyle cut}}{\vdash X, X^\perp \& X^\perp}{\scriptstyle \&}$$

Let us compute the three desequentializations $(\sigma)^\bullet, (\sigma')^\bullet$ and $(\sigma'')^\bullet$.

Both σ and σ'' are associated with the same set of slices $\{\alpha_1, \alpha_2\}$, while σ' is associated with $\{\alpha_{11}, \alpha_{12}, \alpha_2\}$ (see figure 2.14).

Let l_1, l_2, l_{11} and l_{12} be the cuts resp. in α_1, α_2, α_{11} and α_{12}. The sharing equivalences of resp. $(\sigma)^\bullet, (\sigma')^\bullet$ and $(\sigma'')^\bullet$ are as follows:

- in $(\sigma)^\bullet$ we have to define $l_1 \equiv_{(\sigma')^\bullet} l_2$. There is no choice, (σ) has a unique desequentialization;
- in $(\sigma')^\bullet$ we have to define $l_{11} \equiv_{(\sigma')^\bullet} l_{12}$, and $l_{11} \not\equiv_{(\sigma')^\bullet} l_2, l_{12} \not\equiv_{(\sigma')^\bullet} l_2$. σ' has a unique desequentialization too;
- in $(\sigma'')^\bullet$ we are free to define $l_1 \equiv_{(\sigma'')^\bullet} l_2$ or $l_1 \not\equiv_{(\sigma'')^\bullet} l_2$. σ'' has two different desequentializations.

σ'' has two desequentializations since it can be the result of a reduction of a cut in σ as well as of a cut in σ'.

$\sigma \to_\beta \sigma''$, by reducing the commutative additive cut in σ. In this case we justify $l_1 \equiv_{(\sigma'')^\bullet} l_2$, so that σ and σ'' are desequentialized by the same proof structure: $(\sigma)^\bullet = (\sigma'')^\bullet$. In general we expect that a desequentialization is invariant under commutative cut reductions.

$\sigma' \to_\beta \sigma''$, by reducing the cut with premises $X^\perp \& X^\perp, X \oplus X$ in σ'. In this case we set $l_1 \not\equiv_{(\sigma'')^\bullet} l_2$, being l_1 the residual of the l_{11} and l_{12} reductions, so that we have $(\sigma')^\bullet \to_\beta (\sigma'')^\bullet$.

Associating with a unique sequent proof an host of proof nets may be view as a weakness. This is not the case for sliced proof nets, having shown that the $(\ )^\bullet$ indeterminateness only deals with the sharing equivalence of the cut links. It just concerns the way we reduce the slices' cuts, independently one from the other or simultaneously, but not the way we represent the slices, hence the cut-free proofs (for which the sharing equivalence is unique, recall proposition 34)[3].

Actually if we look at the invariants under cut reduction, sliced proof structures give canonical representatives - the cut-free proof structures. For a last example recall the two proof nets in figures 2.5 and 2.6, which are a counter-example to the canonicity of the proof nets based on additive boxes, being β-equivalent. Remark that such proof nets are associated with a unique sliced proof net, of which set of slices $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is in figure 2.15.

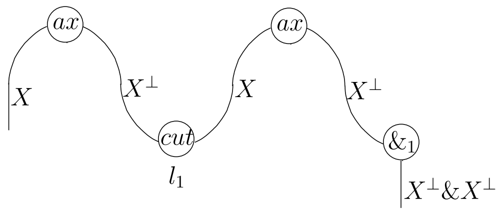
In general we consider the injectivity of the relational semantics for sliced proof nets (theorem 45) a proof of their canonicity.

2.3.2 Correctness criterion for additive proof nets

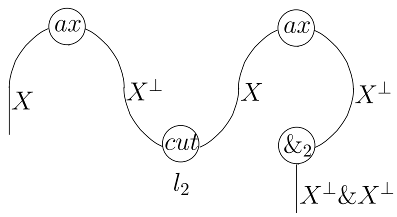
In this subsection we recall the **MALL** correctness criterion by Hughes and van Glabbeek. Such a criterion consists of three conditions, which we call respectively (see definition 56):

1. slice correctness (in [HvG03] called **MLL** correctness),

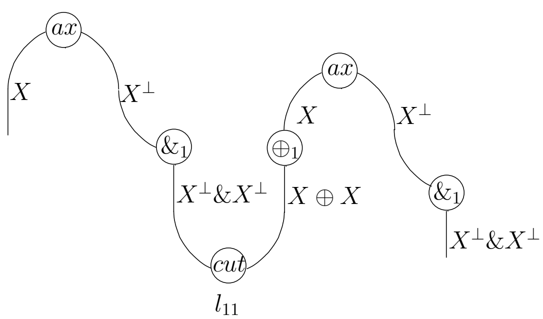
[3] Indeed the proof nets syntax introduced by Girard in [Gir96] associates with a unique sequent proof several proof nets too, but Girard's syntax is ambiguous already at the level of cut-free proofs, not the Hughes and van Glabbeek's one.



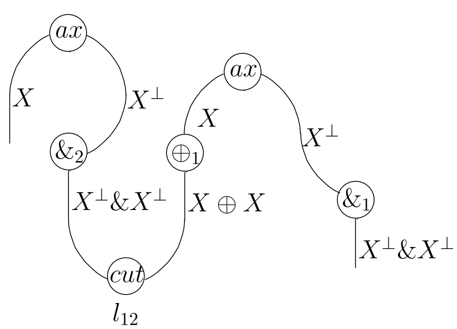
slice α_1



slice α_2



slice α_{11}



slice α_{12}

Figure 2.14: slices α_1, α_2, α_{11} and α_{12} with conclusions $X, X^\perp \& X^\perp$.

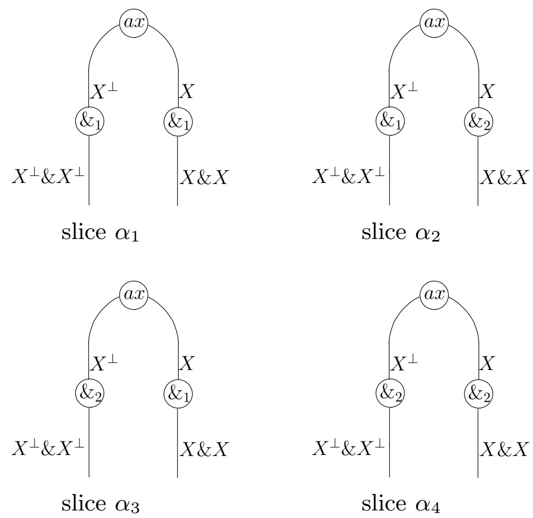


Figure 2.15: slices associated with the additive box based proof nets of figures 2.5 and 2.6.

2. additive acyclicity (in [HvG03] called toggling condition),

3. fullness and compatibility (in [HvG03] called resolution condition).

We have already noticed that Hughes and van Glabbeek use fullness and compatibility already for defining the proof structures. We do not give the proof of the sequentialization theorem (here theorem 58), for which we refer to [HvG03] (see also [HvG05], for a more detailed survey).

A **correctness graph of a slice** α is a subgraph of α which is obtained erasing one premise for each link $\parr$. The first correctness condition we can formulate is the **MLL** correctness:

Definition 53 *A proof structure π is* ***slice correct*** *(resp.* ***slice strong correct****) if for each slice $\alpha \in \pi$, all correctness graphs of α are acyclic (resp. acyclic and connected).*

It is well-known since [Gir87] that the slice correctness is far from characterizing **MALL** proof nets: what we still need is a graph and related paths jumping from one slice to another. For this purpose we associate with any proof structure π a graph G_π, allowing to deal with paths crossing the slices of π.

We recall that the links (resp. edges) of π are the sharing equivalence classes of the links (resp. edges) in the slices of π (see definition 37). **Two slices** α' **and** α'' **of** π **toggle a** $\&$ **w of** π if α' (resp. α'') shares w by means of a link $\&_i$ (resp. $\&_j$) and $i \neq j$.

Let a (resp. w) be an axiom conclusion (resp. link $\&$) of π, a **depends on** w **in** π if there are two slices $\alpha', \alpha'' \in \pi$, such that a is shared by α' but not by α'' and w is the only $\&$ of π toggled by α', α''. An **axiom** l **of** π **depends on** w if at least one conclusion of l depends on w.

Construct the graph G_π from the sharing graph $^{|\pi|}/_{\equiv_\pi}$ by adding for each & link w and each axiom link l depending on w in π, an edge, called **jump**, from w to l.

We adapt the path definition of section 1.1 to G_π. An **oriented edge** is an edge together with a direction *upward*, denoted by $\uparrow a$, or *downward*, denoted by $\downarrow a$. We write $\updownarrow a$ in case we do not want to specify if we mean either $\uparrow a$ or $\downarrow a$. We consider a jump from a & w to an axiom l as a premise of w and a conclusion of l. An **oriented path** (or simply path) from $\updownarrow a_0$ to $\updownarrow a_n$ in a graph G_π is a sequence of G_π oriented edges $<\updownarrow a_0, \ldots, \updownarrow a_n>$ such that for any $i < n$:

- if $\updownarrow a_i = \uparrow a_i$, $\updownarrow a_{i+1} = \uparrow a_{i+1}$, then a_i is conclusion of the link of which a_{i+1} is premise;

- if $\updownarrow a_i = \uparrow a_i$, $\updownarrow a_{i+1} = \downarrow a_{i+1}$, then a_i and a_{i+1} are conclusions of the same link;

- if $\updownarrow a_i = \downarrow a_i$, $\updownarrow a_{i+1} = \downarrow a_{i+1}$, then a_i is the premise of the link of which a_{i+1} is conclusion;

- if $\updownarrow a_i = \downarrow a_i$, $\updownarrow a_{i+1} = \uparrow a_{i+1}$, then a_i and a_{i+1} are premises of the same link;

We say that **a path crosses a link l** if it contains a sequence of two edges having l as a vertex.

A path is **up-oriented** (resp. **down-oriented**) if all its edges are upward (resp. downward) oriented. **An edge** a **is above an edge** b ($a \geq b$) if there is a path down-oriented from $\downarrow a$ to $\downarrow b$.

As always, we denote paths by the Greek letters $\phi, \psi, \ldots$. A path ϕ in π **comes back** if there is an edge a s.t. $\uparrow a, \downarrow a \in \phi$; a **switching edge** of π is a ⅋ or & premise (jumps included); a path ϕ is **switching** if it never comes back and it does not contain two switching edges of a same link. A **switching cycle** is a switching path from $\updownarrow a$ to $\updownarrow a$.

Definition 54 *A proof structure π enjoys **additive acyclicity** if there is a & w toggled by slices in π, such that w is not crossed by any switching cycle in G_π.*

*A proof structure π is **downward additive acyclic** if for each $\pi' \subseteq \pi$, if π' has at least two slices, then π' enjoys additive acyclicity.*

Let w be a non axiom link of G_π. A **branch of w** in G_π is the subgraph of all the edges and non axiom links above one premise of w in G_π. A **&-resolution** $G^{\&}_\pi$ of G_π is a sub-graph of G_π obtained erasing one branch for each binary & in G_π. **A slice** $\alpha \in \pi$ **is in a &-resolution** $\mathbf{G}^{\&}_\pi$, written $\alpha \in G^{\&}_\pi$, if all the α links are in the $G^{\&}_\pi$ links.

Definition 55 *A proof structure π is **compatible** if for each &-resolution $G^{\&}_\pi$ there is at most one slice α s.t. $\alpha \in G^{\&}_\pi$; it is **full** if G_π contains only binary & link and for each &-resolution $G^{\&}_\pi$ there is at least one slice α s.t. $\alpha \in G^{\&}_\pi$.*

The final additive correctness criterion is as follows:

Definition 56 *A proof structure π is* ***correct*** *(resp.* ***strongly correct****) if the following condition holds:*

1. *π is slice correct (resp. slice strongly correct);*
2. *π is downward additive acyclic;*
3. *π is compatible and full.*

First of all, remark that the additive correctness criterion is stable by cut reduction:

Theorem 57 ([HvG03]) *Let $\pi \rightarrow_\beta \pi'$, if π is correct (resp. strongly correct) then so is π'.*

Proof. See [HvG03], [HvG05]. □

Secondly, the correctness corresponds to **MALL** sequentialization:

Theorem 58 ([HvG03]) *Let $\pi \in PS^{ma}$. π is a proof net (resp. a proof net without mix) iff π is correct (resp. strongly correct).*

For the proof details see [HvG03], [HvG05]. We give here just a proof sketch and further definitions which we will use in the following subsection 2.3.3.

The proof of the implication π *proof net* $\Rightarrow \pi$ *correct* is a simple induction on the length of a sequent proof associated with π. Conversely, the proof of π *correct* $\Rightarrow \pi$ *proof net* is quite hard. The key step is the *splitting lemma*, stating that in case π is a correct proof structure with $\invamp/\&$ links then it has a splitting $\invamp/\&$, where a link is splitting if removing it increases the number of the connected components of G_π.

With the splitting lemma in hands the proof of π *correct* $\Rightarrow \pi$ *proof net* reduces to an induction on the number of $\invamp/\&$ in π.

The splitting lemma is proven by the notion of *domination*, which is a binary relation between a $\invamp/\&$ link (the *dominator*) and a general link (the *dominated*).

In the following section we will use the notion of domination, as well as corollary 64. Corollary 64 is a little variant of corollary 1 in [HvG03] (see also corollary 4.35 in [HvG05]). In particular we do not require that π is a proof net and we refer to binary & links instead of $\&/\invamp$ links in general.

In what follows we give the precise definitions and lemmas which allow to prove the corollary 64. Such lemmas are straightforward variants of the ones in [HvG03], thus we omit the proofs.

Let π be a proof structure, we say that a **subset π' of slices of π is saturated** if for each slice $\alpha \in |\pi|/|\pi'|$, $\pi' \cup \{\alpha\}$ toggles more & than π'.

Remark that the downward additive acyclicity (i.e. condition 2 of definition 56) coincides with:

2' for each saturated $\pi' \subseteq \pi$, if π' has at least two slices, then π' enjoys additive acyclicity;

A switching path is a **strong path** if its first edge is not a switching edge of a $\parr/\&$. Let A be a set of edges in G_π. A path is **in A** if each of its edges is in A. We write $a \overset{\phi}{\rightarrow}_A b$ to denote a switching path ϕ from a to b in A and $a \overset{\phi}{\Rightarrow}_A b$ if ϕ is strong. We shall sometimes use slices or set of slices to denote the sets of their edges.

A set A of edges in G_π is an a**-zone**, if for all $b \in A$ there is a strong path $b \overset{\phi}{\Rightarrow}_A a$ such that $\downarrow a \in \phi$. Given a link $\parr/\&$ w and a link l, we define **w dominates l**, denoted by $w \sqsupset l$, if there is a switch edge a of w such that the conclusion of l is in an a-zone. If l is not dominated (resp. not dominated by any $\&$), it is **free** (resp. **&-free**).

Lemma 59 (Properties of domination, [HvG03]) *Let π be a proof structure, then:*

- *SWITCH. If $w \leftarrow l$ is a switch edge then $w \sqsupset l$;*
- *TRANSITIVITY. Domination is transitive;*
- *SELF. A $\parr/\&$ link dominates itself iff it is in a switching cycle;*
- *JUMP-CYCLE. If $w \leftarrow l$ is a jump and l is crossed by a switching cycle ϕ, then w dominates every link crossed by ϕ;*
- *EXTEND. If $w \sqsupset l$ and there is a path ϕ from l to l' which never enters a $\parr/\&$ from above (i.e. $\downarrow a \in \phi$ only if a is not a $\parr/\&$ switching edge), then $w \sqsupset l'$;*
- *FORK. Let a_0, a_n two switching edges of a $\parr/\&$ such that $a_0 \overset{\phi}{\rightarrow} a_n$, then for each link l crossed by ϕ, $w \sqsupset l$;*
- *MEET. If $w \sqsupset l \sqsubset w'$ for distinct free $\parr/\&$ w, w', then exists a switching path $\uparrow w \overset{\phi}{\rightarrow} \downarrow w'$.*

Let w be a binary $\&$ of π, we denote by π^w the proof structure containing all the slices of π which do not share the w right premise. Write $\alpha \overset{w}{=} \alpha'$ if the slices $\alpha, \alpha' \in \pi$ are either equal or w is the only $\&$ toggled by α, α.

It is straightforward to check that:

(S1) if π is saturated and toggles w then π^w is saturated;

(S2) if π is saturated and toggles w and $\alpha \in \pi$ than $\alpha \overset{w}{=} \alpha_w$ for some $\alpha_w \in \pi^w$;

(S3) if π is saturated and toggles w and $\alpha \overset{w'}{=} \alpha'$ then exist $\alpha_w, \alpha'_w \in \pi^w$ s.t.:

$$\begin{array}{ccc} \alpha & \overset{w'}{==} & \alpha' \\ {\scriptstyle w}\,\| & & {\scriptstyle w}\,\| \\ \alpha_w & \overset{w'}{==} & \alpha'_w \end{array}$$

Lemma 60 (from [HvG03]) *Let w be a $\&$ toggled by a saturated proof structure π, and let e be an edge from an axiom l of G_π such that $e \notin G_{\pi^w}$. Then exists a jump $l \rightarrow w$ in G_π.*

Lemma 61 (from [HvG03]) *Let π be a saturated and downward additive acyclic proof structure, then every non-empty union S of switching cycles of G_π has a jump out of it: for some axiom l crossed by S and toggled & w, if w is not crossed by S, then there is a jump $l \to w$ in G_π.*

Lemma 62 (from [HvG03]) *Let π be a saturated and downward additive acyclic proof structure, w be a & toggled in π s.t. $w \sqsupset w$, then exists a & w' toggled in π s.t. $w' \sqsupset w$ and $w' \not\sqsupset w'$.*

Lemma 63 (from [HvG03]) *Let π be a saturated and downward additive acyclic proof structure, every binary & of G_π is either &-free or is dominated by a binary &-free &.*

Corollary 64 (from [HvG03]) *Let π be a saturated, sliced correct and downward additive acyclic proof structure. If G_π has a binary &, then it has a binary &-free &.*

2.3.3 From coherent to hypercoherent semantics

In this subsection we present the state of our research of a surjective semantics for sliced proof nets. The subsection is divided in three paragraphs. In the first one, called *coherent semantics*, we extend coherent spaces to the additives and we study the *Gustave proof structure* - a well-known counter example to the full-completeness of **MALL** coherent spaces.

In the second paragraph, called *hypercoherent semantics*, we recall the hypercoherent spaces introduced by Ehrhard in [Ehr93]. Hypercoherent spaces overcome the Gustave proof structure counter example.

In the third paragraph, called *hypercliques and* **MALL** *correctness* we present our state of knowledge about the correspondence between hypercliques and **MALL** correctness. The main result in this subsection is theorem 68, stating that the interpretation of a correct proof structure is a hyperclique.

Coherent semantics. We extend the coherent semantics defined in subsection 1.2.1 to the additives. Let $\mathcal{X}$ be a coherent space, the **coherent model** $\mathfrak{Coh}^{\mathcal{X}}$ associates with **MALL** formulas coherent spaces defined by induction on the formulas, as follows:

- with X it is associated $\mathcal{X}$;
- with $A^\perp$ it is associated $\mathcal{A}^\perp$ defined as follows: $|\mathcal{A}^\perp| = |\mathcal{A}|$, the coherence of $\mathcal{A}^\perp$ is the incoherence of $\mathcal{A}$, i.e. $x \overset{\frown}{\smile} y \left[\mathcal{A}^\perp\right]$ iff $x \underset{\frown}{\smile} y \,[\mathcal{A}]$;
- with $A \otimes B$ it is associated $\mathcal{A} \otimes \mathcal{B}$ defined as follows: $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ and $< a, b > \overset{\frown}{\smile} < a', b' > [\mathcal{A} \otimes \mathcal{B}]$ iff $a \overset{\frown}{\smile} a' \,[\mathcal{A}]$ and $b \overset{\frown}{\smile} b' \,[\mathcal{B}]$;
- with $A_1 \oplus A_2$ it is associated $\mathcal{A}_1 \oplus \mathcal{A}_2$ defined as follows: $|\mathcal{A}_1 \oplus \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2|$ and $< i, x > \overset{\frown}{\smile} < j, y > [\mathcal{A}_1 \oplus \mathcal{A}_2]$ iff $i = j$ and $x \overset{\frown}{\smile} y \,[\mathcal{A}_i]$.

Of course, the space $\mathcal{A} \parr \mathcal{B}$ is defined by $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$ and $\mathcal{A} \& \mathcal{B}$ is defined by $(\mathcal{A}^\perp \oplus \mathcal{B}^\perp)^\perp$.

As in **MLL** remark that the web associated with a formula A by $\mathfrak{Coh}^{\mathcal{X}}$ is precisely the interpretation of A in $\mathfrak{Rel}^{|\mathcal{X}|}$.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$ the **interpretation of** π in $\mathfrak{Coh}^{\mathcal{X}}$ is the subset $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ of $|\mathcal{C}_1 \wp \ldots \wp \mathcal{C}_n|$ defined exactly in the same way as in the relational semantics (see subsections 1.1.1 and 2.2.1). We have the same definitions concerning the **experiment** e on a proof structure π and its **result**. If π is a **MALL** proof structure, then $[\![\pi]\!]_{\mathfrak{Rel}^{|\mathcal{X}|}} = [\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

In chapter 1 we have seen how coherence provides a semantical notion for multiplicative correctness: for an **MLL** cut-free proof structure π, if π is correct then $[\![\pi]\!]$ is a clique (from Girard's theorem, here theorem 24), as well as if $[\![\pi]\!]$ is a clique then π is correct (from Retoré's theorem, here theorem 25). On the contrary, coherent semantics is far from catching additive correctness: there are **MALL** proof structures which are not correct although their interpretations are cliques.

A well-known example of such proof structures is the so-called *Gustave proof structure*, defined by Hughes and van Glabbeek in [HvG05]. It corresponds with a linear variant of the *Gustave function* studied in [Gir99] and [AM99].

Let $M = X^{\perp} \otimes X^{\perp}$ and let us consider the following formulas:

$$C_1 = (X \& X) \oplus X$$

$$C_2 = (X \& X) \oplus X$$

$$C_3 = (M \& M) \oplus M$$

The **Gustave proof structure** γ is a cut-free proof structure with conclusions C_1, C_2, C_3. γ consists of the five slices $\alpha_1, \ldots, \alpha_5$ defined in figures 2.16 - 2.20.

γ is compatible, full and slice strong correct, but it is not correct, since the graph $G_{\{\alpha_1,\alpha_2,\alpha_3\}}$ is not additive acyclic.

Of course γ is not sequentializable. In fact, if it were, any of its sequentializations should choose one $\oplus$ link to sequentialize first. Now if we consider only α_1 and α_2 we should start from the $\oplus$ above C_1, while if we consider only α_2 and α_3 (resp. α_3 and α_1) the $\oplus$ above C_3 (resp. above C_2) should be chosen first. But the three slices $\alpha_1, \alpha_2, \alpha_3$ together exclude the choice of any $\oplus$.

Nevertheless $[\![\gamma]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is a clique for any coherent space $\mathcal{X}$. Let us show it.

The interpretations of the γ five slices are as follows:

$$[\![\alpha_1]\!] = \quad \{<< 1, x >, < 2, y >, < 3, < x, y >>>, \text{ for any } x, y \in |\mathcal{X}|\}$$

$$[\![\alpha_2]\!] = \quad \{<< 2, x >, < 3, y >, < 1, < x, y >>>, \text{ for any } x, y \in |\mathcal{X}|\}$$

$$[\![\alpha_3]\!] = \quad \{<< 3, x >, < 1, y >, < 2, < x, y >>>, \text{ for any } x, y \in |\mathcal{X}|\}$$

$$[\![\alpha_4]\!] = \quad \{<< 1, x >, < 1, y >, < 1, < x, y >>>, \text{ for any } x, y \in |\mathcal{X}|\}$$

$$[\![\alpha_5]\!] = \quad \{<< 2, x >, < 2, y >, < 2, < x, y >>>, \text{ for any } x, y \in |\mathcal{X}|\}$$

where we consider a generic element of $|(\mathcal{X} \& \mathcal{X}) \oplus \mathcal{X}|$ (resp. of $|(\mathcal{M} \& \mathcal{M}) \oplus \mathcal{M}|$) as $< j, z >$ with $z \in |\mathcal{X}|$ (resp. $z \in |\mathcal{M}|$) and $j = 1, 2, 3$ depending on which

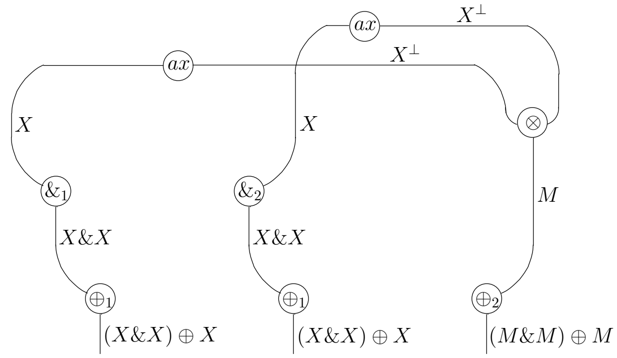


Figure 2.16: slice α_1 of the Gustave proof structure.

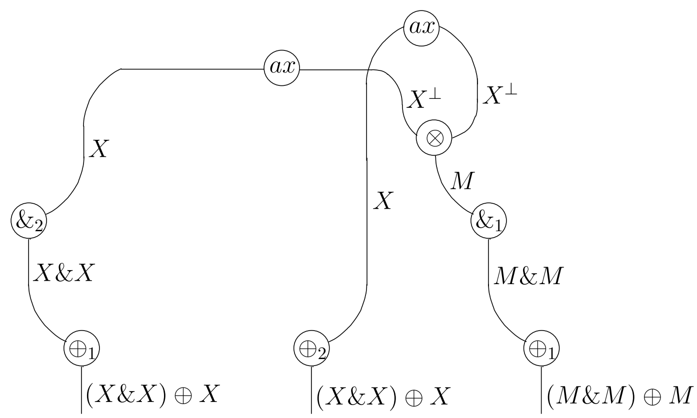


Figure 2.17: slice α_2 of the Gustave proof structure.

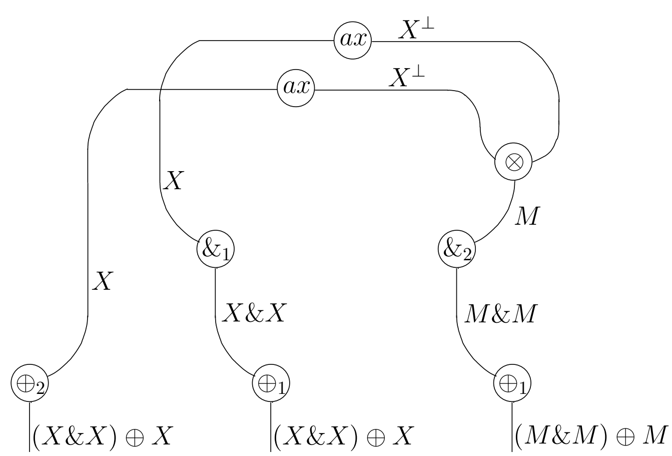


Figure 2.18: slice α_3 of the Gustave proof structure.

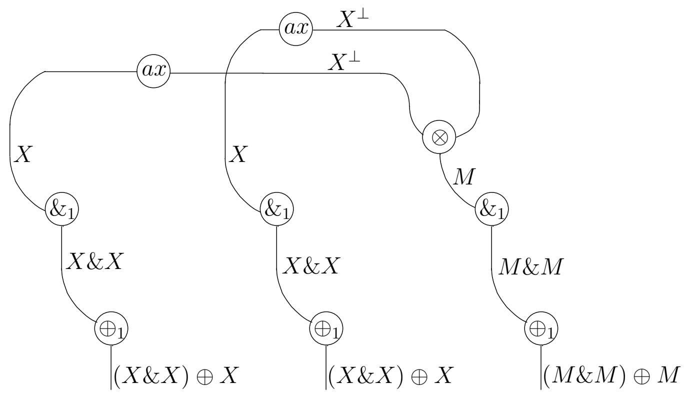


Figure 2.19: slice α_4 of the Gustave proof structure.

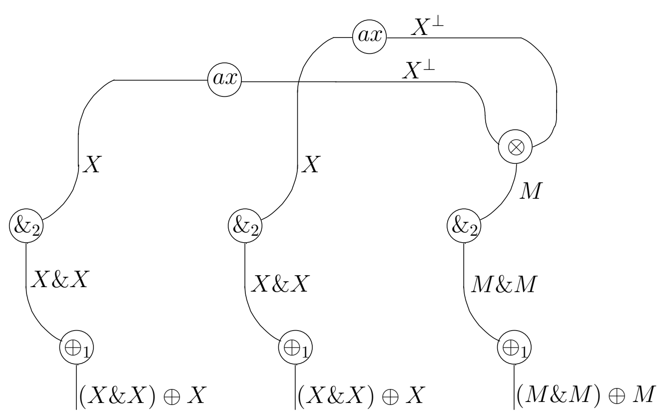


Figure 2.20: slice α_5 of the Gustave proof structure.

additive component z belongs to. Moreover we consider a generic element of $|\mathcal{C}_1 \parr \mathcal{C}_2 \parr \mathcal{C}_3|$ as a triplet.

Now let u, v be two elements of $[\![\gamma]\!]$, we prove $u \frown\!\!\!\smile v\,[\mathcal{C}_1 \parr \mathcal{C}_2 \parr \mathcal{C}_3]$.

If u, v are elements of a single slice interpretation, then of course $u \frown\!\!\!\smile v$, being γ slice correct. Otherwise, suppose u, v belong to different slices interpretations. By definition of the $\parr$ coherence, $u \frown\!\!\!\smile v\,[\mathcal{C}_1 \parr \mathcal{C}_2 \parr \mathcal{C}_3]$ if there is a projection $i = 1, 2, 3$ s.t. $p_i(u) \frown\!\!\!\smile p_i(v)\,[\mathcal{C}_i]$. Choose the projection p_i s.t. both $p_i(u)$ and $p_i(v)$ belong to the left component of the $\oplus$ of $\mathcal{C}_i$. Remark that such a projection always exists, since we deal with at most two elements of $[\![\gamma]\!]$. Finally notice that for such a p_i, $p_i(u)$ and $p_i(v)$ are element of the two distinct components of the & of $\mathcal{C}_i$, hence $p_i(u) \frown\!\!\!\smile p_i(v)\,[\mathcal{C}_i]$, by definition of the $\oplus$ and the & coherence.

Notice that $[\![\gamma]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is a clique because the coherence deals only with pairs of elements: we consider no more than two elements at a time, that is we may look at no more than two slices of γ at a time, hence we do not see the incorrectness among the three slices $\alpha_1, \alpha_2, \alpha_3$ together.

For avoiding such a coherent spaces shortsight, Ehrhard introduces in [Ehr93] the hypercoherence, which is a relation among any finite set of elements, not only pairs.

Hypercoherent semantics. Let $\mathtt{X}$ be a set, we denote by $\wp_{<\omega}(\mathtt{X})$ the set of all the finite subsets of $\mathtt{X}$.

Definition 65 ([Ehr93]) *A **hypercoherent space** $\mathcal{X}$ is a pair $(|\mathcal{X}|, \Gamma_\omega(\mathcal{X}))$, where $|\mathcal{X}|$ is a set, called the **web of** $\mathcal{X}$, and $\Gamma_\omega(\mathcal{X})$ is a subset of $\wp_{<\omega}(|\mathcal{X}|)$ containing all the singletons and called the **hypercoherence of** $\mathcal{X}$.*

*A **hyperclique** of $\mathcal{X}$ is a subset $\mathtt{C} \subseteq |\mathcal{X}|$, such that for each finite subset $\mathtt{C}' \subseteq \mathtt{C}$, $\mathtt{C}' \in \Gamma_\omega(\mathcal{X})$.*

A hypercoherent space $\mathcal{X}$ is identified with a hypergraph, each of whose hyperedges is a finite set of vertices: namely $|\mathcal{X}|$ is the set of vertices and $\Gamma_\omega(\mathcal{X})$ that of hyperedges.

We define $\Gamma^*_\omega(\mathcal{X}) = \Gamma_\omega(\mathcal{X}) / \{\{x\} \mid x \in |\mathcal{X}|\}$. Remark we may define a hypercoherent space $\mathcal{X}$ specifying its web and one between $\Gamma_\omega(\mathcal{X})$ and $\Gamma^*_\omega(\mathcal{X})$.

Hypercoherent spaces provide a semantics for **MALL**. Let $\mathcal{X}$ be a hypercoherent space, a **hypercoherent model on** $\mathcal{X}$ ($\mathfrak{HCoh}^{\mathcal{X}}$) associates with **MALL** formulas hypercoherent spaces, defined by induction on the formulas, as follows:

- with X it is associated $\mathcal{X}$;
- with $A^\perp$ it is associated $\mathcal{A}^\perp$, defined as follows: $|\mathcal{A}^\perp| = |\mathcal{A}|$, $\Gamma^*_\omega(\mathcal{A}^\perp) = \wp_{<\omega}(|\mathcal{A}|) / \Gamma_\omega(\mathcal{A})$;
- with $A \otimes B$ it is associated $\mathcal{A} \otimes \mathcal{B}$, defined as follows: $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$, $\mathtt{C} \in \Gamma_\omega(\mathcal{A} \otimes \mathcal{B})$ iff $p_1(\mathtt{C}) \in \Gamma_\omega(\mathcal{A})$ and $p_2(\mathtt{C}) \in \Gamma_\omega(\mathcal{B})$;
- with $A \oplus B$ it is associated $\mathcal{A} \oplus \mathcal{B}$, defined as follows: $|\mathcal{A} \oplus \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|$, $\mathtt{C} \in \Gamma_\omega(\mathcal{A} \oplus \mathcal{B})$ iff:

  $s_1(\mathtt{C}) = \emptyset$ and $s_2(\mathtt{C}) \in \Gamma_\omega(\mathcal{B})$

  or

  $s_2(\mathtt{C}) = \emptyset$ and $s_1(\mathtt{C}) \in \Gamma_\omega(\mathcal{A})$.

Of course, the space $\mathcal{A}⅋\mathcal{B}$ is defined by $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$ and $\mathcal{A}\&\mathcal{B}$ by $(\mathcal{A}^\perp \oplus \mathcal{B}^\perp)^\perp$.

As for coherent spaces, the web associated with a formula A by $\mathfrak{HCoh}^{\mathcal{X}}$ is precisely the interpretation of A in $\mathfrak{Rel}^{|\mathcal{X}|}$.

Remark that a subset of $|\mathcal{C}|$ can be in $\Gamma_\omega(\mathcal{C})$ without being a hyperclique. For example, let $C = A\&A$ and $\mathtt{C}$ be the subset $\{< 1, x >, < 1, y >, < 2, z >\}$ of $\mathcal{C}$. $\mathtt{C}$ is in $\Gamma_\omega(\mathcal{C})$, since both $s_1(\mathtt{C})$ and $s_2(\mathtt{C})$ are not empty, but $\mathtt{C}$ is a hyperclique only in case $\{< 1, x >, < 1, y >\} \in \Gamma_\omega(\mathcal{C})$, that is only if $\{x, y\} \in \Gamma_\omega(\mathcal{A})$. In general the set of the hypercliques is downward closed, while $\Gamma_\omega(\mathcal{C})$ is not.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$. Like in coherent semantics, the **interpretation of** π in $\mathfrak{HCoh}^{\mathcal{X}}$ is the subset $[\![\pi]\!]_{\mathfrak{HCoh}^{\mathcal{X}}}$ of $|\mathcal{C}_1⅋\ldots⅋\mathcal{C}_n|$ defined exactly in the same way as in relational semantics (see subsection 2.2.1), i.e. if π is a **MALL** proof structure $[\![\pi]\!]_{\mathfrak{HCoh}^{\mathcal{X}}} = [\![\pi]\!]_{\mathfrak{Rel}^{|\mathcal{X}|}}$.

Recall the Gustave proof structure γ. Notice that $[\![\gamma]\!]$ is a clique, but it is not a hyperclique. Let us show that $[\![\gamma]\!]$ is not a hyperclique.

For being a hyperclique, all finite subsets of $[\![\gamma]\!]$ has to be hypercoherent. Let $\mathtt{C}$ be as a set having one element for each $[\![\alpha_1]\!]$, $[\![\alpha_2]\!]$ and $[\![\alpha_3]\!]$. Clearly $\mathtt{C} \subseteq [\![\gamma]\!]$, let us prove that $\mathtt{C} \notin \Gamma_\omega(\mathcal{C}_1⅋\mathcal{C}_2⅋\mathcal{C}_3)$.

By definition of ⅋ hypercoherence, $\mathtt{C} \in \Gamma_\omega(\mathcal{C}_1⅋\mathcal{C}_2⅋\mathcal{C}_3)$ iff there is a projection $i = 1, 2, 3$ s.t. $p_i(\mathtt{C}) \in \Gamma_\omega(\mathcal{C}_i)$. But any projection $p_i(\mathtt{C})$ has elements from both the components of the $\oplus$ of $\mathcal{C}_i$, hence $p_i(\mathtt{C}) \notin \Gamma_\omega(\mathcal{C}_i)$ by definition of $\oplus$ hypercoherence.

The Gustave proof structure example shows that $\mathfrak{HCoh}^{\mathcal{X}}$ has more chance than $\mathfrak{Coh}^{\mathcal{X}}$ for providing a semantical notion of **MALL** correctness. The rest of this subsection presents our ongoing research for outlining a correspondence between hypercliques and additive correctness.

Hypercliques and MALL correctness. The main result of this subsection is theorem 68, stating that:

if π is correct than $[\![\pi]\!]_{\mathfrak{HCoh}^{\mathcal{X}}}$ is a hyperclique for any hypercoherent space $\mathcal{X}$.

Theorem 68 generalizes theorem 24 of chapter 1. As far as we know there is a proof of theorem 68 by De Falco in [Fal05]. However the proof in [Fal05] relies on an *ad hoc* construction (the $\mathbb{B}$-trees). Our proof instead is closer to the notion of switching path, generalizing the proof technique of theorem 24. In particular compare lemma 22 with the present lemma 67.

Lemma 66 *Let π be a compatible and full proof structure, $\pi' \subseteq \pi$ be a saturated subset of slices in π and e be an edge in π' from an axiom l, s.t. there is a slice $\alpha \in \pi'$ not sharing e. There is a & w and a jump j in $G_{\pi'}$ from l to w.*

Proof. Let e be an atomic edge which is not shared by all the slices in π' and l be the axiom with conclusion e. We prove the lemma by induction on the number of & shared by π'.

Let α, α' be two slices such that α shares e but α' does not. By π' compatibility, α, α' toggle at least one & w.

If w is the only & shared by α, α', then l depends on w in π' (α, α' being the toggling pair), i.e. there is a jump j in $G_{\pi'}$ from l to w.

If α, α' toggle more than one &, then let α'' be the slice in π' such that w is the only & toggled by α, α''. Remark that such an α'' exists, since π' is a saturated subset of a full proof structure π. Of course α'' shares the same w premise shared by α'.

If α'' does not share e, then l depends on w in π' (α, α'' being the toggling pair), i.e. there is a jump j in $G_{\pi'}$ from l to w.

If α'' shares e, then consider the proof structure $\pi'' \subset \pi'$ containing all the slices in π' sharing the same w premise shared by α' and α''. π'' is a saturated subset of π toggling less & than π'. Moreover, e is not shared by all the slices in π'' (α' does not shares e, for example), thus by induction hypothesis there is a & w' and a jump j in $G_{\pi''}$ from l to w'. Of course such a jump is also an edge of $G_{\pi'}$. □

Lemma 67 *Let π be a proof structure which is slice correct, compatible and full. Let $e_1 : \alpha_1, \ldots, e_n : \alpha_n$ be experiments on slices of π, and $\pi' \subset \pi$ be a minimal saturated subset of π containing $\alpha_1, \ldots, \alpha_n$. In case there is a binary & w not dominated by any binary & in $G_{\pi'}$, then there is a conclusion $d : D$ of π and a strong path ϕ from w to $\downarrow d$ such that ϕ is in α for any slice $\alpha \in \pi'$ (in particular ϕ does not contain any jump) and $\{e_1(d), \ldots, e_n(d)\} \in \Gamma^*_\omega(\mathcal{D})$.*

Proof. By π' minimality, π' toggles the same &'s then $\{\alpha_1, \ldots, \alpha_n\}$. Let now w be a binary & not dominated by any & in $G_{\pi'}$ (in case it exists).

Firstly we prove that w is shared by all the slices of π'. Suppose there are two slices α, α' in π', such that α shares w but α' does not, let us prove a contradiction. By going down the conclusion of w we eventually meet a link m such that:

1. either m is a cut shared by α but not by α',
2. or m is an additive link, such that the m conclusion is shared by both α and α', while the m premise below w is shared by α and not by α'.

In fact, if nether 1 nor 2 is true, then both α and α' should share w, by the sharing equivalence definition.

In case 1 is true, let us choose an axiom l in α above the m premise which is not below w. Of course l is not shared by α', thus by lemma 66 there is a binary & w' and a jump j from w' to l in $G_{\pi'}$. In this case, $w' \sqsupset w$, so violating the hypothesis about w.

In case 2 is true, remark m cannot be a &, otherwise $m \sqsupset w$, so violating the hypothesis about w. Hence m is a $\oplus$, such that α shares the m premise below w, while α' shares the other one. Let us choose an axiom l in α' above m. Of course l is not shared by α, thus by lemma 66 there is a binary & w' and a jump j from w' to l in $G_{\pi'}$. Hence $w' \sqsupset w$, so violating the hypothesis about w.

We conclude that w is shared by all the slices in π'.

Let f be the w conclusion, we define a switching paths chain $\phi_1 \subset \phi_2 \subset \ldots \subset \phi_k$, s.t. ϕ_1 is $\downarrow f$, ϕ_k starts from $\downarrow f$ and ends in a conclusion $\downarrow d$ of π, and for each ϕ_j among $\phi_1, \ldots, \phi_k$:

1. ϕ_j is a strong path in α, for any $\alpha \in \pi'$;

2. for each edge $a \in \phi_j$, let A be the type of a, if $\downarrow a \in \phi_j$ then $\{e_1(a), \dots, e_n(a)\} \in \Gamma^*_\omega(\mathcal{A})$, if $\uparrow a \in \phi_j$ then $\{e_1(a), \dots, e_n(a)\} \in \Gamma^*_\omega(\mathcal{A}^\perp)$.

Clearly $\phi_1 = \downarrow f$ meets both the conditions above, in fact f is shared by α, for any $\alpha \in \pi'$; moreover if $A\&B$ is the type of f, then $\{e_1(f), \dots, e_n(f)\} \in \Gamma^*_\omega(\mathcal{A}\&\mathcal{B})$, since w is binary in $G_{\pi'}$, hence there are $\alpha_i, \alpha_j \in \{\alpha_1, \dots, \alpha_n\}$ s.t. α_i, α_j toggles w, so that any finite set containing $\{e_i(f), e_j(f)\}$ is in $\Gamma^*_\omega(\mathcal{A}\&\mathcal{B})$.

Let us define ϕ_{j+1} from ϕ_j, which we suppose satisfies conditions 1 and 2. Let a be the last edge of ϕ_j, by hypothesis a is shared by α, for any $\alpha \in \pi'$. Then:

- if $\downarrow a \in \phi_j$, then by hypothesis $\{e_1(a), \dots, e_n(a)\} \in \Gamma^*_\omega(\mathcal{A})$:
  - if a is premise of a $⅋$ with conclusion $c : C$, then c is shared by any $\alpha \in \pi'$ and $\{e_1(c), \dots, e_n(c)\} \in \Gamma^*_\omega(\mathcal{C})$. We define $\phi_{j+1} = \phi_j * \downarrow c$;
  - if a is premise of a $\otimes$ with conclusion c and premises a, b. Clearly, b, c are shared by any $\alpha \in \pi'$. In case $\{e_1(c), \dots, e_n(c)\} \in \Gamma^*_\omega(\mathcal{C})$, we define $\phi_{j+1} = \phi_j * \downarrow c$; otherwise $\{e_1(b), \dots, e_n(b)\} \in \Gamma^*_\omega(\mathcal{B}^\perp)$, in this case we define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is premise of an additive link with conclusion $c : C$, we remark that such additive link is unary in $G_{\pi'}$, since by hypothesis all slices $\alpha \in \pi'$ share the premise a. Hence $\{e_1(c), \dots, e_n(c)\} \in \Gamma^*_\omega(\mathcal{C})$ and obviously c is shared by any $\alpha \in \pi'$. We define $\phi_{j+1} = \phi_j * \downarrow c$;
  - if a is premise of a cut with premises a, b, then b is shared by any $\alpha \in \pi'$ and b is labelled by $A^\perp$. By hypothesis $\{e_1(b), \dots, e_n(b)\} \in \Gamma^*_\omega \mathcal{A}$, so we define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is conclusion of π then we define $\phi_j = \phi_k$.
- if $\uparrow a \in \phi_j$, then by hypothesis $\{e_1(a), \dots, e_n(a)\} \in \Gamma^*_\omega(\mathcal{A}^\perp)$:
  - if a is conclusion of a $⅋$ or a $\otimes$, then it exists a premise $b : B$ such that b is shared by any $\alpha \in \pi'$ and $\{e_1(b), \dots, e_n(b)\} \in \Gamma^*_\omega(\mathcal{B}^\perp)$. Define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is conclusion of a &, remark such a & is unary in $G_{\pi'}$. In fact, being by hypothesis $\{e_1(a), \dots, e_n(a)\} \in \Gamma^*_\omega(\mathcal{A}^\perp)$, all slices $\alpha_1, \dots, \alpha_n$ choose the same premise of the &. Hence by the minimality of π', all the slices in π' choose the same premise of the &. Let $b : B$ be such a premise, b is shared by any $\alpha \in \pi'$ and $\{e_1(b), \dots, e_n(b)\} \in \Gamma^*_\omega(\mathcal{B}^\perp)$. Define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if a is conclusion of a $\oplus$ p, we prove that p is unary in $G_{\pi'}$. In fact, suppose there are $\alpha, \alpha' \in \pi'$, such that α (resp. α') chooses the right (resp. left) premise of p. Let us prove a contradiction.

    Choose one axiom l above one right premise of p. Of course l is not shared by α', thus by lemma 66 there is a & w' and a jump j from w' to l in $G_{\pi'}$. Let us prove that $w' \sqsupset w$. Consider the path ψ from l to p. Of course it is switching (indeed it goes downward until p), moreover ψ and ϕ_j are disjoint, since all the edges in ψ (resp. ϕ_j) are not (resp. are) in α'. Denote by $\overline{\phi_j}$ the inverse path of ϕ_j, which starts from $\downarrow a$ to $\uparrow f$. Consider the path $\psi * \overline{\phi_j}$. It is a strong path from l to w, thus showing $w' \sqsupset w$. Of course this violates the hypothesis of w &-free.

We conclude that p is unary in $G_{\pi'}$. Let $b : B$ be its premise, which is shared by any $\alpha \in \pi'$. Clearly $\{e_1(b), \ldots, e_n(b)\} \in \Gamma^*_\omega(\mathcal{B}^\perp)$. Define $\phi_{j+1} = \phi_j * \uparrow b$;

– if a is conclusion of an axiom l, remark that both the conclusions of l are shared by all slices in π', otherwise w would be dominated by an & w' by a similar argument as in the case above. Let $b : B$ be the conclusion of l other than a. Of course $\{e_1(b), \ldots, e_n(b)\} \in \Gamma^*_\omega(\mathcal{B})$, thus define $\phi_{j+1} = \phi_j * \downarrow b$.

Both conditions are respected by each ϕ_j. In particular, since each ϕ_j is a switching path in any $\alpha \in \pi'$ and since π is slice correct, ϕ_j cannot be a cycle. Hence the sequence $\phi_1, \phi_2, \phi_3, \ldots$ will meet eventually a conclusion d of π, so terminating in a path ϕ_k, satisfying the lemma. □

Theorem 68 *Let π be a proof structure. If π is correct, then $[\![\pi]\!]_{\mathfrak{HCoh}^{\mathcal{X}}}$ is a hyperclique for any hypercoherent space $\mathcal{X}$.*

Proof. Let π be a proof net with conclusions $c_1 : C_1, \ldots, c_k : C_k$ and $e_1 : \alpha_1, \ldots, e_n : \alpha_n$ be experiments on slices in π. We have to prove that:

$$\{< e_1(c_1), \ldots, e_1(c_k) >, \ldots, < e_n(c_1), \ldots, e_n(c_k) >\} \in \Gamma_\omega(\mathcal{C}_1 \parr \ldots \parr \mathcal{C}_k)$$

In case $\alpha_1, \ldots, \alpha_n$ are all the same slice, then the statement is a straightforward extension of theorem 24 to the hypercoherent semantics.

Otherwise let $\pi' \subseteq \pi$ be a minimal saturated subset of slices of π such that $\{\alpha_1, \ldots, \alpha_n\} \subseteq \pi'$. Since π is correct, π' is a saturated, slice correct and additive acyclic proof structure. Hence by corollary 64 there is in $G_{\pi'}$ a binary & w which is not dominated by any & in $G_{\pi'}$. By lemma 67 there is a conclusion c_i of π' such that $\{e_1(c_i), \ldots, e_{n'}(c_i)\} \in \Gamma^*_\omega(\mathcal{C}_i)$, which implies the statement. □

We want to study the converse of theorem 68. Unlucky we clash immediately on a counter-example (see figures 2.21 - 2.24):

Proposition 69 *There is a cut-free, full, compatible and slice correct proof structure π such that $[\![\pi]\!]_{\mathfrak{HCoh}^{\mathcal{X}}}$ is a hyperclique for any hypercoherent space $\mathcal{X}$, but π is not correct.*

Proof. Let π be the cut-free proof structure with conclusions $X^\perp$, $X^\perp$, $(X\&X) \otimes (X\&X)$, $X \oplus X$, $X^\perp$ and slices $\beta_1, \ldots, \beta_4$ defined in figures 2.21 - 2.24

π is compatible, full and slice correct (although it is not strongly slice correct). π is not correct, since there is a switching cycle in G_π crossing both the & (see figure 2.25).

Nevertheless $[\![\pi]\!]$ is a hyperclique. Indeed let us show that any finite set of π experiences results is hypercoherent in the space associated with the π conclusions.

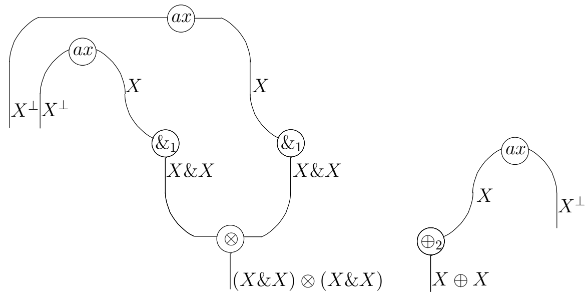


Figure 2.21: slice β_1 with conclusions $X^\perp$, $X^\perp$, $(X\&X)\otimes(X\&X)$, $X\oplus X$, $X^\perp$.

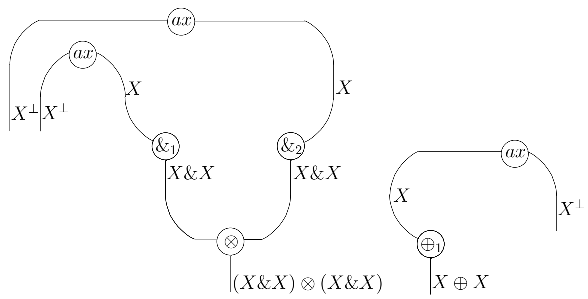


Figure 2.22: slice β_2 with conclusions $X^\perp$, $X^\perp$, $(X\&X)\otimes(X\&X)$, $X\oplus X$, $X^\perp$.

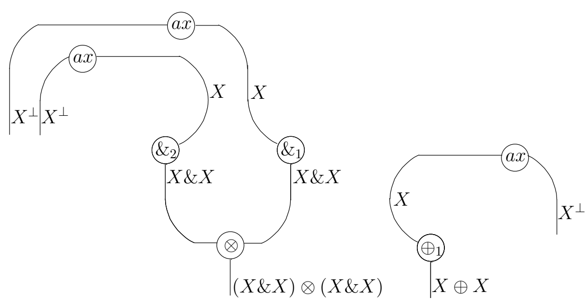


Figure 2.23: slice β_3 with conclusions $X^\perp$, $X^\perp$, $(X\&X)\otimes(X\&X)$, $X\oplus X$, $X^\perp$.

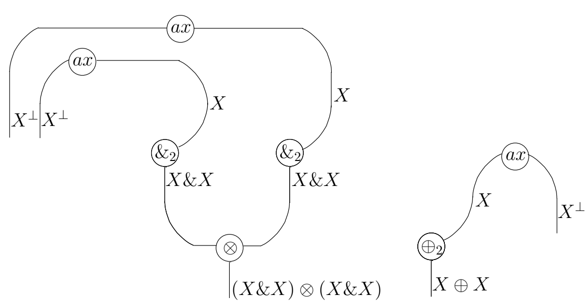


Figure 2.24: slice β_4 with conclusions $X^\perp$, $X^\perp$, $(X\&X)\otimes(X\&X)$, $X\oplus X$, $X^\perp$.

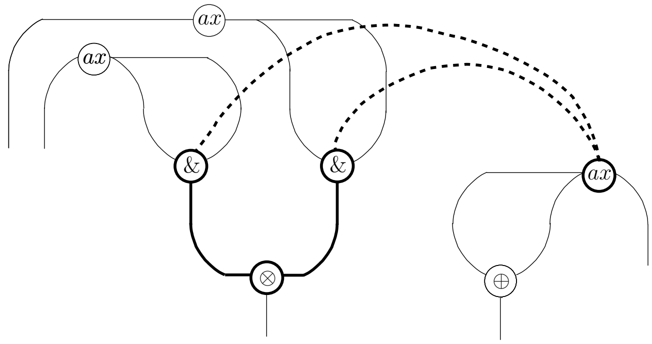


Figure 2.25: $G_{\{\beta_1,\ldots,\beta_4\}}$, the bold lines define a switching cycle crossing both the &.

Let us consider a finite set of π experiences $e_1, \ldots, e_n$.

In case $e_1, \ldots, e_n$ are defined on a unique slice of π, then the $e_1, \ldots, e_n$ results are hypercoherent, being π slice correct.

In case $e_1, \ldots, e_n$ are defined on more than one slice of π, let $a : X\&X$ be the conclusion of a & toggled by the slices on which $e_1, \ldots, e_n$ are defined, c be the π conclusion of type $(X\&X) \otimes (X\&X)$ and b_1, b_2 be the two conclusions of type $X^\perp$ of the axioms predecessor of the two &.

By the & hypercoherent definition, $\{e_1(a), \ldots, e_n(a)\} \in \Gamma^*_\omega(\mathcal{X}\&\mathcal{X})$. Thus $\{e_1(c), \ldots, e_n(c)\} \in \Gamma^*_\omega((\mathcal{X}\&\mathcal{X}) \otimes (\mathcal{X}\&\mathcal{X}))$ or there is an $i \in \{1, 2\}$, such that $\{e_1(b_i), \ldots, e_n(b_i)\} \in \Gamma^*_\omega(\mathcal{X}^\perp)$. In both cases we conclude that the results of $e_1, \ldots, e_n$ are strictly hypercoherent. □

Notice that the counter-example in figures 2.21 - 2.24 depends on the jumps of G_π, which connect the two components of the sharing graph of π. Since the jumps are semantically invisible, hypercoherent spaces do not see the switching cycle crossing both the & in G_π. But what happens if π is strongly slice correct, and not only slice correct?

In [BHS05], Blute, Hamano and Scott study the correspondence between hypercliques and **MALL** correctness in the framework of the proof nets introduced by Girard in [Gir96]. As written in subsection 2.3.1, we have not used such proof nets since they are not canonical. Anyway, Blute, Hamano and Scott prove for Girard's proof nets that the implication $[\![\pi]\!]$ *hyperclique* $\Rightarrow \pi$ *correct* holds in case π is without mix, i.e. in case π is strongly slice correct. Thus we guess:

Conjecture 70 *Let π be a cut-free, full, compatible and strongly slice correct proof structure. If $[\![\pi]\!]_{\mathfrak{HCoh}^{\mathcal{X}}}$ is a hyperclique for any hypercoherent space $\mathcal{X}$, then π is correct.*

Chapter 3

Exponentials

In this chapter we study the proof nets for the multiplicative exponential fragment of linear logic (briefly **MELL**).

In section 3.1 we introduce **MELL** proof nets.

In section 3.2 we recall the multiset based uniform coherent semantics ($\mathfrak{Coh}$) and the non-uniform one ($\mathfrak{nuCoh}$). $\mathfrak{Coh}$ has been introduced by Girard in [Gir91], while $\mathfrak{nuCoh}$ is a more recent semantics defined by Bucciarelli and Ehrhard in [BE01].

In section 3.3 we attack the question of the injectivity of $\mathfrak{Coh}$ for **MELL** proof nets. In subsection 3.3.1, we define a counter-example to the $\mathfrak{Coh}$ injectivity for the polarized fragment of **MELL**, which had been conjectured in [TdF03b]. In subsections 3.3.2, 3.3.3 instead we prove the injectivity of $\mathfrak{Coh}$ for the so-called ($?\parr$)-**MELL** proof nets (theorem 100). Theorem 100 has been proved in [TdF03b], the main novelty of our approach is to provide a different proof by means of lemma 98, based on Girard's notion of longtrip.

In section 3.4 we solve the open question of characterizing those proof structures whose interpretation is a clique in $\mathfrak{nuCoh}$ (theorems 103, 104). Such a characterization provides a new geometric criterion on **MELL** proof structures: the *weak correctness* (definition 102).

The formulas of **MELL** are defined by the following grammar:

$$F ::= \quad X \quad | \quad X^{\perp} \quad | \quad F \parr F \quad | \quad F \otimes F \quad | \quad ?F \quad | \quad !F$$

As always we set $(?F)^{\perp} = !F^{\perp}$ and $(!F)^{\perp} = ?F^{\perp}$.

The rules of the **MELL** sequent calculus are those for **MLL** extended by the following rules for the exponentials:

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma}\ c$$

where $?\Gamma$ means a multiset of ?-formulas. The top rule is called **of course** rule, the bottom ones are called respectively (from left to right) **weakening**, **dereliction** and **contraction**.

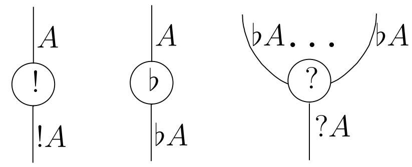


Figure 3.1: **MELL** links.

The exponentials have the crucial rôle of introducing the structural rules in linear logic: weakening and contraction. By such rules linear logic preserve the expressive power of classical logic, still keeping its constructive feature.

The **MELL** proof nets provide a common and powerfull framework for analyzing both intuitionistic (i.e. typed λ-calculus) and classical logic.

3.1 Proof structures and proof nets

In this section we introduce the **MELL** proof nets. The section is divided in three paragraphs. In the first one, called *proof structures*, we define the **MELL** proof structures - morally inductive frames of multiplicative proof structures. In the second paragraph, called *cut reduction*, we introduce the reduction rule for the exponential cut. Such a reduction is not local, allowing to erase or duplicate broad pieces of a proof structure. In the third paragraph, called *proof nets*, we define the **MELL** proof nets and the extension to **MELL** of the Danos-Regnier's correctness criterion.

Proof structures. By following [LTdF04], we introduce the $\flat$-formulas, which allow a sharper definition of the exponential links. A **$\flat$-formula** is a **MELL** formula prefixed by the symbol $\flat$, as for example $\flat A$. The $\flat$-formulas do not compose with the logical connectives, they just label the premises of a link ?.

We will often use induction on the complexity of a formula. We precise that we consider a $\flat$-formula $\flat A$ more complex than A, but simpler than $?A$.

The **MELL** links are defined extending the **MLL** ones with the following **exponential links** (figure 3.1):

1. the **of course** (! link), which has one premise and one conclusion. If the premise is labelled by a formula A, then the conclusion is labelled by the formula $!A$;

2. the $\flat$ link, which has one premise and one conclusion. If the premise is labelled by a formula A, then the conclusion is labelled by the $\flat$-formula $\flat A$;

3. the **why not** (? link), which has n unordered premises ($n \geq 0$) labelled by a same $\flat$-formula $\flat A$, and one conclusion labelled by $?A$.

To sum up, the **MELL** links are divided in three groups: the structural links (axiom and cut), the multiplicative links ($⅋$ and $\otimes$) and the exponential ones ($!$, $\flat$ and $?$).

Remark that the link ? gathers in a unique link the rules of weakening, dereliction and contraction. We keep however the names of such rules, calling **weakening** a ? link with arity 0, **dereliction** a ? link with arity 1, and **contraction** a ? link with arity greater than 1.

Notice that every formula F is conclusion of a unique link introducing F, in particular recall that compound formulas do not label conclusions of axioms.

A set of **MELL** links π is a **proof structure** if the following conditions hold:

linearity: every edge is conclusion of exactly one link and premise of at most one link. The edges which are not any link premise are the **conclusions** of the proof structure;

exponential box: with every ! link o is associated a unique subgraph π^o of π satisfying the linearity condition and s.t. one π^o conclusion is the o premise and all further π^o conclusions are labeled by $\flat$-formulas. π^o is called the **exponential box of o** (or simply the box of o) and it is represented by a rectangular frame. The o conclusion is called π^o **principal door**, while the π^o conclusions labelled by $\flat$-formulas are called π^o **auxiliary doors**;

nesting: two exponential boxes are either disjoint or included one in the other.

Remark that a ? conclusion cannot be an auxiliary door of an exponential box. Indeed the notion of $\flat$-formulas allows to push the ? links below the frames of the exponential boxes.

The tricks of pushing down the ? and of gathering in a unique link weakening, dereliction and contraction are a well-known way for providing a more canonical representation of the exponential rules. Such tricks are due to Danos and Regnier (see for example [Reg92]) and they are mentioning as the *nouvelle syntax*.

Let us briefly recall the **MLL** notation. Proof structures are denoted by Greek letters: $\pi, \sigma, \ldots$, the edges by initial Latin letters: $a, b, c \ldots$ and the links by middle-position Latin letters: $l, m, n, o \ldots$. We write $a : A$ if a is an edge labeled by the formula A.

We define by $\mathbf{PS^{me}}$ the set of **MELL** proof structures.

Formally **an edge** a **is above another edge** b (denoted $a \geq b$) if a is equal or above a premise of the link of which b is conclusion.

A link l of π is **terminal** if:

- in case l is not a link !, then all the conclusions of l are conclusions of π;
- in case l is a link !, then all the doors of the box associated with l are conclusions of π.

The **depth of a link** in a proof structure is the number of boxes in which it is contained. The exponential depth of an edge a is 0 in case a is a conclusion of the proof structure, otherwise it is the depth of the link whose premise is a. Remark that an edge conclusion of a link $\flat$ at depth n and premise of a link ? at depth $m \leq n$ has depth m.

The **depth of a proof structure** is the maximal depth of its links.

Cut reduction. A proof structure without cuts is called **cut free**.

The **MELL cut reduction rules** are an extension of the **MLL** ones. Remark that an **MELL** cut l can be:

- an axiom cut, whose premises are labeled by dual atomic formulas X and $X^\perp$;
- a $⅋/\otimes$ cut, whose premises are labeled by dual multiplicative formulas $A ⅋ B$ and $A^\perp \otimes B^\perp$;
- a $!/?$ cut, whose premises are labeled by dual exponential formulas $!A$ and $?A^\perp$.

In case l is an axiom or a $⅋/\otimes$ cut, we reduce l as in the **MLL** proof structures (see subsection 1.1, figures 1.2-1.4).

In case l is a $!/?$ cut, let o be the $!$ link of which the conclusion is the premise of l labeled by $!A$ and let w be the $?$ link of which the conclusion is the premise of l labeled by $?A^\perp$. We reduce l only in case no auxiliary conclusion of the box of o is a premise of w.

Let $a : A$ be the o premise, π^o be the o box and $b_1 : \flat B_1, \ldots, b_k : \flat B_k$ be the π^o auxiliary doors ($k \geq 0$). Remark that each w premise, in case it exists any, is conclusion of a $\flat$ link. Thus, let $a'_1 : A, \ldots, a'_n : A$ ($n \geq 0$) be the premises of the $\flat$ links of which conclusion is a premise of w. Suppose that no a'_i, for $i \leq n$, is a π^o auxiliary door. Under this hypothesis the cut l is reduced in three steps (see figure 3.2):

1. erase l, w, their premises, the $\flat$ links immediately above w, o and its box π^o;
2. for each i, $1 \leq i \leq n$, define π^o_i as a copy of π^o with conclusions $a_i : A, b_{i,1} : \flat B_1, \ldots, b_{i,k} : \flat B_k$. For each $j \leq k$ set $b_{j,k}$ be the premise of the same $?$ link with premise b_k in π;
3. for each i, $1 \leq i \leq n$, put π^o_i in the boxes containing a'_i, increasing their auxiliary doors by the π^o_i conclusions $b_{i,1}, \ldots, b_{i,k}$. Finally, joint π^o_i with a'_i by adding a cut link l_i with premises $a_i : A$ and $a'_i : A^\perp$.

We write $\pi \rightsquigarrow_\beta \pi'$ if π' is the result of a reduction of a cut in π. As always, $\rightarrow_\beta$ is the reflexive and transitive closure of $\rightsquigarrow_\beta$ and $=_\beta$ is the symmetrical closure of $\rightarrow_\beta$.

The reduction $\rightsquigarrow_\beta$ is not defined on the cuts whose premises come from the same exponential box, as for example in figure 3.3. We call such cuts **deadlocks**.

Remark that $\rightarrow_\beta$ is not confluent at the level of **MELL** proof structures. For example consider the proof structure of figure 3.4. By reducing the cut with premises $!A$, $?A^\perp$ we get a deadlock with premises $!B, ?B^\perp$; vice-versa, by reducing the cut with premises $!B, ?B^\perp$ we get a deadlock with premises $!A, ?A^\perp$.

Proof nets. The proofs of **MELL** sequent calculus can be translated into proof structures by means of a desequentialization function, denoted by $(\ )^\bullet$. The **MELL desequentialization** is an immediate extension of the **MLL** one.

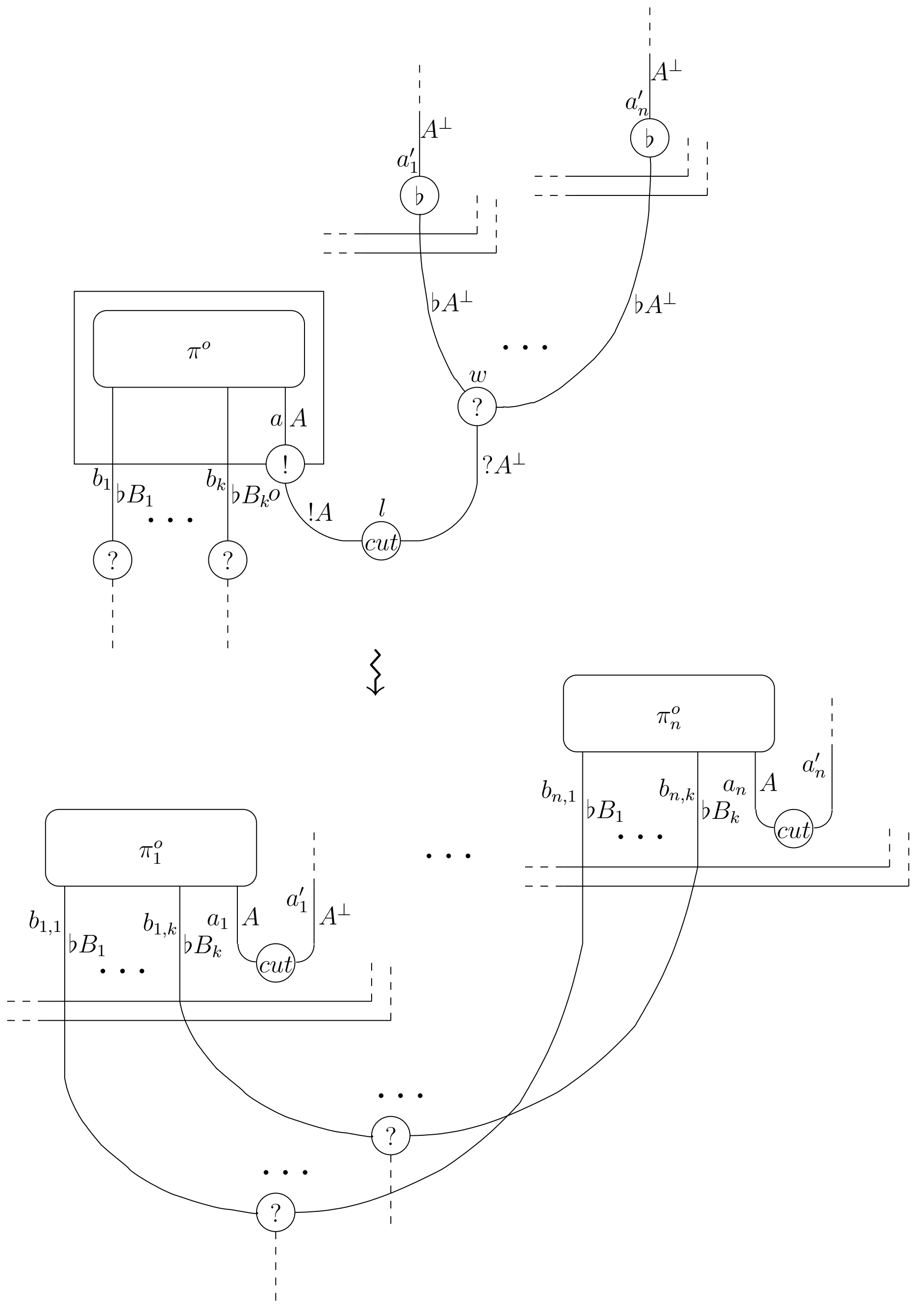


Figure 3.2: !/? cut reduction.

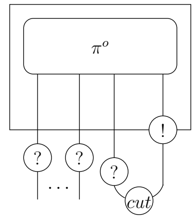


Figure 3.3: example of deadlock.

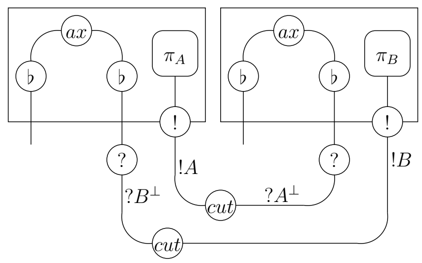


Figure 3.4: counter-example to the confluence of cut reduction for **MELL** proof structures.

If σ is a sequent proof then $(\sigma)^\bullet$ is defined by induction on σ. In case σ ends with a **MLL** rule then $(\sigma)^\bullet$ is defined as in section 1.2. In case σ ends in an exponential rule then $(\sigma)^\bullet$ is defined as follows:

- if σ ends in a !-rule having as premise the subproof σ', let $\vdash ?B_1, \dots, ?B_k, A$ ($k \geq 0$) be the σ' conclusion and $b_1 :?B_1, \dots, b_k :?B_k, a : A$ be the conclusions of $(\sigma')^\bullet$. For each $i \leq k$, let $b_{i,1} : \flat B_i, \dots, b_{i,n_i} : \flat B_i$ ($n_i \geq 0$) be the premises of the ? link with conclusion b_i. $(\sigma)^\bullet$ is obtained from $(\sigma')^\bullet$ in three steps:

  - erase the $(\sigma')^\bullet$ edges $b_1, \dots, b_k$ and the ? links of which they are conclusions. Call π' the graph so obtained;
  - add a ! link with premise a and conclusion a new edge $a' :!A$, set π' as the exponential box associated with the added ! link;
  - for each $i \leq k$, add a new ? link with premises $b_{i,1}, \dots, b_{i,n_i}$ and conclusion a new edge $b_i :?B_i$;

- if σ ends in a weakening rule having as premise the subproof σ', then $(\sigma)^\bullet$ is obtained by adding to $(\sigma')^\bullet$ the weakening link correspondent to the sequent rule;

- if σ ends in a dereliction rule having as premise the subproof σ', let $\vdash \Gamma, A$ be the σ' conclusion and $\Gamma, a : A$ be the conclusions of $(\sigma')^\bullet$. $(\sigma)^\bullet$ is obtained by adding to $(\sigma')^\bullet$ a $\flat$ link with premise a and conclusion a new edge $a' : \flat A$ and a ? link with premise a' and conclusion a new edge $a'' :?A$;

- if σ ends in a contraction rule having as premise the subproof σ', let $\vdash \Gamma, ?A, ?A$ be the conclusion of σ' and $\Gamma, a_1 :?A, a_2 :?A$ be the conclusions of $(\sigma')^\bullet$. For $i = 1, 2$, let $a_{i,1} : \flat A, \ldots, a_{i,n_i} : \flat A$ ($n_i \geq 0$) be the premises of the ? link with conclusion a_i. $(\sigma)^\bullet$ is obtained from $(\sigma')^\bullet$ in two steps:
  - erase the $(\sigma')^\bullet$ edges a_1, a_2 and the ? links of which they are conclusions;
  - add a ? link with premises $a_{1,1}, \ldots, a_{1,n_1}, a_{2,1}, \ldots, a_{2,n_2}$ and conclusion a new edge $a :?A$.

Remark that the desequentialization pushes the weakening and contraction rules down the !-rules: this is the peculiarity of the proof structures nouvelle syntax.

A **MELL proof net** π is a proof structure associated with a sequent proof, moreover π is said **without mix** if an associated sequent proof does not contain the mix rule.

We denote by $\mathbf{PN^{mex}}$ (resp. $\mathbf{PN^{me}}$) the set of proof nets (resp. of proof nets without mix). Of course:

$$PN^{me} \subset PN^{mex} \subset PS^{me}$$

The sets PN^{me} and PN^{mex} are not easily characterizable by a correctness criterion, because of the weakening link. We do not enter in the details of the problem, for which we refer to [TdF03a] and [TdF00]. Instead we recall a simple extension of the **MLL** correctness criterion, characterizing the proof structures sequentializable in **MELL** sequent calculus enlarged with the following **daimon rule**:

$$\frac{}{\vdash ?F}\ dai$$

We denote by $\mathbf{PN^{mexd}}$ the set of proof nets sequentializable in **MELL** sequent calculus enlarged with the daimon.

For extending the criterion introduced in definition 19, we adapt the definition of the oriented paths to the framework of exponential boxes.

We do not consider paths crossing edges of different exponential depths: if a path enters in a box π^o through a door $\uparrow a$, it have to exit immediately through a door $\downarrow b$. Stated otherwise, for a path a box is a node, whose incident edges are the doors of the box.

An **oriented edge** is an edge together with a direction *upward*, denoted by $\uparrow a$, or *downward*, denoted by $\downarrow a$. We write $\updownarrow a$ in case we do not want to specify if we mean either $\uparrow a$ or $\downarrow a$. An **oriented path** (or simply path) from $\updownarrow a_0$ to $\updownarrow a_n$ in a proof structure π is a sequence of oriented edges $<\updownarrow a_0, \ldots, \updownarrow a_n>$ such that for any $i < n$, $\updownarrow a_i, \updownarrow a_{i+1}$ have the same depth and:

- if $\updownarrow a_i =\uparrow a_i$, $\updownarrow a_{i+1} =\uparrow a_{i+1}$, then a_i is conclusion of the link of which a_{i+1} is premise;
- if $\updownarrow a_i =\uparrow a_i$, $\updownarrow a_{i+1} =\downarrow a_{i+1}$, then a_i and a_{i+1} are conclusions of the same link, or they are doors of the same exponential box;
- if $\updownarrow a_i =\downarrow a_i$, $\updownarrow a_{i+1} =\downarrow a_{i+1}$, then a_i is the premise of the link of which a_{i+1} is conclusion;

- if $\updownarrow a_i = \downarrow a_i$, $\updownarrow a_{i+1} = \uparrow a_{i+1}$, then a_i and a_{i+1} are premises of the same link;

morally $\updownarrow a_i = \uparrow a_i$ (resp. $\updownarrow a_i = \downarrow a_i$) when the path crosses the edge a_i from the link it is conclusion (resp. premise) to the link it is premise (resp. conclusion). We say that **a path crosses a link l** if it contains a sequence of two edges having l as a vertex.

We denote paths by Greek letters $\phi, \tau, \psi, \ldots$. We write $\updownarrow a \in \phi$ to mean that $\updownarrow a$ occurs in ϕ, sometimes we write simply $a \in \phi$ for meaning that $\uparrow a$ or $\downarrow a$ occurs in ϕ. We denote by $\psi \sqsubseteq \phi$ when ψ is a subpath of ϕ. We may denote a path $<\updownarrow a_0, \ldots, \updownarrow a_n>$ by a simple succession of oriented edges, i.e. $\updownarrow a_0 \ldots \updownarrow a_n$.

A path ϕ **comes back** if there is an edge a s.t. $\uparrow a, \downarrow a \in \phi$. A **cycle** is a path from $\updownarrow a$ to $\updownarrow a$.

A **switching edge** is a premise of a link $\parr$ or ?. A path ϕ is **switching** if ϕ never comes back and it does not contain two switching edges of a same link.

Definition 71 *A* **MELL** *proof structure is* ***correct*** *if it does not contain any switching cycle.*

It is well-known that such a correctness criterion characterizes PN^{mexd} (see for example [TdF00]):

Theorem 72 *Let* $\pi \in PS^{me}$. $\pi \in PN^{mexd}$ *iff* π *is correct.*

The correctness guarantees also nice properties with respect to cut reduction[1]:

Theorem 73 (Stability) *Let* $\pi \rightarrow_\beta \pi'$, *if* π *is correct then* π' *is correct.*

Theorem 74 (Confluence) *If* π_1 *is a correct proof structure s.t.* $\pi_1 \rightarrow_\beta \pi_2$ *and* $\pi_1 \rightarrow_\beta \pi_3$, *then there is a correct proof structure* π_4, *s.t.* $\pi_2 \rightarrow_\beta \pi_4$ *and* $\pi_3 \rightarrow_\beta \pi_4$.

Theorem 75 (Strong normalization) *For every correct proof structure* π, *there is no infinite sequence of proof structures* $\pi_0, \pi_1, \pi_2, \ldots$ *s.t.* $\pi_0 = \pi$ *and* $\pi_i \rightsquigarrow_\beta \pi_{i+1}$.

Before concluding the section, let us stress the fact that in absence of weakening, both PN^{mex} and PN^{me} are easily characterizable by extending the notion of correctness graph to **MELL**.

A **correctness graph** of a **MELL** proof structure π is a graph obtained from π in two steps:

- for each ! link o at depth 0, make the auxiliary doors of π^o new conclusions of o and erase all other edges and links of π^o;
- for each ? and $\parr$ link at depth 0, erase all its premises except one.

Of course a proof structure π is correct in the sense of definition 71 if and only if the correctness graph of π as well as the correctness graphs of its boxes are acyclic.

[1] For the proofs of the following theorems we refer to [Dan90].

Definition 76 *A* **MELL** *proof structure π is* ***strongly correct*** *if the correctness graph of π as well as the correctness graphs of its boxes are acyclic and connected.*

Then we have the following theorem:[2]

Theorem 77 *Let π be a* **MELL** *proof structure without weakening. $\pi \in PN^{mex}$ (resp. $\pi \in PN^{me}$) iff π is correct (resp. strongly correct).*

3.2 MELL coherent spaces

The exponentials change the web of a space from a set of points to a set of sets (or multisets) of points. That is the web of a space associated with a formula $!A$ is composed by sets (or multisets) of elements of the web associated with A. In this way the semantics interprets the fact that $!A$ stands potentially for $n \geq 0$ copies of A, in the sense that the reduction of a cut may duplicate or erase the occurrences of $!A$.

Moreover, a semantics can memorize the fact that such $!A$ copies morally come from a single occurrence of $!A$, or it may forget it. In the former case, a set (or a multiset) in the web of $!A$ must be composed by elements of the web of A which are in some sense *uniform*, so we speak of a *uniform semantics*. Instead if any set (or multiset) of elements of the web of A is in the web of $!A$, then we speak of a *non-uniform semantics*.

In this section we define both the uniform and non-uniform coherent semantics for **MELL**. We will deal only with semantics based on multisets, omitting the definition of the set-based coherent spaces.[3]

The main difference between uniform and non-uniform coherent semantics is precisely in the definition of the web of $!\mathcal{A}$. The non-uniform web of $!\mathcal{A}$ contains all finite multisets of elements in $\mathcal{A}$, while the uniform web of $!\mathcal{A}$ contains only those finite multisets whose elements are pairwise coherent in $\mathcal{A}$.

The uniform coherent semantics based on multisets has been introduced by Girard in [Gir91], while the non-uniform one has been defined by Bucciarelli and Ehrhard in [BE01]. Actually we will deal with a variant of Bucciarelli and Ehrhard's semantics, which is due to Boudes (see [Bou02]).

3.2.1 Uniform coherent spaces

The coherent spaces defined in subsection 1.2.1 provide a semantics for **MELL**. Let $\mathcal{X}$ be a coherent space, a **coherent model on** $\mathcal{X}$, denoted by $\mathfrak{Coh}^{\mathtt{X}}$, associates with **MELL** formulas coherent spaces, defined by induction on the formulas:

- with X it is associated $\mathcal{X}$;
- with $A^{\perp}$ it is associated $\mathcal{A}^{\perp}$ defined as follows: $|\mathcal{A}^{\perp}| = |\mathcal{A}|$, the coherence of $\mathcal{A}^{\perp}$ is the incoherence of $\mathcal{A}$, i.e. $x \frown\!\!\!\smile y \, [\mathcal{A}^{\perp}]$ iff $x \smile\!\!\!\frown y \, [\mathcal{A}]$;

[2] For a proof see [TdF00].

[3] Actually, it is well-known that the set-based non-uniform coherent spaces do not provide a semantics for **MELL**.

- with $A \otimes B$ it is associated $\mathcal{A} \otimes \mathcal{B}$ defined as follows: $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ and $< a, b > \frown\!\!\!\!\smile < a', b' > [\mathcal{A} \otimes \mathcal{B}]$ iff $a \frown\!\!\!\!\smile a' [\mathcal{A}]$ and $b \frown\!\!\!\!\smile b' [\mathcal{B}]$
- with $!A$ it is associated the following $!\mathcal{A}$. The web of $!\mathcal{A}$ is so defined:

$$|!\mathcal{A}| = \{v \in M_{fin}(|\mathcal{A}|) \mid Supp(v) \text{ is a clique of } \mathcal{A}\}$$

the strict incoherence of $!\mathcal{A}$ is the following: $v \smile u \, [!\mathcal{A}]$ iff $\exists a \in v$ and $\exists a' \in u$, s.t. $a \smile a' \, [\mathcal{A}]$.

Of course, the space $\mathcal{A} \parr \mathcal{B}$ is defined by $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$ as well as $?\mathcal{A}$ is defined by $(!\mathcal{A}^\perp)^\perp$. We associate with a $\flat$-formula $\flat A$ the space $?\mathcal{A}$.

The web of $!\mathcal{A}$ expels those multisets whose support is not a clique of $\mathcal{A}$. Actually, such a condition is necessary for guaranteeing the anti-reflexivity of $\smile$ in $!\mathcal{A}$. Indeed, let a,b be two elements of $\mathcal{A}$ s.t. $a \smile b \, [\mathcal{A}]$. Consider the multiset $v = [a, b]$: if v were in the web of $!\mathcal{A}$, then $\smile$ would not be anti-reflexive, being $v \smile v \, [!\mathcal{A}]$.

The coherent model $\mathfrak{Coh}^{\mathsf{X}}$ is called *uniform*, in the sense that any multiset in the web of $!\mathcal{A}$ is composed by pairwise coherent, i.e. *uniform*, elements of $\mathcal{A}$. Such a uniformity gives a mark to the way coherence spaces interpret **MELL** proof nets. We will deal with this question in section 3.3.

For each proof structure π, we define the **interpretation of** π **in** $\mathfrak{Coh}^{\mathsf{X}}$, denoted by $[\![\pi]\!]_{\mathfrak{Coh}^{\mathsf{X}}}$, where the index $\mathfrak{Coh}^{\mathsf{X}}$ is omitted when it is clear which is the model we refer to.

In case π has no conclusion, let $[\![\pi]\!]$ set as undefined. Otherwise, let $c_1 : C_1, \ldots, c_n : C_n$ be the conclusions of π, $[\![\pi]\!]$ is a subset of $\mathcal{C}_1 \parr \ldots \parr \mathcal{C}_n$, defined by using an extension of the **MLL** experiments introduced in definition 4.

We define an experiment e on π by induction on the depth of π. Remark that the following definition of **MELL** experiment is slightly different from the usual one (see for example [TdF03b]), namely e is defined only on the edges at depth 0 of π.

Definition 78 *A* $\mathfrak{Coh}^{\mathsf{X}}$ ***experiment*** **e** ***on a*** **MELL** ***proof structure*** π*, denoted by* $e : \pi$*, is a function associating with every* $!$ *link* o *at depth* 0 *a multiset* $[e_1^o, \ldots, e_k^o]$ *of experiments on* π^o*, and with every edge* $a : A$ *at depth* 0 *an element of* $\mathcal{A}$*, such that the following conditions are respected:*

axiom: *if* a, b *are the conclusions of an axiom at depth* 0*, then* $e(a) = e(b)$*;*

cut: *if* a, b *are the premises of a cut at depth* 0*, then* $e(a) = e(b)$*;*

multiplicative: *if* c *is the conclusion of a* $\parr$ *or* $\otimes$ *at depth* 0 *with premises* a *and* b*, then* $e(c) =< e(a), e(b) >$*;*

flat: *if* c *is the conclusion of a* $\flat$ *at depth* 0 *with premise* a*, then* $e(c) = [e(a)]$*;*

why not: *if* c *is the conclusion of a* $?$ *at depth* 0 *with premises* $a_1, \ldots, a_n$*, then* $e(c) = e(a_1) + \ldots + e(a_n)$*. In case* $n = 0$*, then* $e(c) = \emptyset$*;*

exponential doors: *if* c *is a door of a box associated with a* $!$ *link* o *at depth* 0*, let* a *be the* o *premise and* $e(o) = [e_1^o, \ldots, e_k^o]$*. If* c *is the principal door then* $e(c) = [e_1^o(a), \ldots, e_k^o(a)]$*, if* c *is an auxiliary door then* $e(c) = e_1^o(c) + \ldots + e_k^o(c)$*;*

uniformity condition: *if c is an edge at depth 0 labelled by a formula $!\mathcal{C}$, either $\flat\mathcal{C}^\perp$ or $?\mathcal{C}^\perp$, then Supp(e(c)) is a clique of $\mathcal{C}$.*

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$ and $e : \pi$ be an experiment, then the **result of e**, denoted by $|e|$, is the element $< e(c_1), \ldots, e(c_n) >$ of $\mathcal{C}_1 \parr \ldots \parr \mathcal{C}_n$. The **interpretation of** π **in** $\mathfrak{Coh}^{\mathcal{X}}$ is the set of the results of all the experiments on π:

$$[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = \left\{ < e(c_1), \ldots, e(c_n) > \mid e \text{ is a } \mathfrak{Coh}^{\mathcal{X}} \text{ experiment on } \pi \right\}$$

The $\mathfrak{Coh}^{\mathcal{X}}$ interpretation of a proof structure is invariant under cut-reduction:

Theorem 79 (Soundness of $[\![\ ]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$) *For every π, π' proof structures, $\pi =_\beta \pi'$ implies $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi']\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.*

Proof. The proof is an immediate extension of the original proof given by Girard in [Gir87] of the soundness of $[\![\ ]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ for the cut reduction on proof nets. □

3.2.2 Non-uniform coherent spaces

In this subsection we recall the non-uniform coherent spaces, which provide a non-uniform semantics for **MELL**.

Definition 80 ([BE01]) *A **non-uniform coherent space** $\mathcal{X}$ is a triple $(|\mathcal{X}|, \frown\!\!\!\!\smile, \smile\!\!\!\!\frown)$, where $|\mathcal{X}|$ is a set, while $\frown\!\!\!\!\smile$ and $\smile\!\!\!\!\frown$ are two binary symmetric relations on $|\mathcal{X}|$, such that for every $x, y \in \mathcal{X}$, $x \frown\!\!\!\!\smile y$ or $x \smile\!\!\!\!\frown y$.*

*A **clique** of $\mathcal{X}$ is a subset $\mathtt{C}$ of $|\mathcal{X}|$ such that for every $x, y \in \mathtt{C}$, $x \frown\!\!\!\!\smile y$.*

Remark the difference with the uniform coherent spaces: we do not require $\frown\!\!\!\!\smile$ to be also reflexive.

$|\mathcal{X}|$ is the **web** of $\mathcal{X}$, while $\frown\!\!\!\!\smile$ (resp. $\smile\!\!\!\!\frown$) is its **coherence** (resp. **incoherence**). We will write $x \frown\!\!\!\!\smile y\,[\mathcal{X}]$ and $x \smile\!\!\!\!\frown y\,[\mathcal{X}]$ if we want to explicit the coherent space $\frown\!\!\!\!\smile$ and $\smile\!\!\!\!\frown$ refer to. We introduce the following notation, well-known in the framework of coherent spaces:

neutrality: $x \equiv y\,[\mathcal{X}]$, if $x \frown\!\!\!\!\smile y\,[\mathcal{X}]$ and $x \smile\!\!\!\!\frown y\,[\mathcal{X}]$;

strict coherence: $x \frown y\,[\mathcal{X}]$, if $x \frown\!\!\!\!\smile y\,[\mathcal{X}]$ and $x \not\equiv y\,[\mathcal{X}]$;

strict incoherence: $x \smile y\,[\mathcal{X}]$, if $x \smile\!\!\!\!\frown y\,[\mathcal{X}]$ and $x \not\equiv y\,[\mathcal{X}]$.

Remark that $\equiv$ is the intersection of $\frown\!\!\!\!\smile$ and $\smile\!\!\!\!\frown$, $\frown$ is the opposite of $\smile\!\!\!\!\frown$, and $\smile$ the opposite of $\frown\!\!\!\!\smile$. Therefore we may define a non-uniform coherent space specifying its web and two well chosen relations among $\equiv$, $\frown\!\!\!\!\smile$, $\frown$, $\smile\!\!\!\!\frown$, $\smile$.

Let $\mathcal{X}$ be a non-uniform coherent space, a **non-uniform coherent model on** $\mathcal{X}$ ($\mathfrak{nuCoh}^{\mathsf{X}}$) associates with formulas non-uniform coherent spaces, by induction on the formulas, as follows:

- with X it is associated $\mathcal{X}$;

- with $A^\perp$ it is associated $\mathcal{A}^\perp$, defined as follows: $|\mathcal{A}^\perp| = |\mathcal{A}|$, the neutrality and coherence of $\mathcal{A}^\perp$ are the following:
  - $a \equiv a' \left[\mathcal{A}^\perp\right]$ iff $a \equiv a' \, [\mathcal{A}]$,
  - $x \stackrel{\frown}{\smile} y \left[\mathcal{A}^\perp\right]$ iff $x \stackrel{\smile}{\frown} y \, [\mathcal{A}]$;
- with $A \otimes B$ it is associated $\mathcal{A} \otimes \mathcal{B}$, defined as follows: $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$, the neutrality and coherence of $\mathcal{A} \otimes \mathcal{B}$ are the following:
  - $< a, b > \equiv < a', b' > [\mathcal{A} \otimes \mathcal{B}]$ iff $a \equiv a' \, [\mathcal{A}]$ and $b \equiv b' \, [\mathcal{B}]$,
  - $< a, b > \stackrel{\frown}{\smile} < a', b' > [\mathcal{A} \otimes \mathcal{B}]$ iff $a \stackrel{\frown}{\smile} a' \, [\mathcal{A}]$ and $b \stackrel{\frown}{\smile} b' \, [\mathcal{B}]$;
- with $!A$ it is associated $!\mathcal{A}$, defined as follows: $|!\mathcal{A}| = M_{fin}(|\mathcal{A}|)$, the strict incoherence and neutrality of $!\mathcal{A}$ are the following:
  - $v \smile u \, [!\mathcal{A}]$ iff $\exists a \in v$ and $\exists a' \in u$, s.t. $a \smile a' \, [\mathcal{A}]$,
  - $v \equiv u \, [!\mathcal{A}]$ iff not $v \smile u \, [!\mathcal{A}]$ and there is a v (resp. u) enumeration $v = [a_1, \ldots, a_n]$ (resp. $u = [a'_1, \ldots, a'_n]$), s.t. for each $i \leq n$, $a \equiv a' \, [\mathcal{A}]$.

Of course, the space $\mathcal{A} \parr \mathcal{B}$ is defined by $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$ as well as $?\mathcal{A}$ is defined by $(!\mathcal{A}^\perp)^\perp$. We associate with a $\flat$-formula $\flat A$ the space $?\mathcal{A}$.

Remark that a non-uniform coherent space may have elements strictly incoherent with themselves, i.e. $\smile$ is not anti-reflexive. Indeed recall the example in the preceding subsection 3.2.1: let a, b be two elements on $\mathcal{A}$ s.t. $a \smile b \, [\mathcal{A}]$. The multiset $[a, b]$ is an element of the non-uniform space $!\mathcal{A}$ s.t. $[a, b] \smile [a, b] \, [!\mathcal{A}]$.

For each proof structure π, we define the **interpretation of π in $\mathfrak{nuCoh}^{\mathsf{X}}$**, denoted by $[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathsf{X}}}$, where the index $_{\mathfrak{nuCoh}^{\mathsf{X}}}$ is omitted if it is clear which model we refer to.

In case π has no conclusion, let $[\![\pi]\!]$ set as undefined. Otherwise, let $c_1 : C_1, \ldots, c_n : C_n$ be the conclusions of π, $[\![\pi]\!]$ is a subset of $\mathcal{C}_1 \parr \ldots \parr \mathcal{C}_n$, defined by using the notion of the $\mathfrak{nuCoh}^{\mathsf{X}}$ experiment.

The **$\mathfrak{nuCoh}^{\mathsf{X}}$ experiments** are defined exactly in the same way as in definition 78, but for the uniformity condition, which is now omitted.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$ and $e : \pi$ be an experiment, then the **result of e**, denoted by $|e|$, is the element $< e(c_1), \ldots, e(c_n) >$ of $\mathcal{C}_1 \parr \ldots \parr \mathcal{C}_n$. The $\mathfrak{nuCoh}^{\mathcal{X}}$ interpretation of π is the set of the results of all the $\mathfrak{nuCoh}^{\mathcal{X}}$ experiments on π:

$$[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} = \left\{ < e(c_1), \ldots, e(c_n) > \mid e \text{ is a } \mathfrak{nuCoh}^{\mathcal{X}} \text{ experiment on } \pi \right\}$$

Like for $\mathfrak{Coh}^{\mathcal{X}}$, the interpretation of $\mathfrak{nuCoh}^{\mathcal{X}}$ is invariant under cut reduction:

Theorem 81 (Soundness of $[\![\;]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$) *For every π, π' proof structures, $\pi =_\beta \pi'$ implies $[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} = [\![\pi']\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$.*

In the end, remark that the uniform interpretation of a proof structure is related with its non-uniform interpretation as follows:

Fact 82 *Let π be a proof structure with conclusions Γ, $|\Gamma|_{\mathfrak{Coh}^{\mathcal{X}}}$ be the web of the uniform coherent space associated by $\mathfrak{Coh}^{\mathcal{X}}$ with the conclusions of π. Then:*

$$[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} \cap |\Gamma|_{\mathfrak{Coh}^{\mathcal{X}}}$$

3.3 Injectivity and uniformity

In this section we address the question of injectivity of coherent semantics for **MELL** proof nets.

Such a question is much harder than the one for **MLL** as well as for **MALL**. Namely the exponential links introduce subtle differences between proof nets, which are hard to read in their interpretations.

More precisely, contrary to the **MLL** case, the type of the conclusions of an **MELL** cut-free proof net is far from characterizing the proof structure up to the linkings of the axioms. Firstly, because from the type of the conclusion of a ? link we do not infer the number of its premises. Secondly, because from the type of a $\flat$-edge we do not know the exponential boxes of which it is an auxiliary door. These two information must be recovered from the semantics.

Actually the question of **MELL** injectivity can be addressed in $\mathfrak{Coh}$ as well as in $\mathfrak{nuCoh}$. In the sequel we will deal mainly with $\mathfrak{Coh}$.

As noticed in fact 82, the interpretation in $\mathfrak{Coh}$ of a proof net contains only the *uniform* elements of its interpretation in $\mathfrak{nuCoh}$. Hence two proof nets can be distinguished by $\mathfrak{nuCoh}$ but not by $\mathfrak{Coh}$, while the vice-versa does not hold.

Let us look at an example, taken from [TdF03b]. Consider the cut-free proof nets π_1 and π_2 defined respectively in figures 3.5 and 3.6. The difference between π_1 and π_2 is in the location of the sub-proof net σ. We show that $[\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} \neq [\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ but $[\![\pi_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

$\mathfrak{nuCoh}$ reads easily the difference between π_1 and π_2. In fact take an experiment e_1 on π_1 such that $e_1(a_1) = [\emptyset]$ and $e_1(a_2) = \emptyset$. The result of such an experiment is:

- $e_1(a) = [\emptyset, [\emptyset]]$;
- $e_1(b) =< [x], \emptyset >$, for a $x \in [\![\sigma']\!]$;
- $e_1(c) = [y]$, for a $y \in [\![\sigma]\!]$.

On the other hand any experiment e_2 on π_2 such that $e_2(b) = e_1(b)$, gives $e_2(c) = \emptyset$. That is, there is no experiment on π_2 with same result as e_1, hence $[\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} \neq [\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$.

Instead $\mathfrak{Coh}$ is not able two read the difference between π_1 and π_2. In fact the uniformity condition on a requires that $e_1(a_1) = e_1(a_2)$, so forbidding the unique way for $\mathfrak{Coh}$ to express the fact that σ is in a given box and not in the other one.

The couple of proof nets π_1 and π_2 is a counter-example to the injectivity of $\mathfrak{Coh}^{\mathcal{X}}$ for **MELL**. The aim of this section is to understand better up to where the uniform coherent semantics is able to read the differences between proof nets. The feeling is that the amount of information that $\mathfrak{Coh}^{\mathcal{X}}$ reads from a proof net is strictly related with the degree of connectedness of the correctness graphs of the proof net.

The section proceed in this way. In subsection 3.3.1 we deal with another example of two proof nets distinguished by $\mathfrak{nuCoh}$ but not by $\mathfrak{Coh}$. The novelty of such an example is that the two proof nets are polarized in the sense defined in [Lau99]. In subsection 3.3.2 we recall a result of [TdF03b], reducing the problem of distinguishing two proof nets in $\mathfrak{Coh}^{\mathcal{X}}$ with the one of the existence of $\mathfrak{Coh}$ injective experiments. Finally in subsection 3.3.3 we prove the existence

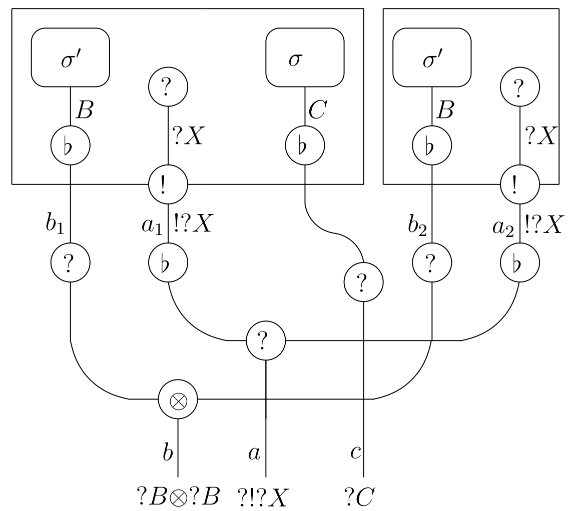


Figure 3.5: counter-example to the injectivity of $\mathfrak{Coh}$: proof net π_1.

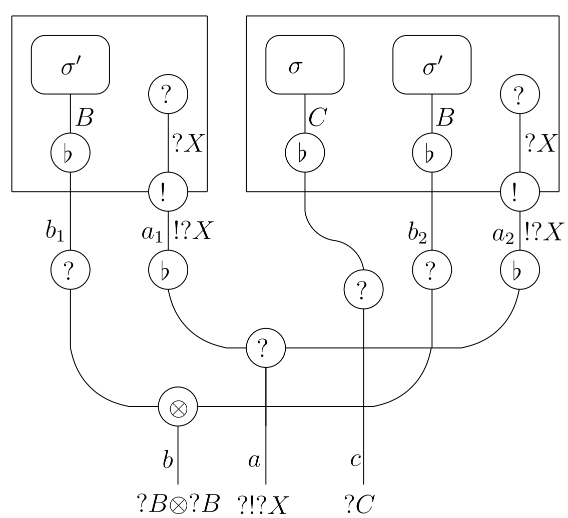


Figure 3.6: counter-example to the injectivity of $\mathfrak{Coh}$: proof net π_2.

of $\mathfrak{Coh}$ injective experiments for the so-called (?⅋)-**MELL** proof nets (theorem 99). Theorem 99 has been already proved by Tortora in [TdF03b]. The main novelty of our approach is to provide a simpler proof of theorem 99 by using Girard's notion of longtrip.

3.3.1 A polarized example

In this subsection we present another example of two proof nets π_1 and π_2 such that $[\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} \neq [\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ but $[\![\pi_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

The novelty of such an example is that π_1 and π_2 are polarized proof nets, in the sense defined in [Lau99]. The example gives a negative answer to the open question of the injectivity of $\mathfrak{Coh}^{\mathcal{X}}$ for polarized linear logic, which instead was conjectured in [TdF03b].

The content of this subsection is due to a joint work with Damiano Mazza.

The proof nets π_1 and π_2 are defined respectively in figures 3.7 and 3.8, where δ_1 and δ_2 are proof nets with conclusion P and such that $[\![\delta_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} \neq [\![\delta_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

The difference between π_1 and π_2 is in the boxes associated with the ! links o_4 and o_5: π_1 has δ_2 in the o_4 box and δ_1 in the o_5 box, conversely π_2 has δ_1 in the o_4 box and δ_2 in the o_5 box.

The two proof nets morally correspond with the two PCF terms:

$$\lambda x.\mathtt{if}x\mathtt{then}(\mathtt{if}x\mathtt{then}\delta_1\mathtt{else}\delta_2)\mathtt{else}(\mathtt{if}x\mathtt{then}\delta_1\mathtt{else}\delta_2)$$

$$\lambda x.\mathtt{if}x\mathtt{then}(\mathtt{if}x\mathtt{then}\delta_1\mathtt{else}\delta_1)\mathtt{else}(\mathtt{if}x\mathtt{then}\delta_2\mathtt{else}\delta_2)$$

which are a well-known example of terms distinguished by $\mathfrak{nuCoh}$ but not by $\mathfrak{Coh}$ (see [BE01]).

Moreover, π_1 and π_2 morally correspond also with the two $\lambda\mu$-terms:

$$\lambda x\mu\alpha\,[\alpha]\,\mu\beta\,[\alpha]\,x(\mu\nu\,[\alpha]\,x(\mu\nu\,[\beta]\,\delta_1)(\mu\nu\,[\beta]\,\delta_2))(\mu\nu\,[\alpha]\,x(\mu\nu\,[\beta]\,\delta_1)(\mu\nu\,[\beta]\,\delta_2))$$

$$\lambda x\mu\alpha\,[\alpha]\,\mu\beta\,[\alpha]\,x(\mu\nu\,[\alpha]\,x(\mu\nu\,[\beta]\,\delta_1)(\mu\nu\,[\beta]\,\delta_1))(\mu\nu\,[\alpha]\,x(\mu\nu\,[\beta]\,\delta_2)(\mu\nu\,[\beta]\,\delta_2))$$

which are a variant[4] of the counter-example to the syntactic separability of the $\lambda\mu$-calculus (see [DP01a]).

Non-uniform interpretation. We prove that $[\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} \neq [\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$, by defining an experiment e_1 on π_1, such that $|e_1| \in [\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} / [\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$.

We start from a box at depth 2: consider π^{o_4} associated with the ! link o_4. Let $e_1^{o_4}$ be an experiment on π^{o_4} and suppose $e_1^{o_4}(b_4) = [y]$, for an $y \in [\![\delta_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} / [\![\delta_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$. Then define the experiment $e_1^{o_1}$ on π^{o_1} as follows:

- $e_1^{o_1}(o_3) = \emptyset$;
- $e_1^{o_1}(o_4) = [e_1^{o_4}, e_1^{o_4}]$.

[4] The two inseparable $\lambda\mu$-terms defined in [DP01a] are actually distinguished by $\mathfrak{Coh}$, since coherent spaces are sound w.r.t. the mix rule.

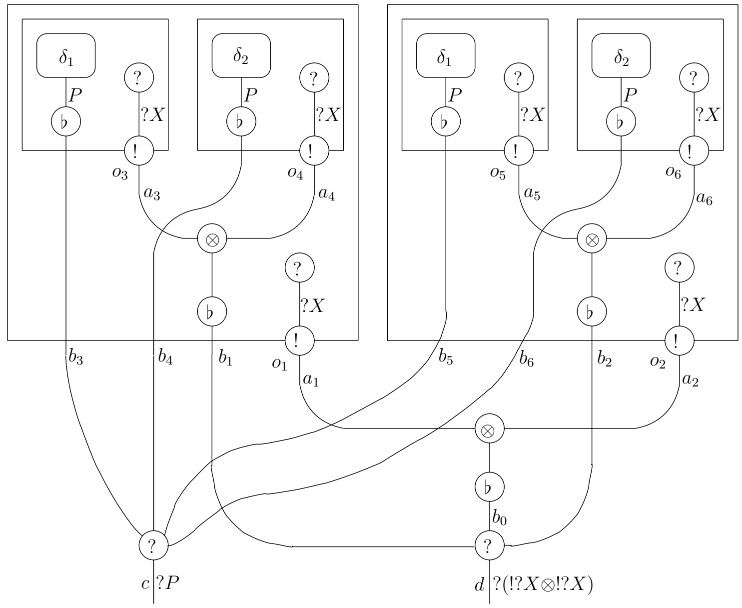


Figure 3.7: polarized counter-example to the injectivity of $\mathfrak{Coh}$: proof net π_1.

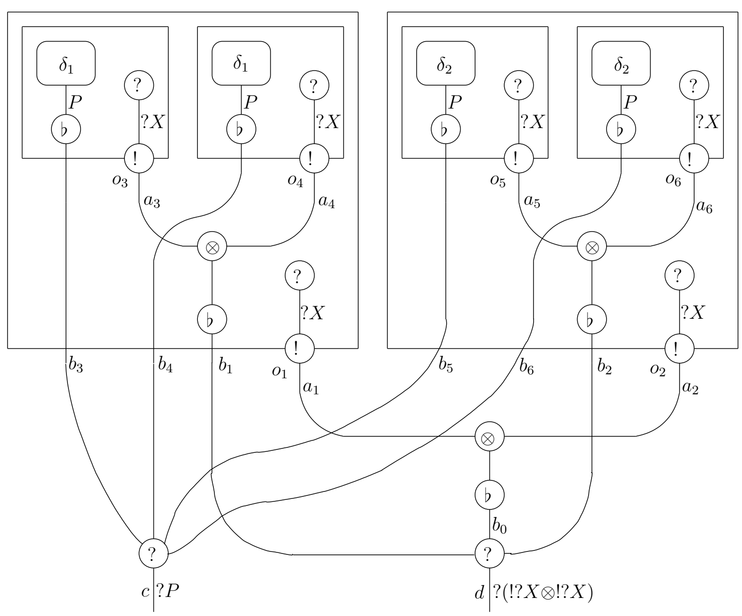


Figure 3.8: polarized counter-example to the injectivity of $\mathfrak{Coh}$: proof net π_2.

Finally define the experiment e_1 as follows:

- $e_1(o_1) = [e_1^{o_1}]$;
- $e_1(o_2) = \emptyset$.

Just for joking, let us compute the values of e_1 on the doors of π^{o_1} and π^{o_2}:

$e_1(a_1) = [\emptyset]$	$e_1(b_1) = [<\emptyset, [\emptyset, \emptyset]>]$	$e_1(b_3) = \emptyset$	$e_1(b_4) = [y, y]$
$e_1(a_2) = \emptyset$	$e_1(b_2) = \emptyset$	$e_1(b_5) = \emptyset$	$e_1(b_6) = \emptyset$

Here are the values of e_1 on the conclusions of π_1:

- $e_1(c) = [y, y]$;
- $e_1(d) = [<\emptyset, [\emptyset, \emptyset]>, <[\emptyset], \emptyset>]$.

The result of e_1 is an element of $[\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ but not of $[\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$. In fact, suppose e_2 is an experiment on π_2 s.t. $e_2(c) = e_1(c)$ and $e_2(d) = e_1(d)$. In this case we have two possibility:

1. $e_2(b_0) = [<\emptyset, [\emptyset, \emptyset]>]$ and $e_2(b_2) = [<[\emptyset], \emptyset>]$, or
2. $e_2(b_0) = [<[\emptyset], \emptyset>]$ and $e_2(b_1) = [<\emptyset, [\emptyset, \emptyset]>]$.

The case 1 is not possible, since $e_2(b_0) = [<\emptyset, [\emptyset, \emptyset]>]$ implies that $e_2(b_2)$ has two elements and not only one.

In case 2 instead we have $e_2(c) = e_2(b_4)$, hence $e_2(b_4) = [y, y]$, which is impossible, having we supposed $y \notin [\![\delta_1]\!]$.

We conclude that $|e_1| \notin [\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$, hence:

$$[\![\pi_1]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}} \neq [\![\pi_2]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$$

Uniform interpretation. We prove that $[\![\pi_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

First of all, remark that the $\mathfrak{nuCoh}^{\mathcal{X}}$ experiment e_1 above defined is not a $\mathfrak{Coh}^{\mathcal{X}}$ experiment. Indeed, $e_1(d)$ is not in the uniform interpretation of $?(!?\mathcal{X} \otimes !?\mathcal{X})$, since $<\emptyset, [\emptyset, \emptyset]> \frown <[\emptyset], \emptyset> [!?\mathcal{X} \otimes !?\mathcal{X}]$.

More generally, let us prove that $[\![\pi_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$.

Let e_1 be a generic $\mathfrak{Coh}^{\mathcal{X}}$ experiment on π_1. Let us try to precise the result of e_1.

Firstly, let us compute $e_1(d)$. We focus on the edges above d. Clearly $e_1(a_1) = n[\emptyset]$ and $e_2(a_2) = m[\emptyset]$ for $n, m \geq 0$, i.e. $e_1(b_0) = [<n[\emptyset], m[\emptyset]>]$.

Suppose $u \in e_1(b_1)$. Of course $u = <n'[\emptyset], m'[\emptyset]>$ for $n', m' \geq 0$. Since $e_1(b_1) + e_1(b_0)$ belongs to the web of $?(!?\mathcal{X} \otimes !?\mathcal{X})$, we deduce $u \overset{\smile}{\frown} <n[\emptyset], m[\emptyset]> [!?\mathcal{X} \otimes !?\mathcal{X}]$. This last condition is true only in case $u = <n[\emptyset], m[\emptyset]>$. We conclude that $e_1(b_1) = ne_1(b_0)$. By similar arguments we deduce $e_1(b_2) = me_1(b_0)$. Thus:

$$e_1(d) = (n + m + 1) [< n [\emptyset] , m [\emptyset] >]$$

Now, let us compute $e_1(c)$. Firstly remark that $Supp(e_1(b_3)) = \{x\}$ for an $x \in [\![\delta_1]\!]$. In fact, suppose $x, x' \in Supp(e_1(b_3))$. Since $x, x' \in [\![\delta_1]\!]$, we deduce[5] $x \stackrel{\frown}{\smile} x' [\mathcal{B}]$. On the other hand by the uniformity of $?\mathcal{B}$, we known that $x \stackrel{\smile}{\frown} x' [\mathcal{B}]$, thus $x = x'$.

In the same way we may argue that $Supp(e_1(b_4))$, $Supp(e_1(b_5))$ and $Supp(e_1(b_6))$ are all singleton. We conclude that:

- $e_1(b_3) = n^2 [x]$, for an $x \in [\![\delta_1]\!]$;
- $e_1(b_4) = nm [y]$, for an $y \in [\![\delta_2]\!]$;
- $e_1(b_5) = mn [x']$, for an $x' \in [\![\delta_1]\!]$;
- $e_1(b_6) = m^2 [y']$, for an $y' \in [\![\delta_2]\!]$.

That is:

$$e_1(c) = n^2 [x] + nm [y] + mn [x'] + m^2 [y']$$

Let us consider now a generic $\mathfrak{Coh}^{\mathcal{X}}$ experiment e_2 on π_2. By similar consideration as for e_1, we deduce that:

$$e_2(d) = (n + m + 1) [< n [\emptyset] , m [\emptyset] >]$$

for numbers $n, m \geq o$, and:

- $e_2(b_3) = n^2 [x]$, for an $x \in [\![\delta_1]\!]$;
- $e_2(b_4) = nm [x']$, for an $x' \in [\![\delta_1]\!]$;
- $e_2(b_5) = mn [y]$, for an $y \in [\![\delta_2]\!]$;
- $e_2(b_6) = m^2 [y']$, for an $y' \in [\![\delta_2]\!]$.

That is:

$$e_2(c) = n^2 [x] + nm [x'] + mn [y] + m^2 [y']$$

Of course by commutation on the sum between multisets and the product between numbers, we conclude that e_1 and e_2 have the same result, hence:

$$[\![\pi_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$$

[5] We are using the well-known theorem stating that the interpretation of a proof net is a clique. Indeed such a theorem can be deduced as a corollary from the stronger theorem 103 of section 3.4.

A third way. We have considered both the $\mathfrak{nuCoh}$ and the $\mathfrak{Coh}$ interpretation of π_1 and π_2. The first one distinguishes the two proof nets, the second one does not.

It is worth of mentioning that such proof nets correspond with two designs in the system of ludics with repetition, introduced by Maurel in [Mau04]. For separating the two designs in a some sense uniform way, Maurel introduces a calculus in which the sum, hence the product, between multiset is no more commutative. In our terms it means that $e_2(c) \neq e_1(c)$, since $nm\,[y]+mn\,[x'] \neq nm\,[x'] + mn\,[y]$.

From a sintactical point of view, the commutation of the sum between multisets corresponds with the fact that the ? premises are unordered. Forbidding such a commutation means introducing a kind of order among the ? premises. Such an approach may be interesting, but at the moment we do not known any satisfactory definition of such a non-commutative link ?.

3.3.2 From injectivity to the injective experiment

In this subsection we recall a result in [TdF03b], reducing for several **MELL** fragments the question of the injectivity of $\mathfrak{Coh}$ to the one of the existence of a $\mathfrak{Coh}$ injective experiment.

The idea is to understand what $\mathfrak{Coh}$ is able to read from a proof net π by trying to reconstruct π itself from $[\![\pi]\!]_{\mathfrak{Coh}}$. Proposition 84 reduces in several cases the reconstruction of π to the one of the *linear proof structure* of π (definition 83). Theorem 89 shows that the linear proof structure of π can be reconstructed from the result of particular experiments, called *injective n-obsessional experiments* (definition 88). So the reconstruction of π turns in the problem of the existence of such experiments. Proposition 91 reduces the existence of injective n-obsessional experiments to the one of injective experiments on proof nets without boxes. This latter problem will be the object of the following subsection 3.3.3.

From now on by coherent semantics we will mean uniform coherent semantics, moreover we will denote $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ simply by $[\![\pi]\!]$.

Open the boxes

Definition 83 ([TdF03b]) *Let π be a proof structure. The* ***linear proof structure of*** *π, denoted by $LPS(\pi)$, is a weighted graph obtained from π by erasing the boxes frames and by labelling each $\flat$-edge with the exponential depth of the $\flat$ link it descends from.*

Remark that several proof structures can be associated with a unique LPS. For example the proof nets in figures 3.5 and 3.6 are both associated with the linear proof structure in figure 3.9. Actually the $LPS(\pi_{1,2})$ is exactly what coherent spaces read from $\pi_{1,2}$. The same remark holds for the couple of proof nets in figures 3.7 and 3.8.

But in case $LPS(\pi)$ is a connected graph, then π is unique:

Proposition 84 ([TdF03b]) *Let π be a proof structure s.t. $LPS(\pi)$ is a connected graph. If π' is a proof structure s.t. $LPS(\pi) = LPS(\pi')$, then $\pi = \pi'$.*

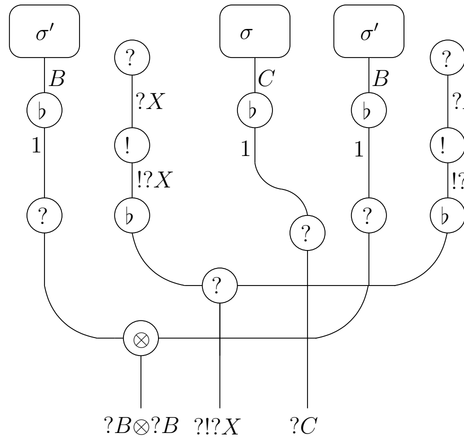


Figure 3.9: linear proof structure of the proof nets in figures 3.5 and 3.6.

Proof. Let $LPS(\pi)$ be a connected graph. We reconstruct the frames of the exponential boxes from $LPS(\pi)$ by declaring for each ! link which are its auxiliary doors.

Let o be a ! link and a be an edge conclusion of a $\flat$ link. a is an o auxiliary door if and only if there is a path ϕ in $LPS(\pi)$ from a to o, s.t. each edge $a' \in \phi$ conclusion of a $\flat$ link has exponential depth strictly greater than the one labelling a. □

Reading the linear proof structures

The linear proof structures can be reconstructed from the result of a special kind of experiments, called *injective n-obsessional experiments*. They satisfy two properties: n-obsessionality and injectivity.

An n-obsessional experiment is an experiment which takes for each ! link exactly n copies of a unique (n-obsessional) experiment on the box associated with the ! (definition 85).

An injective n-obsessional experiment is an n-obsessional experiment which takes different values on different axioms (definition 88).

Theorem 89 states that if it exists for π an injective n-obsessional experiment, for n arbitrary large, then for any proof net π' s.t. $[\![\pi]\!] = [\![\pi']\!]$ we have $LPS(\pi) = LPS(\pi')$. We do not give here the proof of the theorem, for which we refer to [TdF03b]. Such a proof consists in a procedure for reconstructing $LPS(\pi)$ from the result of an injective n-obsessional experiment on π. The idea is that the n-obsessionality allows us to read both the depth and the number of the premises of a link ?, while the injectivity allows us to read the linkings of the axioms.

Definition 85 ([TdF03b]) *Let $n \in \mathbb{N}$, an experiment $e : \pi$ is n-**obsessional** if for every ! link o at depth 0, $e(o) = n\,[e^o]$, for some n-obsessional experiment e^o on the box associated with o.*

We have defined the experiments just on the edges and ! links at depth 0 of a proof structure, by taking advantage of the inductive frame introduced by boxes. Nevertheless for extending to **MELL** the notion of injective experiment it will be usefull computing experiments also on deeper edges, so that we introduce the following definition:

Definition 86 *Let e be an experiment on π and a be an edge. The* ***experiments in* e *associated with* a** *are the elements of the multiset $Ex^e(a)$ defined by induction on the depth of a:*

- *if a is at depth 0 then $Ex^e(a) = [e]$,*
- *if a is in a box associated with a ! link o at depth 0 and $e(o) = [e_1^o, \dots, e_n^o]$, then $Ex^e(a) = \sum_{i \leq n} Ex^{e_i^o}(a)$.*

An n-obsessional experiment e is both regular and powerfull. It is regular in the sense that it associates with an edge a at any depth of π many copies of a unique experiment $e' \in Ex^e(a)$, so that we may speak about *the* experiment associated in e with a. It is powerfull because the number of e' copies associated with a codifies the exponential depth of a:

Proposition 87 ([TdF03b]) *Let $e : \pi$ be an n-obsessional experiment, a be an edge at depth d. Then, $Ex^e(a) = n^d\,[e']$, for some n-obsessional experiment e' on the smaller box containing a.*

Moreover, if $a : \flat A$ is conclusion of a $\flat$ link at depth d, then $e(a) = n^d\,[e'(a)]$.

PROOF. By induction on d. If a is at depth 0 then by definition $Ex^e(a) = [e]$. If a is in a box associated with a ! link o at depth 0 and $e(o) = n\,[e^o]$, then by definition of Ex^e, $Ex^e(a) = nEx^{e^o}(a)$. By induction we have $Ex^{e^o}(a) = n^{d-1}\,[e']$, so $Ex^e(a) = n^d\,[e']$.

Of course if a is an auxiliary door with depth d, then $e(a) = n^d\,[e'(a)]$. □

Definition 88 ([TdF03b]) *An n-obsessional experiment $e : \pi$ is* ***injective*** *if for every pair of atomic edges $a : X, a' : X$, with $e_a, e_{a'}$ as associated experiments in e, $e_a(a) \neq e_{a'}(a')$.*

Theorem 89 ([TdF03b]) *Let π be a cut-free proof net. If there is an injective n-obsessional $\mathfrak{Coh}^{\mathcal{X}}$ experiment on π, for n arbitrary large, then for any cut-free proof net π' with same conclusions as π, $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi']\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ implies $LPS(\pi) = LPS(\pi')$.*

Theorem 89 does not hold if we substitute $\mathfrak{Coh}^{\mathcal{X}}$ with $\mathfrak{nuCoh}^{\mathcal{X}}$. In fact the proof makes use of the uniformity of the injective n-obsessional $\mathfrak{Coh}^{\mathcal{X}}$ experiment in a crucial passage. More precisely, only in $\mathfrak{Coh}^{\mathcal{X}}$ the n-obsessionality of an experiment e can be read from its result $|e|$. Stated otherwise: if two $\mathfrak{Coh}^{\mathcal{X}}$ experiments $e : \pi, e' : \pi'$ have the same result, then e is n-obsessional iff e' is n-obsessional. On the other hand, if e, e' are $\mathfrak{nuCoh}^{\mathcal{X}}$ experiments such a statement is false.

Moreover, remark that theorem 89 deals with proof nets and not with proof structures in general. Indeed the above statement is false also in case π or π' are not correct proof structures.

Theorem 89 and proposition 84 turn the problem of injectivity for several fragments of **MELL** into a problem of existence of injective n-obsessional experiments.

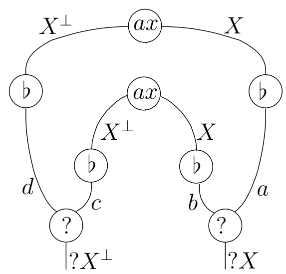


Figure 3.10: example of a proof net without $\mathfrak{Coh}$ injective experiments.

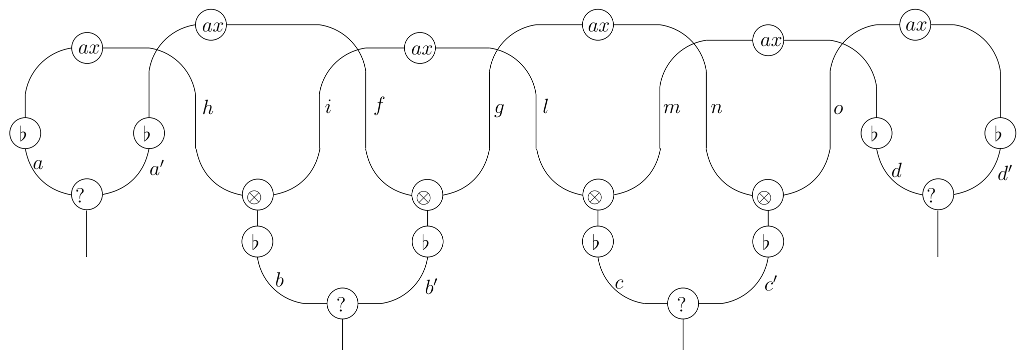


Figure 3.11: example of a proof net without $\mathfrak{Coh}$ injective experiments.

Do $\mathfrak{Coh}^{\mathcal{X}}$ injective n-obsessional experiments exist?

In uniform coherent semantics, not every choice of values on the axioms and ! links determines an experiment on a cut-free proof net because of the uniformity condition (see definition 78): the elements associated with edges of types $\flat C^\perp$, $?C^\perp$ and $!C$ must be in the web of $|!\mathcal{C}|$, i.e. their support must be a clique of $\mathcal{C}$.

For example, let π be the proof net in figure 3.10. We may not define a $\mathfrak{Coh}$ injective experiment on π. In fact by the uniformity of $?\mathcal{X}$ we need satisfy $e(a) \underset{\frown}{\smile} e(b)\,[\mathcal{X}]$, while by the uniformity of $?\mathcal{X}^\perp$ we need satisfy $e(c) \underset{\frown}{\smile} e(d)\,\left[\mathcal{X}^\perp\right]$, but $e(a) = e(c)$ and $e(b) = e(d)$, hence $e(a) = e(b)$.

Let us consider a more complex example. Let π be the proof net in figure 3.11. Suppose e is an injective experiment on π, let us check if e can meet all the uniformity conditions required by its links ?. Remark that since e is injective and π is without weakening, for any formula A, e associates with different edges labelled by A different elements of $\mathcal{A}$.

The uniformity conditions are:

- $e(a) \smile e(a')$
- $e(b) \smile e(b')$
- $e(c) \smile e(c')$
- $e(d) \smile e(d')$

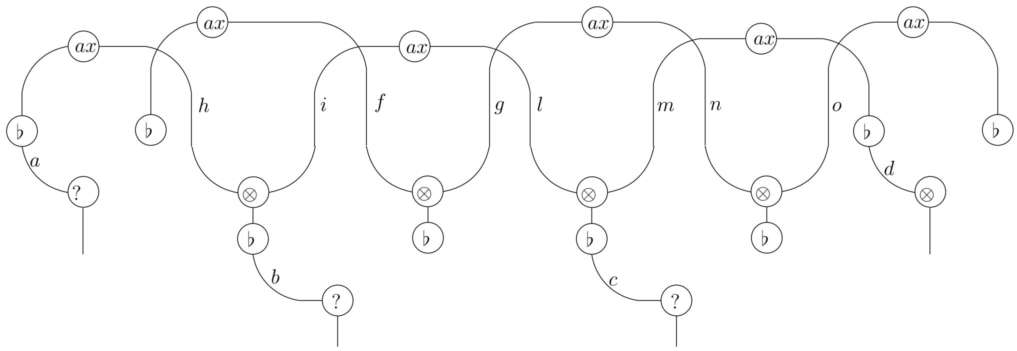


Figure 3.12: correctness graph of the proof net in figure 3.11.

Now if we suppose $e(a) \smile e(a')$, by crossing the axioms above resp. a and a', we deduce $e(f) \frown e(h)$. On the other hand we must satisfy $e(b) \smile e(b')$, hence we need $e(g) \smile e(i)$. By crossing the axioms above g, i the last incoherence sets $e(n) \frown e(l)$. As before, since we need $e(c) \smile e(c')$, we deduce $e(m) \smile e(o)$, which imposes $e(d) \frown e(d')$, so violating a uniform condition. Thus on π are not definable injective experiments.

Remark that for checking the uniformity conditions, we have drawn the following two switching paths in π:

$$\uparrow a \quad \rightarrow \quad \downarrow h \quad \rightarrow \quad \uparrow i \quad \rightarrow \quad \downarrow l \quad \rightarrow \quad \uparrow m \quad \rightarrow \quad \downarrow d$$

$$\uparrow a' \quad \rightarrow \quad \downarrow f \quad \rightarrow \quad \uparrow g \quad \rightarrow \quad \downarrow n \quad \rightarrow \quad \uparrow o \quad \rightarrow \quad \downarrow d'$$

which are in two disjoint components of a correctness graph of π (figure 3.12).

Now, what about if we deal with a proof net whose correctness graphs are all connected? The two switching paths drawn for checking the uniformity conditions should eventually meet, so showing a solution for satisfying such conditions.

Actually, what we guess is that injective (n-obsessional) experiments always exist on those proof nets whose correctness graphs are connected (see conjecture 101).

A possibly proof of conjecture 101 should link in a very interesting way the injectivity of a uniform semantics with the connectedness of the correctness graphs. Unluckily all our efforts for proving conjecture 101 did not succeed. Until now, what we known is theorem 99, stating that for the so-called (?⅋)-**MELL** proof nets without mix (see subsection 3.3.3) there exist $\mathfrak{Coh}^{\mathcal{X}}$ injective (n-obsessional) experiments. The proof of theorem 99 is a stimulating example of how from a connected correctness graph we may define a $\mathfrak{Coh}^{\mathcal{X}}$ injective experiment.

Before turning to the proof of theorem 99, let us simplify further the problem of the existence of injective n-obsessional experiments.

Remark that we never deal with the uniformity conditions on the doors of the boxes. Indeed a valuable property of n-obsessional experiments is that they satisfy straightforwardly that kind of conditions. In other and more precise words:

Definition 90 ([TdF03b]) *Let π be a cut-free proof structure. The* ***linearized of*** *π, denoted by $L(\pi)$, is the proof structure obtained from π erasing every box frame and every ! link, and changing the type of the edges by substituting every ! subformula $!A$ with A.*

Notice that in case π is a proof net, so is $L(\pi)$. The definition of $L(\pi)$ is justified by the following proposition:

Proposition 91 ([TdF03b]) *Let π be a cut-free proof net and n be a number greater than the maximum arity of the ? links in π. If there is an injective experiment on $L(\pi)$, then there is an injective n-obsessional experiment on π.*

In a cut-free linearized proof net can occur only links axiom, $\parr$, $\otimes$, $\flat$ and ?. Since we have not boxes, the $\flat$ links are useless too, so that we can reduce further our syntax to proof nets composed only by links axiom, $\parr$, $\otimes$ and ?. Of course in this no-$\flat$ syntax the premises of a link ? are edges labelled by simple formulas A, instead of $\flat A$.

3.3.3 Injective experiments in $(?\parr)$-MELL

A $(?\parr)$-**MELL** formula is defined in [TdF03b] by the following grammar:

$$F ::= \quad X \quad | \quad X^\perp \quad | \quad F \parr F \quad | \quad ?F \parr F \quad | \quad F \parr ?F \quad | \quad F \otimes F \quad | \quad !F$$

A $(?\parr)$-**MELL proof net** is a **MELL** proof net without mix in which any edge is labeled by a $(?\parr)$-**MELL** formula.

We do not consider $(?\parr)$-**MELL** a fragment of **MELL** in a strict sense, since it is not closed by orthogonality: $!X$ is a $(?\parr)$-**MELL** formula while $?X^\perp$ is not. Nevertheless, $(?\parr)$-**MELL** includes several interesting **MELL** fragments, such as the weakly polarized fragment of **MELL** (see [TdF00]), in which we may translate the simply typed λ-calculus (see [Gir87]).

The set of the $(?\parr)$-**MELL** proof nets is the largest set of proof nets for which we have the proof of the $\mathfrak{Coh}$ injectivity (theorem 100).

In the preceding subsection we have reduced the question of the $\mathfrak{Coh}$ injectivity with the one of the existence of $\mathfrak{Coh}$ injective experiments. In this subsection, we give a positive answer to such a question for $(?\parr)$-**MELL** proof nets (theorem 99).

Theorem 99 has been proven in [TdF03b] by a proof based on the notion of correctness graph. The novelty of our approach is to provide an alternative proof based on the Girard's notion of longtrip.

Let us be more precise. Suppose π is a $(?\parr)$-**MELL** proof net without exponential boxes. The proof of theorem 99 in [TdF03b] associates with a connected correctness graph of π an injective experiment $e : \pi$, which satisfies all the uniformity conditions required by the links ?. More particularly, e is defined by approximations, i.e. the proof provides a sequence of injective experiments $e_1, e_2, e_3, \ldots$, satisfying more and more uniformity conditions, eventually obtaining the desired experiment e.

Our proof of theorem 99 instead defines in a single step the injective experiment e on π, so increasing our control on e. The improvement of our approach is due to the fact that we refer to a longtrip of π instead of a correctness graph of π. A longtrip has more information than a correctness graph, since it is an oriented path inside a correctness graph, so it takes with itself a visiting order on the edges of the correctness graph (such an order is explicated in definition 95).

The subsection is divided in three paragraph. The first one is called *a trip on a proof structure*, and it is devoted to introduce briefly the notion of trip in the framework of ($?\parr$)-**MELL** proof nets. The second paragraph is called *existence of injective experiments*, and it contains our proof of theorem 99. The last paragraph is called *the injectivity of* **MELL** *without weakening and mix*, and it proposes the conjecture 101 about the existence of injective experiments on **MELL** proof nets without weakening and mix.

From now on, by a proof structure (resp. proof net) π we will mean a ($?\parr$)-**MELL** proof structure (resp. proof net without mix) without cut, ! and $\flat$ links.

A trip on a proof structure. We introduce the notion of *switchings* for the links $\otimes$, $\parr$ and ? (see figure 3.13). A switching for a link l determines the way a path τ crosses l. More precisely, let l be a link with premises $a_1, \dots, a_n$ and conclusion c. τ may arrive in l by $\uparrow c$, or by $\downarrow a_i$ for an $i \leq n$, and it may leave l by $\downarrow c$, or by $\uparrow a_i$ for an $i \leq n$. A switching is a bijection between the arriving possibilities ($\{\uparrow c, \downarrow a_1, \dots, \downarrow a_n\}$) and the leaving ones ($\{\downarrow c, \uparrow a_1, \dots, \uparrow a_n\}$):

$\otimes$ **switchings:** let l be a $\otimes$ link with premises a, b and conclusion c. We say that τ **respects the switching** T_1 **for** l, if a, b, c may occur in τ only in the sequences $\uparrow c \uparrow a$, $\downarrow a \uparrow b$ and $\downarrow b \uparrow c$.

We say that τ **respects the switching** T_2 **for** l, if a, b, c may occur in τ only in the sequences $\uparrow c \uparrow b$, $\downarrow b \uparrow a$ and $\downarrow a \downarrow c$.

$\parr$ **switchings:** let l be a $\parr$ link with premises a, b and conclusion c. We say that τ **respects the switching** P_1 **for** l, if a, b, c may occur in τ only in the sequences $\uparrow c \uparrow a$, $\downarrow a \downarrow c$ and $\downarrow b \uparrow b$.

We say that τ **respects the switching** P_2 **for** l, if a, b, c may occur in τ only in the sequences $\uparrow c \uparrow b$, $\downarrow b \downarrow c$ and $\downarrow a \uparrow a$.

? **switchings:** let l be a ? link with has premises $a_1, \dots, a_n$, for $n > 0$, and conclusion c. We say that τ **respects the** W_i **switching for** l (for $i \leq n$), if $a_1, \dots, a_n, c$ may occur in τ only in the sequences $\uparrow c \uparrow a_i$, $\downarrow a_i \downarrow c$ and for any $j \neq i$, $\downarrow a_j \uparrow a_j$.

A **switching for a proof structure** π is a function S associating with each link $\otimes$, $\parr$ and ? one among its switchings.

Definition 92 (from [Gir87]) *A* ***trip on a proof structure*** π *is an oriented path* τ *s.t.:*

1. τ *is a cycle;*
2. *if* c *is a conclusion of* π*, then* c *may occur in* τ *only in a sequence* $\downarrow c \uparrow c$*;*
3. *if* a, b *are the conclusions of an axiom, then* a, b *may occur in* τ *only in the sequences* $\uparrow a \downarrow b$ *and* $\uparrow b \downarrow a$*;*

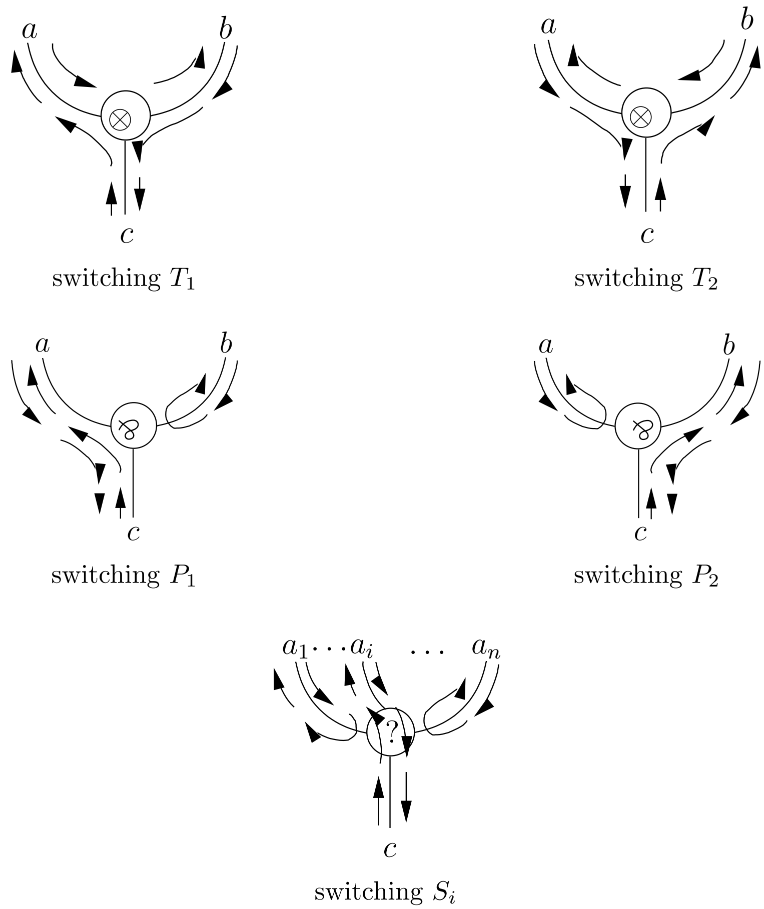


Figure 3.13: switchings for $\otimes$, $\parr$ and ?.

4. *if a, b are the premises of a cut, then a, b may occur in τ only in the sequences $\downarrow a \uparrow b$ and $\downarrow b \uparrow a$;*

5. *if c is a conclusion of a weakening, then c may occur in τ only in the sequence $\uparrow c \downarrow c$;*

6. *there is a switching S for π, s.t. for any $\otimes$, $⅋$ or $?$ link l of π, τ respects the $S(l)$ switching for l.*

*A trip τ is a **longtrip** if each edge a occurs in τ exactly twice, once as $\uparrow a$ and once as $\downarrow a$.*

In the sequel we will denote a trip by the Greek letter τ.
The notion of longtrip provides Girard's correctness criterion:

Theorem 93 ([Gir87]) *A* **MLL** *proof structure is a proof net without mix iff all its trips are longtrips.*

Let $\updownarrow a, \updownarrow b \in \tau$, we denote by $\updownarrow a - \updownarrow b$ the section of τ from $\updownarrow a$ to $\updownarrow b$. In particular we denote by $\tau(a)$ the set of edges occurring in the section $\uparrow a - \downarrow a$.

Existence of injective experiments. Finally we may attack the question of the existence of injective experiments.

($?⅋$)-**MELL** proof nets are not characterizable by the longtrip criterion, because they have weakenings. But in ($?⅋$)-**MELL** proof nets a weakening conclusion is premise of a $⅋$ link of which the other premise is surely not a weakening conclusion. This restriction allows to define in a ($?⅋$)-**MELL** proof net a trip which is nearly a longtrip, in the sense that it meets almost all the properties of a usual longtrip:

Proposition 94 [6] *Let π be a ($?⅋$)-**MELL** proof net without mix and exponential boxes. If τ is a trip on π containing at least one axiom and s.t.:*

(*) *if l is a link with conclusion $A⅋B$: in case A is not a $?$-formula, τ respects the switching P_1 for l; in case A is a $?$-formula τ respects the switching P_2 for l,*

then τ meets the following properties:

1. *for each π edge a, if a is not a weakening conclusion then a occurs in τ exactly twice, once as $\uparrow a$, once as $\downarrow a$;*

2. *if a, b are two edges, $\uparrow b \in \uparrow a - \downarrow a$ iff $\downarrow b \in \uparrow a - \downarrow a$;*

3. *if a, b are two conclusions of an axiom, then $\tau(a) \cap \tau(b) = \{a, b\}$ and $\tau(a) \cup \tau(b) = \{c \mid \updownarrow c \in \tau\}$;*

4. *if a, b are the two premises of a $⅋$ or $?$ link, then $b \in \tau(a)$ and $a \in \tau(b)$;*

5. *if a, b are the two premises of a $\otimes$ link, then $\tau(a) \cap \tau(b) = \emptyset$;*

[6] The condition (*) of proposition 94 corresponds with the operation of *par-mutilation* defined by Tortora in [TdF03b].

6. if c is conclusion of a ? link which is not a weakening, then $\downarrow c \uparrow c \sqsubseteq \tau$.

PROOF. By induction on a sequentialization of π. We do only the ⅋ and $\otimes$ cases, the others being similar or straightforward.

par: if the last rule of the π sequentialization is a ⅋-rule, let l be the π link associated with such a rule. Let $c : A⅋B$ be the conclusion of l and $a : A$ and $b : B$ be its premises. Let τ be a trip of π respecting condition (*).

Define π_c as the proof net obtained from π erasing l and c.

In case τ respects the P_1 switching for l (the case τ respects P_2 is similar). Remark that since τ meets condition (*), then a is not conclusion of a weakening, thus:

$$\tau = \downarrow a \downarrow c \uparrow c \uparrow a \ldots\ldots$$

where $\uparrow a - \downarrow a$ contains at least one axiom of π, hence of π_c. Define τ_c as $\uparrow a - \downarrow a$, and remark that τ_c is a trip of π_c satisfying (*) and containing at least one π_c axiom. By induction hypothesis τ_c meets the properties 1-6, which straightforwardly implies that τ meets 1-6 too.

tensor: if the last rule of the π sequentialization is a $\otimes$-rule, let l be the π link associated with such a rule. Let $c : A \otimes B$ be the conclusion of l and $a : A$, $b : B$ be its premises. Let τ be a trip of π respecting condition (*).

Since l is associated with the last sequent rule, l is splitting π in two subproof nets π_a and π_b with conclusions respectively $a : A, \Pi'$ and $b : B, \Pi''$, supposing $c : A \otimes B, \Pi', \Pi''$ to be the π conclusions.

Suppose τ respects the T_1 switching for l (the case τ respects T_2 is similar). Then, being c a π conclusion and l splitting:

$$\tau = \uparrow c \uparrow a \ldots\ldots \downarrow a \uparrow b \ldots\ldots \downarrow b \downarrow c$$

Define $\tau_a = \uparrow a - \downarrow a$ (resp. $\tau_b = \uparrow b - \downarrow b$). Of course τ_a (resp. τ_b) is a trip on π_a (resp. of π_b) satisfying condition (*). By induction hypothesis both τ_a and τ_b meet properties 1-6. We leave to the reader checking that under such hypothesis τ meets 1-6 too.

□

From now on, let us fix a trip τ on a proof net π, satisfying condition (*) of proposition 94 and containing at least one axiom of π.

Let s be any conclusion of π. Since s is a conclusion, $\downarrow s \uparrow s \sqsubseteq \tau$. If we cut the cycle τ between $\downarrow s$ and $\uparrow s$, we obtain an oriented line starting from $\uparrow s$ and ending in $\downarrow s$. By this line we define a linear order on the edges of π:

Definition 95 *Let τ be a trip of π, s be a conclusion of π, a, b be two edges. We write $a <_{\tau,s} b$ if a is the first edge between a and b which we meet in τ starting from $\uparrow s$, without taking care if we meet $\uparrow a$ or $\downarrow a$.*

Remark that by property 1 of proposition 94, for any edges a, b which are not conclusion of weakenings, $a <_{\tau,s} b$ or $b <_{\tau,s} a$.

Definition 96 *An unordered pair of π edges (a, b) is a **candidate** if $a <_{\tau,s} b$ implies $b \in \tau(a)$.*

Proposition 97 *Let $a : X, a' : X^{\perp}$ (resp. $b : X, b' : X^{\perp}$) be the two conclusions of an axiom link. (a, b) is a candidate if and only if (a', b') is not a candidate.*

Proof. Suppose (a, b) is candidate, we prove (a', b') is not candidate. We may suppose $a <_{\tau,s} b$ and $b \in \tau(a)$ (recall that $a <_{\tau,s} b$ or $b <_{\tau,s} a$). Since a, a' (resp. b, b') are linked by an axiom, the first occurrence of a (resp. of b) in τ is contiguous with the first occurrence of a' (resp. of b'). So that we deduce $a' <_{\tau,s} b'$. Moreover, since $\tau(a) \cap \tau(a') = \{a, a'\}$, we conclude $b' \notin \tau(a')$, hence (a', b') is not candidate.

Suppose (a, b) is not candidate, we prove (a', b') is candidate. The case is symmetrical to the preceding one: suppose $a <_{\tau,s} b$ and $b \notin \tau(a)$. By $a <_{\tau,s} b$ we conclude that $a' <_{\tau,s} b'$, by $b \notin \tau(a)$ that $b \in \tau(a')$, hence (a', b') is candidate.
□

Proposition 97 shows that we may use the candidates for defining the pairwise strict incoherence between the values of an experiment e on the atomic edges. The following lemma 98 proves a crucial property of such an experiment e:

Lemma 98 *Let $e : \pi$ be an experiment s.t. for any pair a, b of atomic edges with same type: $e(a) \smile e(b)$ if and only if (a, b) is a candidate.*

Let c, c' be two π edges of same type C, if (c, c') is a candidate then $e(c) \smile e(c')$.

Proof. Suppose (c, c') is a candidate, the proof is by induction on C:

atom: if $C = X, X^{\perp}$ the statement is immediate;

par: if $C = A \invamp B$, let $a : A$, $b : B$ (resp. $a' : A$, $b' : B$) be the premises of the $\invamp$ l (resp. l') with conclusion $c : C$ (resp. $c' : C$). Since τ meets condition (*) of proposition 94, τ respects the same switching for l and l' (they have the same type).

Suppose such a switching is P_1 (the case it is P_2 is similar), so that $\uparrow c \uparrow a$, $\downarrow a \downarrow c$, $\uparrow c' \uparrow a'$, $\downarrow a' \downarrow c' \sqsubseteq \tau$. In particular, by $c <_{\tau,s} c'$ we deduce that $a <_{\tau,s} a'$: in fact the first occurrence of c (resp. c') is contiguous in τ with the first occurrence of a (resp. a'). Moreover, by $c' \in \tau(c)$, we infer $a' \in \tau(a)$. We conclude that (a, a') is a candidate.

By definition of the $\invamp$ P_1 switching, $\downarrow b \uparrow b, \downarrow b' \uparrow b' \sqsubseteq \tau$, i.e. $b \in \tau(b')$ and $b' \in \tau(b)$, which straightforwardly implies that (b, b') is a candidate.

By induction hypothesis on (a, a') and (b, b'), we deduce $e(a) \smile e(a')$ and $e(b) \smile e(b')$, thus $e(c) \smile e(c')$.

tensor: if $C = A \otimes B$, let $a : A$, $b : B$ (resp. $a' : A, b' : B$) be the premises of the $\otimes$ l (resp. l') with conclusion $c : C$ (resp. $c' : C$). Suppose the switching respected by τ for l is T_1 (the case it is T_2 is similar), that is:

$$\tau = \uparrow c \uparrow a \ldots\ldots \downarrow a \uparrow b \ldots\ldots \downarrow b \downarrow c \ldots\ldots$$

Since $\tau(c) = \tau(a) \cup \tau(b) \cup \{c\}$, by $c' \in \tau(c)$ we deduce $c' \in \tau(a)$ or $c' \in \tau(b)$. Suppose $c' \in \tau(a)$ (the case $c' \in \tau(b)$ is similar). By proposition

94 condition 2, both $\uparrow c'$ and $\downarrow c'$ are in $\uparrow a - \downarrow a$. Since one occurrence of a' is contiguous with $\uparrow c'$ and $\downarrow c'$ (depending on the l' switching), we deduce $a' \in \uparrow a - \downarrow a$, i.e. $a' \in \tau(a)$. Moreover, by $c <_{\tau,s} c'$, it follows $a <_{\tau,s} a'$. We conclude that (a, a') is a candidate. By induction hypothesis we deduce $e(a) \smile e(a')$, i.e. $e(c) \smile e(c')$.

why not: if $C = ?A$. Since (c, c') is a candidate, then neither c nor c' are weakening conclusions. Let $a_1, \ldots, a_n$ (resp. $a'_1, \ldots, a'_m$) be the premises of the ? link l (resp. l') with conclusion $c : C$ (resp. $c' : C$). Suppose that both the l, l' switchings are S_1 (i.e. order the l, l' premises s.t. the first ones are the switched ones).

Since $c <_{\tau,s} c'$, we deduce $a_1 <_{\tau,s} a'_1$, and since $c' \in \tau(c)$, we deduce $a' \in \tau(a)$, thus (a_1, a'_1) is a candidate. Moreover for any $i, 1 < i \leq n$ and $j, 1 < j \leq m$ have that $a'_j \in \tau(a_i)$ as well as $a_i \in \tau(a'_j)$, so that (a_i, a_j) is a candidate. But for concluding our proof we have to prove that also (a_1, a'_j) and (a_i, a'_1) are candidates. We may prove that (a_1, a'_j) and (a_i, a'_1) by means of condition 6 of proposition 94. In fact by condition 6 and the l, l' switchings, $\downarrow a_1 \downarrow c \uparrow c \uparrow a_1, \downarrow a'_1 \downarrow c' \uparrow c' \uparrow a'_a \sqsubseteq \tau$. We so deduce $a'_j \in \tau(a_1)$ and $a_i \in \tau(a'_1)$, i.e. (a_1, a'_j) and (a'_1, a_i) are candidates.

To sum up, we have proven that for each $i \leq n, j \leq m$, (a_i, a'_j) is a candidate. By induction hypothesis $e(a_i) \smile e(a'_j)$, thus $e(c) \smile e(c')$.

□

By means of lemma 98 we prove straightforwardly the existence of injective experiments in uniform coherent spaces:

Theorem 99 ([TdF03b]) *Let π be a cut-free* $(?\wp)$-**MELL** *proof net without weakening, mix and exponential boxes. There is a coherent space $\mathcal{X}$ and a $\mathfrak{Coh}^{\mathcal{X}}$ injective experiment on π.*

Proof. Let e be an injective experiment on π s.t. for any pair a, b of atomic edges with same type: $e(a) \smile e(b)$ if and only if (a, b) is a candidate.

We have to prove that e respects the uniformity condition, i.e. for any premises a, b of a ? link, $e(a) \smile e(b)$. By proposition 94 condition 4 $a \in \tau(b)$ as well as $b \in \tau(a)$, so that (a, b) is a candidate. By lemma 98 we conclude that $e(a) \smile e(b)$. □

Theorem 100 ([TdF03b]) *Let π_1, π_2 be two* $(?\wp)$-**MELL** *cut-free proof nets without mix. If for all coherent spaces $\mathcal{X}$, $[\![\pi_1]\!]_{\mathfrak{Coh}^{\mathcal{X}}} = [\![\pi_2]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$, then $\pi_1 = \pi_2$.*

Proof. Let π_1, π_2 be two $(?\wp)$-**MELL** cut-free proof nets. By theorem 99 there is an injective experiment on $L(\pi_1)$. By proposition 91 there is an injective n-obsessional experiment on π_1, for any number n greater than the maximum arity of the ? links in π_1. By theorem 89, $LPS(\pi_1) = LPS(\pi_2)$. Finally by proposition 84, $\pi_1 = \pi_2$. □

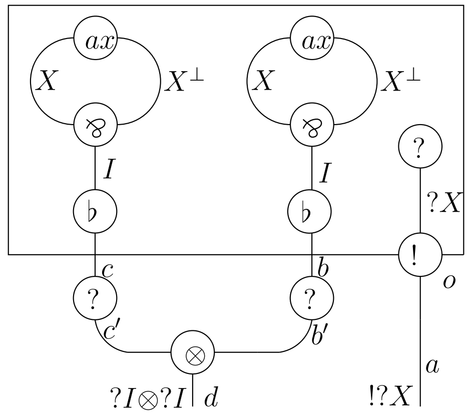


Figure 3.14: example of switching cycle invisible by coherent spaces.

The injectivity of MELL without weakening and mix. By following [TdF03b], we guess that the existence of injective experiments is deeply linked with the connectedness of the correctness graphs of π, i.e. with the existence of a longtrip in π.

We have already noticed that if π is a proof structure without weakening, then π is a proof net without mix iff π is strongly correct, otherwise stated all the trips of π are longtrips. Thus the link between the existence of $\mathfrak{Coh}^{\mathcal{X}}$ injective experiments and the one of a longtrip should be clarified by a proof of the following conjecture:

Conjecture 101 *Let π be a* **MELL** *cut-free proof net without weakening, mix and exponential boxes. There is a coherent space $\mathcal{X}$ and a $\mathfrak{Coh}^{\mathcal{X}}$ injective experiment on π.*

3.4 Exponential acyclicity and cliques

In **MLL** we have a perfect correspondence between switching acyclicity and clique, in the sense of theorems 24 and 25, stating the following:

(*) let π be a **MLL** cut-free proof structure. π is correct if and only if $[\![\pi]\!]_{\mathfrak{Coh}^{\mathcal{X}}}$ is a clique for every coherent space $\mathcal{X}$.

Does the statement (*) hold in presence of exponentials, i.e. for **MELL** proof structures too?

Such a question has been stated by Di Giamberardino in [Gia04], and negatively answered by the following example.

Consider the proof structure π in figure 3.14. Of course π contains the switching cycle $\uparrow c \downarrow b \downarrow b' \uparrow c' \uparrow c$, so it is not correct. Nevertheless $[\![\pi]\!]$ is a clique in both uniform and non-uniform coherent spaces.

Let us show it. Let e_1, e_2 be two experiments on π, let us show that $|e_1| \frown\!\!\!\smile |e_2|\ [(?\mathcal{I}\otimes?\mathcal{I})\parr!?\mathcal{X}]$, where $I = X\parr X^{\perp}$.

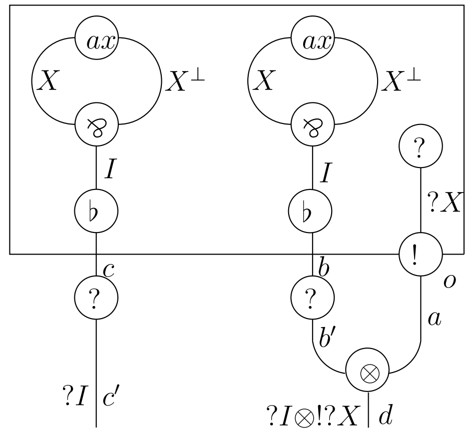


Figure 3.15: example of switching cycle visible by coherent spaces.

Suppose $e_1(o) = [e_1^1, \ldots, e_1^n]$ and $e_2(o) = [e_2^1, \ldots, e_2^m]$. Remark that for any experiments e_i^l, e_j^h, $e_i^l(c) \stackrel{\frown}{\smile} e_j^h(c)\,[?I]$ as well as $e_i^l(b) \stackrel{\frown}{\smile} e_j^h(b)\,[?I]$. There are two cases, depending if either $n = m$ or $n \neq m$.

In case $n = m$, then we deduce $e_1(c) \stackrel{\frown}{\smile} e_2(c)$ and $e_1(b) \stackrel{\frown}{\smile} e_2(b)$, hence $e_1(d) \stackrel{\frown}{\smile} e_2(d)$. Of course $e_1(a) \stackrel{\frown}{\smile} e_2(a)$, thus $|e_1| \stackrel{\frown}{\smile} |e_2|\,[(?\mathcal{I}\otimes?\mathcal{I})\parr!?\mathcal{X}]$.

In case $n \neq m$, then $e_1(a) \frown e_2(a)$, thus $|e_1| \stackrel{\frown}{\smile} |e_2|\,[(?\mathcal{I}\otimes?\mathcal{I})\parr!?\mathcal{X}]$.

The failure of the correspondence between switching acyclicity and coherent spaces shows that these last ones read the exponential boxes in a different way as switching paths do. Indeed the cycle $\uparrow c \downarrow b \downarrow b' \uparrow c' \uparrow c$ is due to the box associated with o: if we erase o and the frame of its box, we would get a correct proof structure. Coherent spaces do not read the boxes as switching paths do, but it is not true that they do not read the boxes at all. For example, consider the proof structure π' in figure 3.15.

π' has the switching cycle $\downarrow a \uparrow b' \uparrow b \downarrow a$, which is due to the box of o, as in the example before. However in this case the cycle is visible by coherent spaces, i.e. $[\![\pi']\!]$ is not a clique. Let us show it.

Let e_1, e_2 be two experiments on π', s.t. $e_1(o) = \emptyset$ and $e_2(o) = [e']$, for an experiment e' on the o box. Clearly $e_1(c') \smile e_2(c')\,[?\mathcal{I}]$ and $e_1(b') \smile e_2(b')\,[?\mathcal{I}]$. By the last one we deduce $e_1(d) \smile e_2(d)\,[?\mathcal{I}\otimes!?\mathcal{X}]$, i.e. $|e_1| \smile |e_2|\,[?\mathcal{I}\parr(?\mathcal{I}\otimes!?\mathcal{X})]$.

In this section we define the visible paths (definition 102). Such a definition will induce a new geometric criterion, which we call *weak correctness*, characterizing those proof structures whose interpretation is a clique.

We have defined in section 3.2 two different kinds of coherent spaces: $\mathfrak{Coh}$ and $\mathfrak{nuCoh}$. The general question of characterizing the cycles visible by a semantics can be set both in $\mathfrak{Coh}$ and in $\mathfrak{nuCoh}$. In the uniform case however, such a question gets mixed with the uniformity problem, for which our tools are yet too weak. For such a reason we will deal only with $\mathfrak{nuCoh}$.

From now on, by coherent spaces we mean precisely non-uniform multiset based coherent spaces.

Let ϕ be a path in a proof structure and π^o be an exponential box associated with a ! link at the same depth of ϕ. A **passage of ϕ through π^o** is a sequence $\uparrow a \downarrow b \sqsubseteq \phi$ for a, b doors of π^o.

Notice that a switching path can pass through an exponential box by means of any pair of its doors; with the following definition we forbid instead some of such passages:

Definition 102 *Let π be a proof structure. By induction on the depth of π, we define its **visible paths**:*

- *if π has depth 0, then a visible path in π is a switching path;*
- *if π has depth $n+1$, let π^o be a box associated with a link $!\, o$, a, b be doors of π^o we say that:*
    - *a **is in the orbit of** o if either a is the principal door or there is a visible path in π^o from the premise of o to a;*
    - *a **leads to** b if either b is in the orbit of o or there is a visible path in π^o from a to b;*

  then a visible path in π is a switching path s.t. for any passage $\uparrow a \downarrow b$ through an exponential box, a leads to b.

*A proof structure is **weakly correct** whenever it does not contain any visible cycle.*

Visible paths introduce two noteworthy novelties with respect to the switching paths:

1. they partly break the black box principle: the admissible passages through an exponential box depend on what is inside the box, i.e. changing the contents of a box may alter the visible paths outside it;
2. they are sensitive to the direction: if ϕ is visible from a to b, the same path done in the opposite direction from b to a may be no longer visible. For example recall the proof structure of figure 3.14: the path $\uparrow b \downarrow a$ is visible, but $\uparrow a \downarrow b$ isn't, since a does not lead to b.

Of course if π is correct then it is also weakly correct, but the converse does not hold. For example recall the proof structure of figure 3.14, which is weakly correct although it contains a switching cycle.

The weakly correctness characterizes those proof structures whose interpretation is a clique, in the following sense:

Theorem 103 *Let π be a* **MELL** *proof structure, $\mathcal{X}$ be any non-uniform coherent space.*

If π is weakly correct, then $[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ is a clique.

Theorem 104 *Let π be a cut-free* **MELL** *proof structure, $\mathcal{X}$ be a non-uniform coherent space with $x, y, z \in |\mathcal{X}|$ such that $x \frown y\, [\mathcal{X}]$, $x \smile z\, [\mathcal{X}^{\perp}]$ and $x \equiv x\, [\mathcal{X}]$.*

If $[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ is a clique, then π is weakly correct.

The following subsection 3.4.1 (resp. 3.4.2) is dedicated to the proof of theorem 103 (resp. 104).

3.4.1 Proof of theorem 103

Theorem 103 is a straightforward consequence of the following lemma:

Lemma 105 *Let π be a weakly correct proof structure. If $d : D$ is a conclusion of π and e_1, e_2 are two experiments such that $e_1(d) \smile e_2(d)\,[\mathcal{D}]$, then there is a visible path ϕ from d to a conclusion $d' : D'$ such that $e_1(d') \frown e_2(d')\,[\mathcal{D}']$.*

Proof. Let $e_1(d) \smile e_2(d)\,[\mathcal{D}]$. We prove the lemma by induction on the exponential depth of π. Remark that the base of induction corresponds with lemma 22 in chapter 1.

We define a sequence of visible paths $\phi_1 \subset \phi_2 \subset \ldots \subset \phi_k$, such that ϕ_1 is exactly $\uparrow d$, ϕ_k starts from $\uparrow d$ and ends in $\downarrow d'$, for a conclusion d' of π, and for each ϕ_j among $\phi_1, \ldots, \phi_k$:

1. ϕ_j is a visible path at depth 0;
2. for every edge $c : \mathcal{C}$, if $\uparrow c \in \phi_j$ then $e_1(c) \smile e_2(c)\,[\mathcal{C}]$, if $\downarrow c \in \phi_j$ then $e_1(c) \frown e_2(c)\,[\mathcal{C}]$.

Let us define ϕ_{j+1} from ϕ_j, this last one supposed satisfying conditions 1 and 2. Let $c : \mathcal{C}$ be the last edge of ϕ_j. Then:

- in case $\downarrow c \in \phi_j$, by hypothesis c is an edge of π at depth 0 and $e_1(c) \frown e_2(c)\,[\mathcal{C}]$:
  - if c is a premise of a $⅋$ with conclusion $b : B$, then $e_1(b) \frown e_2(b)\,[\mathcal{B}]$. We define $\phi_{j+1} = \phi_j * \downarrow b$;
  - if c is a premise of a $\otimes$ with conclusion $b : C \otimes A$ and premises $c : C, a : A$, in case $e_1(b) \frown e_2(b)\,[\mathcal{C} \otimes \mathcal{A}]$, we define $\phi_{j+1} = \phi_j * \downarrow b$; otherwise $e_1(a) \smile e_2(a)\,[\mathcal{A}]$, in this case we define $\phi_{j+1} = \phi_j * \uparrow a$;
  - if c is a premise of a $\flat$ with conclusion $b : \flat C$, then $e_1(b) \frown e_2(b)\,[?\mathcal{C}]$. We define $\phi_{j+1} = \phi_j * \downarrow b$;
  - if c is a premise of a $?$ with conclusion b, then c (resp. b) is of type $\flat B$ (resp. $?B$) for a formula B, and $e_1(c) \subseteq e_1(b)$, $e_2(c) \subseteq e_2(b)$. Since $e_1(c) \frown e_2(c)\,[?B]$, we deduce $e_1(b) \frown e_2(b)\,[?B]$. We define $\phi_{j+1} = \phi_j * \downarrow b$;
  - if c is a premise of a cut with premises $c : C, b : C^{\perp}$, then $e_1(b) \smile e_2(b)\,\left[\mathcal{C}^{\perp}\right]$, so let $\phi_{j+1} = \phi_j * \uparrow b$;
  - if c is a conclusion of π, then we define ϕ_j as ϕ_k.

  Notice that c cannot be a door of an exponential box, being at depth 0. Clearly ϕ_{j+1} satisfies condition 2. Let us prove that it is visible.

  Since in any case the edge b added to ϕ_{j+1} is not a door of an exponential box, all the ϕ_{j+1} passages through exponential boxes are already in ϕ_j. Thus we only have to prove that ϕ_{j+1} still is a switching path for deducing that it is visible. Now let us suppose that b is premise of a $⅋/?$ link already crossed by ϕ_j and let us prove a contradiction. Call c the conclusion of the $⅋/?$ link, of course $c \in \phi_j$. Since $e_1(b) \frown e_2(b)$ we deduce $e_1(c) \frown e_2(c)$. Since ϕ_j meets condition 2, $\downarrow c \in \phi_j$. Thus ϕ_j has the following shape:

$$\phi_j = \phi'_j * \downarrow c \; * \phi''_j * \downarrow b$$

but then $\downarrow c \; * \phi''_j * \downarrow b \downarrow c$ should be a visible cycle in π, so violating the weak correctness of π. Thus we conclude that b cannot be the premise of a $⅋/?$ link already crossed by ϕ_j, so proving that ϕ_{j+1} still is switching.

- in case $\uparrow c \in \phi_j$, by hypothesis c is an edge of π at depth 0 and $e_1(c) \smile e_2(c)\,[\mathcal{C}]$:
  - if c is the conclusion of an axiom with conclusions $c : C, b : C^{\perp}$, then $e_1(b) \frown e_2(b)\,\left[\mathcal{C}^{\perp}\right]$, thus we define $\phi_{j+1} = \phi_j * \downarrow b$;
  - if c is the conclusion of a $⅋$ or a $\otimes$, then exists a premise $b : B$ s.t. $e_1(b) \frown e_2(b)\,[\mathcal{B}]$. We define $\phi_{j+1} = \phi_j * \uparrow b$;
  - if c is the conclusion of a ! link o, let $C = !A$, π^o be the o box and $a : A$ be the o premise. Since $e_1(c) \smile e_2(c)\,[!\mathcal{A}]$, there are $e_1^o \in e_1(o)$, $e_2^o \in e_2(o)$ such that $e_1^o(a) \smile e_2^o(a)\,[\mathcal{A}]$. By induction hypothesis on π^o, e_1^o, e_2^o, there is a π^o conclusion $b : \flat B$ and a visible path on π^o from $\uparrow a$ to $\downarrow b$, such that $e_1^o(b) \frown e_2^o(b)\,[?B]$.
    Since $e_1^o(b) \subseteq e_1(b)$ and $e_2^o(b) \subseteq e_2(b)$, we deduce $e_1(b) \frown e_2(b)\,[?B]$. We thus define $\phi_{j+1} = \phi_j * \downarrow b$. Remark that c leads to b, hence the passage $\uparrow c \downarrow b$ is allowed to the visible paths;
  - if c is an auxiliary conclusion of an exponential box π^o associated with a ! link o, let $b : !B$ be the o conclusion and $a : B$ its premise. We split in two cases:
    - ∗ in case $e_1(b) \not\equiv e_2(b)\,[!\mathcal{B}]$, then $e_1(b) \frown e_2(b)$ or $e_1(b) \smile e_2(b)$.
      If $e_1(b) \frown e_2(b)\,[!\mathcal{B}]$, we set $\phi_{j+1} = \phi_j * \downarrow b$. Remark that c leads to b, being this last one in the o orbit.
      If $e_1(b) \smile e_2(b)\,[!\mathcal{B}]$, then there is $e_1^o \in e_1(o)$, $e_2^o \in e_2(o)$ s.t. $e_1^o(a) \smile e_2^o(a)\,[\mathcal{B}]$. By induction hypothesis on π^o, e_1^o, e_2^o, there is a π^o conclusion $b' : \flat B'$ and a π^o visible path from $\uparrow a$ to $\downarrow b'$, s.t. $e_1^o(b') \frown e_2^o(b')\,[?\mathcal{B}']$. Since $e_1^o(b') \subseteq e_1(b')$ and $e_2^o(b') \subseteq e_2(b')$, we deduce $e_1^o(b') \frown e_2^o(b')\,[?\mathcal{B}']$. Remark that since by hypothesis $e_1(c) \smile e_2(c)\,[\mathcal{C}]$, we are sure that b' and c are different π^o auxiliary conclusions, moreover c leads to b', being this last one in the o orbit. Define $\phi_{j+1} = \phi_j * \downarrow b'$.
    - ∗ in case $e_1(b) \equiv e_2(b)\,[!\mathcal{B}]$, then by definition of ! neutrality there is an enumeration $e_1^1, \ldots, e_1^l$ (resp. $e_2^1, \ldots, e_2^l$) of the π^o experiments associated with o by e_1 (resp. e_2), s.t. for each $i \leq l$, $e_1^i(a) \equiv e_2^i(a)\,[\mathcal{B}]$. Remark that $l > 0$, otherwise $e_1(c) \equiv e_2(c)$.
      On the other hand, since $e_1(c) \smile e_2(c)\,[\mathcal{C}]$ and $e_1(c) = e_1^1(c) + \ldots e_1^l(c)$, $e_2(c) = e_2^1(c) + \ldots e_2^l(c)$, there is an $h \leq l$ s.t. $e_1^h(c) \smile e_2^h(c)\,[\mathcal{C}]$. Now we apply the induction hypothesis on π^o, e_1^h, e_2^h, so obtaining a π^o conclusion $b' : B'$ and a visible path from $\uparrow c$ to $\downarrow b'$ s.t. $e_1^h(b') \frown e_2^h(b')\,[\mathcal{B}']$. Remark that $b' \neq a$, since we are in the hypothesis that $e_1^h(a) \equiv e_2^h(a)\,[\mathcal{B}]$. Thus in particular $B' = ?D$ for a formula D. By $e_1^h(b') \frown e_2^h(b')\,[?\mathcal{D}]$, we deduce $e_1(b') \frown e_2(b')\,[?\mathcal{D}]$. Hence we set $\phi_{j+1} = \phi_j * \downarrow b'$, remarking that c leads to b', existing a visible path from c to b'.

- if c is the conclusion of a $\flat$ link with premise $b : B$, then $e_1(b) \smile e_2(b)\,[\mathcal{B}]$. We define $\phi_{j+1} = \phi_j * \uparrow b$;
- if c is the conclusion of a ? l, remark that l is not a weakening, since $e_1(c) \smile e_2(c)\,[\mathcal{C}]$.
  Let $C = ?B$ and $b_1 : \flat B, \ldots, b_k : \flat B$ be the premises of l. Recall that $e_1(c) = e_1(b_1) + \ldots + e_1(b_k)$ and $e_2(c) = e_2(b_1) + \ldots + e_2(b_k)$. Since $e_1(c) \smile e_2(c)\,[?\mathcal{B}]$, it exists $i \leq k$ s.t. $e_1(b_i) \smile e_2(b_i)\,[?B]$. Hence we define $\phi_{j+1} = \phi_j * \uparrow b_i$.

Of course ϕ_{j+1} meets condition 2, let us prove that it is a visible path. In each case we have added a box passage to ϕ_{j+1} (i.e. in the ! and auxiliary conclusion cases) we have also proved that such a new passage is admitted by visible paths. Thus we only have to prove that ϕ_{j+1} is a switching path.

We give the proof only in one case, the most crucial one, being the proof in the other cases similar or easier. Let us recall the case c is a conclusion of ! link o. We have extended ϕ_{j+1} by adding $\downarrow b$ for a π^o auxiliary door. Since b has a $\flat$ type, it is premise of a ? link l. We have to prove that ϕ_j does not contain any l premise. Let $d : ?B$ be the l conclusion, let us prove $d \notin \phi_j$, which implies that no l premise is in ϕ_j.

Since $e_1(b) \subseteq e_1(d)$ and $e_2(b) \subseteq e_2(d)$ and $e_1(b) \frown e_2(b)\,[?B]$, we deduce $e_1(d) \frown e_2(d)\,[?B]$. Thus, by condition 2 on ϕ_j, $\uparrow d \notin \phi_j$. On the other hand, suppose $\downarrow d \in \phi_j$, that is $\phi_j = \phi'_j * \downarrow d * \phi''_j * \uparrow c$. In this case $\downarrow d * \phi''_j * \uparrow c \downarrow b \downarrow d$ should be a visible cycle, violating the π^o weak correctness. We conclude $d \notin \phi_j$, i.e. ϕ_{j+1} is switching.

Since π is weakly correct, each ϕ_j is not a visible cycle. Thus the sequence $\phi_1, \phi_2, \phi_3, \ldots$ will meet eventually a conclusion c' of π, so terminating in a path ϕ_k satisfying the lemma. □

Proof of theorem 103. Recall the statement of theorem 103:

> Let π be a **MELL** proof structure, $\mathcal{X}$ be any non-uniform coherent space.
>
> If π is weakly correct, then $[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ is a clique.

Proof. Let π be a **MELL** proof structure, $\mathcal{X}$ be any non-uniform coherent space and e_1, e_2 be two experiment on π. By lemma 105, $|e_1| \frown\!\!\smile |e_2|$, hence $[\![\pi]\!]_{\mathfrak{nuCoh}^{\mathcal{X}}}$ is a clique. □

3.4.2 Proof of theorem 104

The proof of theorem 104 is based on the key lemma 110. In some sense lemma 110 is the converse of lemma 105: lemma 105 associates with two experiments e_1, e_2 a visible path proving $|e_1| \frown\!\!\smile |e_2|$, lemma 110 instead associates with a visible cycle (morally) two experiments s.t. $|e_1| \smile |e_2|$.

However lemma 110 has to take care of a typical difficulty of ? links. For proving the lemma we need to manage the coherence/incoherence relationship between the values of e_1 and e_2. Unfortunately ? links soon make such a relationship unmanageable. In fact, if l is a ? link with conclusion c and premises

$a_1, \ldots, a_n$, the incoherence $e_1(c) \stackrel{\smile}{\frown} e_2(c)$ holds if and only if for each $i, j \leq n$, $e_1(a_i) \stackrel{\smile}{\frown} e_2(a_j)$. The incoherence on one edge (the conclusion of l) is linked with the incoherence on n^2 pairs of edges (the premises of l): such an explosion of the number of edges soon becomes unmanageable.

Remark that a similar problem is at the origin of the difficulty of the conjecture 101.

Luckily, we have found a way for avoiding the problem in the proof of lemma 110. Namely we have noticed that one of the two experiments e_1, e_2 which we want to associate with a visible path can be chosen to be very simple, i.e. e_1 can be a (x, n)-*simple experiment* (see definition 108). If x is an element of a coherent space $\mathcal{X}$ and $n \in \mathbb{N}$, the unique (x, n)-simple experiment on a proof structure π is the n-obsessional experiment (definition 85) taking the constant value x on the axioms of π (definition 108). The key property of a (x, n)-simple experiment is that all of its possible values on an arbitrary edge of type A are semantically characterized by the definition 106. To be precise, they are (x, n)-*simple elements of* $\mathcal{A}$ *with degree less or equal to* wn^d, where d is the exponential depth of π and w is the maximal arity of the ? links in π (proposition 109). Once we have such a semantical characterization, we may define the second experiment e_2 not by looking at the particular value that the (x, n)-simple experiment e_1 takes on an edge of type A, but by looking at all the possible values e_1 can take on edges of type A, i.e. by referring to the (x, n)-simple elements of $\mathcal{A}$ with degree less or equal to wn^d. In this way, if we are considering the premises $a_1 : \flat A, \ldots, a_n : \flat A$ of a ? link, instead of proving that for each $i, j \leq n$, $e_1(a_i) \stackrel{\smile}{\frown} e_2(a_j)$, we reduce to check that for each $i \leq n$ and (x, n)-simple element $v \in ?\mathcal{A}$ with degree less or equal to wn^d, $v \stackrel{\smile}{\frown} e_2(a_j)$.

Definition 106 *Let $n \in \mathbb{N}$, x be an element of a non-uniform coherent space $\mathcal{X}$ and $\mathcal{C}$ the $\mathfrak{nuCoh}^{\mathcal{X}}$ interpretation of a formula C. An element $v \in \mathcal{C}$ is a* $(\mathbf{x}, \mathbf{n})$***-simple element with degree*** $\mathbf{d(v)}$ *if:*

- *in case $C = X, X^{\perp}$, $v = x$ and $d(v) = 0$;*
- *in case $C = A \otimes B, A \parr B$, $v = < v', v'' >$, for v' (resp. v'') (x, n)-simple element in $\mathcal{A}$ (resp. in $\mathcal{B}$), and $d(v) = max(d(v'), d(v''))$;*
- *in case $C = !A$, $v = n\,[v']$, for v' (x, n)-simple element of $\mathcal{A}$, and $d(v) = d(v')$;*
- *in case $C = ?A$, $v = [v_1, \ldots, v_m]$, for $m \geq 0$, each v_i (x, n)-simple element of $\mathcal{A}$, and $d(v) = max(m, d(v_1), \ldots, d(v_m))$.*

Remark that in general an element can be (x, n)-simple in $\mathcal{C}$ but not in $\mathcal{C}^{\perp}$, for example the empty multiset is a (x, n)-simple element in $?\mathcal{C}$ but not in $!\mathcal{C}^{\perp}$, if $n \neq 0$.

Proposition 107 *Let $\mathcal{X}$ be a non-uniform coherent space, $x \in \mathcal{X}$ s.t. $x \equiv x\,[\mathcal{X}]$, $\mathcal{C}$ be the $\mathfrak{nuCoh}^{\mathcal{X}}$ interpretation of a formula C. For any $n \in \mathbb{N}$ and v, v' (x, n)-simple elements of $\mathcal{C}$, $v \stackrel{\smile}{\frown} v'\,[\mathcal{C}]$.*

Proof. By an easy induction on C:

atom: if $C = X, X^{\perp}$, the case is immediate;

tensor: if $C = A \otimes B, A \parr B$, then $v =< w, u >$, $v' =< w', u' >$, for w, w' (resp. u, u') (x, n)-simple elements of $\mathcal{A}$ (resp. of $\mathcal{B}$). By induction hypothesis $w \asymp w'\,[\mathcal{A}]$ and $u \asymp u'\,[\mathcal{B}]$, hence $v \asymp v'\,[\mathcal{C}]$;

of course: if $C = !B$, then $v = n\,[w]$, $v' = n\,[w']$ for w, w' (x, n)-simple elements of $\mathcal{B}$. By induction hypothesis $w \asymp w'\,[\mathcal{B}]$, hence $v \asymp v'\,[\mathcal{C}]$;

why not: if $C = ?B$, then $v = [v_1, \ldots, v_m]$, $v' = [v'_1, \ldots, v'_h]$, for $m, h \geq 0$ and each v_i, v'_j (x, n)-simple elements of $\mathcal{B}$. By induction hypothesis for each $i \leq m$, $j \leq h$, we have $v_i \asymp v'_j\,[\mathcal{B}]$, hence $v \asymp v'\,[\mathcal{C}]$.

□

Definition 108 *Let π be a proof structure, $n \in \mathbb{N}$, x be an element of a non-uniform coherent space $\mathcal{X}$.* ***The* $(\mathbf{x}, \mathbf{n})$*-simple experiment on*** *π, denoted by $e^{\pi}_{(x,n)}$, is defined as follows:*

- *for each edge $a : X$ at depth 0, $e^{\pi}_{(x,n)}(a) = x$;*
- *for each ! link o at depth 0, let π^o be the o box, $e^{\pi}_{(x,n)}(o) = n\left[e^{\pi^o}_{(x,n)}\right]$.*

Proposition 109 *Let π be a proof structure, d be the depth of π and w be the maximal arity of the links ? in π. Let $e^{\pi}_{(x,n)}$ be the (x, n)-simple experiment on π. For any edge $c : C$ at depth 0, $e^{\pi}_{(x,n)}(c)$ is a (x, n)-simple element of $\mathcal{C}$ with degree at most wn^d.*

Proof. By an easy induction on C. For the degree of $e^{\pi}_{(x,n)}(c)$ remark that a (x, n)-simple experiment is a particular case of n-obsessional experiment, thus use proposition 87. □

The key lemma for the proof of theorem 104 is the following lemma 110:

Lemma 110 *Let $\mathfrak{nuCoh}^{\mathcal{X}}$ be defined from a coherent space $\mathcal{X}$ s.t. $\exists x, y, z \in \mathcal{X}$, $x \equiv x\,[\mathcal{X}]$, $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$.*

Let π be a cut-free proof structure, k be the maximal number of doors of a box of π. Let ϕ be a visible path of π at depth 0 from a conclusion $\uparrow a$ to a conclusion $\downarrow b$, s.t. ϕ is not a cycle.

For any $n, m \in \mathbb{N}$, $m \geq n \geq k$, there is an experiment e_ϕ on π, s.t. for any π edge $c : \mathcal{C}$ at depth 0 and any (x, n)-simple element v in $\mathcal{C}$ with degree less or equal m:

1. *if $\exists c' \geq c$, $c' \in \phi$, then $e_\phi(c) \not\equiv v\,[\mathcal{C}]$;*
2. *if $\downarrow c \notin \phi$, then $e_\phi(c) \asymp v\,[\mathcal{C}]$.*

Proof.

Once for all we fix $x, y, z \in |\mathcal{X}|$ s.t. $x \equiv x\,[\mathcal{X}]$, $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$.

The proof of the lemma is by induction on the π exponential depth.

Firstly we define e_ϕ on the conclusions of the π axioms at depth 0. Let $a : X$ be an edge at depth 0:

- if $\uparrow a \in \phi$, $e_\phi(a) = z$;
- if $\downarrow a \in \phi$, $e_\phi(a) = y$;
- otherwise $e_\phi(a) = x$.

Secondly we define e_ϕ on the π ! links at depth 0. Let o be a ! link at depth 0, π^o the o box and $\uparrow a_1 \downarrow b_1, \ldots, \uparrow a_h \downarrow b_h$ be the ϕ passages through π^o $(h \geq 0)$. Remark that by definition $h \leq k \leq n$, where k is the maximal number of doors of a box of π.

Notice that, being ϕ visible, for each $i \leq h$, a_i leads to b_i. We associate with each passage $\uparrow a_i \downarrow b_i$ an experiment $e^o_{\phi_i}$ on π^o as follows:

- if $\downarrow b_i$ is the principal door, then $e^o_{\phi_i} = e^{\pi^o}_{(x,n)}$;
- if $\downarrow b_i$ is an auxiliary door in the orbit of o, then let ϕ_i be a visible path in π^o from the o premise to $\downarrow b_i$. By induction we may define an experiment e_{ϕ_i} on π^o satisfying condition 1,2 with respect to π^o and ϕ_i;
- if $\downarrow b_i$ is an auxiliary door not in the orbit of o, then let ϕ_i be a visible path in π^o from $\uparrow a_i$ to $\downarrow b_i$. By induction we may define an experiment e_{ϕ_i} on π^o satisfying condition 1,2 with respect to π^o and ϕ_i.

Finally we define e_ϕ on o:

- if ϕ does not pass through the orbit of o:

$$e_\phi(o) = [e_{\phi_1}, \ldots, e_{\phi_h}] + (n-h)\left[e^{\pi^o}_{(x,n)}\right]$$

- if ϕ passes through the orbit of o:

$$e_\phi(o) = [e_{\phi_1}, \ldots, e_{\phi_h}] + (m+1-h)\left[e^{\pi^o}_{(x,n)}\right]$$

Now, let us prove that e_π satisfies conditions 1, 2. Let $c : C$ be an edge of π at depth 0, we prove 1, 2 by induction on C.

Atom: in case $C = X, X^\perp$, both 1, 2 are immediate.

Par: in case $C = A \parr B$, let $a : A$, $b : B$ be the premises of the $\parr$ with conclusion c, $v =< v', v'' >$ be a (x,n)-simple element of $\mathcal{C}$ with degree less or equal m:

1. if $\exists c' \geq c$, $c' \in \phi$, then $\exists a' \geq a$, $a' \in \phi$ or $\exists b' \geq b$, $b' \in \phi$, thus by induction $e_\phi(a) \not\equiv v'$ or $e_\phi(b) \not\equiv v''$. In both cases $e_\phi(c) \not\equiv v$;
2. if $\downarrow c \notin \phi$, then $\downarrow a \notin \phi$ and $\downarrow b \notin \phi$, thus by induction $e_\phi(a) \asymp v'$ and $e_\phi(b) \asymp v''$. Hence we deduce $e_\phi(c) \asymp v$.

Tensor: in case $C = A \otimes B$, let $a : A$, $b : B$ be the premises of the $\otimes$ with conclusion c, $v =< v', v'' >$ be a (x,n)-simple element of $\mathcal{C}$ with degree less or equal m:

1. if $\exists c' \geq c$, $c' \in \phi$, then $e_\phi(c) \not\equiv v$ by the same argument as in the $⅋$ case;
2. if $\downarrow c \notin \phi$, we split in three cases.

   In case $\downarrow a \in \phi$, then $\uparrow b \in \phi$. Of course $\downarrow b \notin \phi$, hence by induction hypothesis 2, $e_\phi(b) \underset{\frown}{\smile} v''$. Moreover, by induction hypothesis 1 $e_\phi(b) \not\equiv v''$, thus $e_\phi(b) \smile v''$. By symmetrical arguments, if $\downarrow b \in \phi$, we deduce $e_\phi(a) \smile v'$. In both cases we have $e_\phi(c) \smile v$.

   In case both $\downarrow a, \downarrow b \notin \phi$, then by induction $e_\phi(a) \underset{\frown}{\smile} v'$ and $e_\phi(b) \underset{\frown}{\smile} v''$, which implies $e_\phi(c) \underset{\frown}{\smile} v$.

Of course: in case $C = !B$, let o be the $!$ link with conclusion $c : !B$, π^o the o box and $b : B$ the o premise. Let $v = n\,[v']$ be a (x, n)-simple element of $!\mathcal{B}$ with degree less or equal m:

1. if $\exists c' \geq c$, $c' \in \phi$, then clearly $c = c'$ (ϕ is a path crossing only edges at depth 0). In this case ϕ passes through the o orbit, so $e_\phi(c)$ has $m + 1$ elements. Since v has n elements and $n \leq m$, we deduce $e_\phi(c) \not\equiv v$;
2. if $\downarrow c \notin \phi$. We split in two cases, depending if ϕ passes or not through the o orbit:
   - in case ϕ passes through the o orbit, then it exists a visible path ϕ_i associated with a ϕ passage through the o orbit. Remark that $\uparrow b \in \phi_i$ (being $\downarrow c \notin \phi$), hence by definition of the experiment e_{ϕ_i} associated with ϕ_i, we have both $e_{\phi_i}(b) \not\equiv v'$ (by 1) and $e_{\phi_i}(b) \underset{\frown}{\smile} v'$ (by 2), i.e. $e_{\phi_i}(b) \smile v'$. Since $e_{\phi_i}(b) \in e_\phi(c)$, we deduce $e_\phi(c) \smile v$;
   - in case ϕ does not pass through the o orbit, then let $\phi_1, \dots, \phi_h$ ($h \geq 0$) be the visible paths associated with the ϕ passages through o. Since ϕ does not pass through the o orbit, for each $i \leq h$, $b \notin \phi_i$. Hence by definition of the experiment e_{ϕ_i} associated with ϕ_i, we have $e_{\phi_i}(b) \underset{\frown}{\smile} v'$. Moreover recall that $e^{\pi^o}_{(x,n)}$ is the (x, n)-simple experiment on π^o. By proposition 109, $e^{\pi^o}_{(x,n)}(b)$ is a (x, n)-simple element of $\mathcal{B}$, hence by proposition 107, $e^{\pi^o}_{(x,n)}(b) \underset{\frown}{\smile} v'$. Finally, since $e_\phi(c) = [e_{\phi_1}(b), \dots, e_{\phi_h}(b)] + (n - h)\left[e^{\pi^o}_{(x,n)}(b)\right]$, we deduce $e_\phi(c) \underset{\frown}{\smile} v$.

Why not: in case $C = ?B$, let v be a (x, n)-simple element of $\mathcal{C}$ with degree less or equal to m:

1. if $\exists c' \geq c$, $c' \in \phi$, then c is not conclusion of a weakening. Let $b_1 : \flat B, \dots, b_h : \flat B$ be the premises of the $?$ link with conclusion c. Notice there is an $i \leq h$, $\exists b'_i \geq b_i$, $b'_i \in \phi$, so by induction hypothesis $e_\phi(b_i) \not\equiv v'$, for any (x, n)-simple element v' of $\mathcal{C}$ with degree less or equal m.

   Now, suppose $e_\phi(c) \equiv v$ and let us prove a contradiction. Since $e_\phi(b_i) \subseteq e_\phi(c)$, there should be a subset $v' \subseteq v$ s.t. $e_\phi(b_i) \equiv v'$, but we have just proven $e_\phi(b_i) \not\equiv v'$, for any (x, n)-simple element v' of $\mathcal{C}$ with degree less or equal m. Hence we conclude $e_\phi(c) \not\equiv v$;

2. if $\downarrow c \notin \phi$, in case c is conclusion of a weakening then it is immediate that $e_\phi(c) \smile\frown v$.

   Otherwise, let $b_1 : \flat B, \ldots, b_h : \flat B$ be the premises of the ? link with conclusion c. Of course for each $i \leq h$, $\downarrow b_i \notin \phi$, hence by induction hypothesis $e_\phi(b_i) \smile\frown v$. Since $e_\phi(c) = e_\phi(b_1) + \ldots + e_\phi(b_h)$, we deduce $e_\phi(c) \smile\frown v$.

$\flat$-formula: in case $C = \flat B$, then c is conclusion of a $\flat$ link at depth 0 or it is an auxiliary door of a ! box. In the first case it is very simple proving 1, 2. Let us deal with the second case.

Let c be an auxiliary door of a box π^o associated with a ! link o at depth 0. Let v be a (x,n)-simple element of $\mathcal{C}$ with degree less or equal m:

1. if $\exists c' \geq c$, $c' \in \phi$, then clearly $c' = c$ (ϕ is a path crossing only edges at depth 0). In this case there is a π^o door d s.t. $\uparrow c \downarrow d$ or $\uparrow d \downarrow c$ is a ϕ passage through o. We split in two cases, depending if ϕ passes or not through the o orbit:
   - in case ϕ passes through the o orbit, then $e_\phi(c)$ has at least $m+1$ elements, while v has at most m elements, being of degree less or equal m. Thus $e_\phi(c) \not\equiv v$;
   - in case ϕ does not pass through the o orbit, let ϕ_i be the visible path in π^o between c and d. Of course $c \in \phi_i$, thus $e_{\phi_i}(c) \not\equiv v'$, for any (x,n)-simple element v' of $\mathcal{C}$ with degree less or equal m. Since $e_{\phi_i}(c) \subseteq e_\phi(c)$, we conclude $e_\phi(c) \not\equiv v$, by the same argument as in point 1 case why not.
2. if $\downarrow c \notin \phi$, let $\phi_1, \ldots, \phi_h$ (for $h \geq 0$) be the visible paths in π^o associated with the ϕ passages through o. Since $\downarrow c \notin \phi$, then for each $i \leq h$, $\downarrow c \notin \phi_i$, thus by ϕ_i definition $e_{\phi_i}(c) \smile\frown v$. Moreover recall that $e^{\pi^o}_{(x,n)}$ is the (x,n)-simple experiment on π^o. By proposition 109, $e^{\pi^o}_{(x,n)}(c)$ is a (x,n)-simple element of $\mathcal{C}$, hence by proposition 107, $e^{\pi^o}_{(x,n)}(c) \smile\frown v$. Finally, since $e_\phi(c) = e_{\phi_1}(c) + \ldots + e_{\phi_h}(c) + (n - h)\left[e^{\pi^o}_{(x,n)}(c)\right]$, we deduce $e_\phi(c) \smile\frown v$.

$\square$

Lemma 111 *Let $\mathfrak{nuCoh}^{\mathcal{X}}$ be defined from a coherent space $\mathcal{X}$ s.t. $\exists x, y, z \in \mathcal{X}$, $x \equiv x\,[\mathcal{X}]$, $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$.*

Let π be a cut-free proof structure with conclusions Π, k be the maximal number of doors of an exponential box in π. If π is not weakly correct then for any $n, m \in \mathbb{N}$, $m \geq n \geq k$, there is an experiment $e : \pi$, such that for any (x,n)-simple element v in $\wp\Pi$ with degree less or equal to m, $|e| \smile v\,[\wp\Pi]$.

Proof. Let us fix two numbers m, n, $m \geq n \geq k$, and let us suppose π is not weakly correct. We prove by induction on the number of links of π, that there is an experiment $e : \pi$, s.t. for any (x,n)-simple element v in $\wp\Pi$ with degree less or equal to m, $|e| \smile v\,[\wp\Pi]$.

Base of induction: if π has only terminal axioms, then π is weakly correct, which is contrary to the hypotheses.

Par: if π has a terminal $\parr$ link l with conclusion $c : A \parr B$ and premises $a : A$, $b : B$, define π' from π by erasing l and its conclusion. Suppose $\Pi = A \parr B, \Pi''$, hence $\Pi' = A, B, \Pi''$ are the conclusions of π'. Of course π' is not weakly correct, thus by induction hypothesis there is an experience $e' : \pi'$, s.t. for any (x, n)-simple element v' in $\parr \Pi'$ with degree less or equal m, $|e'| \smile v'$.

We define $e : \pi$ as the straightforward extension of $e' : \pi'$ to the missing edge c, i.e. for any π edge d at depth 0:

$$e(d) = \begin{cases} e'(d) & \text{if } d \in \pi' \\ < e'(a), e'(b) > & \text{if } d = c \end{cases}$$

Let now v be a (x, n)-simple element in $\parr \Pi$ with degree less or equal to m. Since $\Pi = A \parr B, \Pi''$, we may write $v =< v_1, v_2, v_3 >$, where v_1, v_2 and v_3 are (x, n)-simple elements resp. in $\mathcal{A}$, $\mathcal{B}$ and Π'' with degree less or equal to m. By hypothesis $|e'| \smile v$, hence of course $|e| \smile v$.

Tensor: if π has a terminal $\otimes$ link l with conclusion $c : A \otimes B$ and premises $a : A$, $b : B$, define π' from π by erasing l and its conclusion. Suppose $\Pi = A \otimes B, \Pi''$, hence $\Pi' = A, B, \Pi''$ are the conclusions of π'.

In case π' is not weakly correct, then the assertion follows by induction hypothesis like in the $\parr$ case.

In case π' is weakly correct, then all the visible cycles of π crosses the link l erased in π'. In this case there is a visible path in π' from $\uparrow a$ to $\downarrow b$ or from $\uparrow b$ to $\downarrow a$. Let us suppose the first case (the second being similar). By lemma 110 there is an experiment $e' : \pi'$ such that for any (x, n)-simple elements v_1 and v_3 resp. in $\mathcal{A}$ and $\parr \Pi'$ with degree less or equal m: $e'(a) \smile v_1$, and $< e'(c_1), \ldots, e'(c_k) > \overset{\smile}{\frown} v_3$ (where $c_1, \ldots, c_k$ are the conclusions of π' different from a, b).

We define $e : \pi$ as the straightforward extension of $e' : \pi'$ to the missing edge c, i.e. for any π edge d at depth 0:

$$e(d) = \begin{cases} e'(d) & \text{if } d \in \pi' \\ < e'(a), e'(b) > & \text{if } d = c \end{cases}$$

Let now v be a (x, n)-simple element in $\parr \Pi$ with degree less or equal to m. Since $\Pi = A \otimes B, \Pi''$, we may write $v =<< v_1, v_2 >, v_3 >$, where v_1, v_2 and v_3 are a (x, n)-simple element resp. in $\mathcal{A}$, $\mathcal{B}$ and Π'' with degree less or equal to m. By the hypothesis on e' and the incoherence definition in the $\otimes$ space, we deduce $|e| \smile v$

Why not: if π has a terminal ? link l with conclusion c. Let $b_1 : \flat B, \ldots, b_h : \flat B$ $(h \geq 0)$ be the l premises. Define π' from π by erasing l and c. Suppose $\Pi = ?B, \Pi''$, then π' has conclusions $\Pi' = \flat B, \ldots, \flat B, \Pi''$.

Of course π' is not weakly correct, hence by induction there is an experiment $e' : \pi'$, s.t. for any (x, n)-simple element v' in $\parr \Pi'$ with degree less or equal to m, $|e'| \smile v'$.

We define $e : \pi$ as the immediate extension of $e' : \pi'$ to the missing edge c, i.e. for any π edge d at depth 0:

$$e(d) = \begin{cases} e'(d) & \text{if } d \in \pi' \\ e'(b_1) + \ldots + e'(b_h) & \text{if } d = c \end{cases}$$

Let now v be a (x, n)-simple element in $\wp\Pi$ with degree less or equal to m. Since $\Pi = ?B, \Pi''$, we may write $v = < v_1, v_2 >$, where v_1 (resp. v_2) is a (x, n)-simple element in $?\mathcal{B}$ (resp. in $\wp\Pi'$) with degree less or equal to m.

Firstly, let us prove $|e| \asymp v\,[\wp\Pi]$. Define $v' = < v_1, \ldots, v_1, v_2 >$, which is a (x, n)-simple element in $\wp\Pi'$ with degree less or equal to m. By hypothesis $|e'| \smile v'$. Hence we deduce $|e| \asymp v$.

Secondly, let us prove $|e| \not\equiv v$. Suppose $|e| \equiv v$ and let us prove a contradiction. Under such a supposition, for each $i \leq h$, $\exists v'_i \subseteq v_1$, $e^o(b_i) \equiv v'_i$. Define $v' = < v'_1, \ldots, v'_h, v_2 >$ and remark that v' is a (x, n)-simple element in $\wp\Pi^o$ with degree less or equal m. By the $\wp$ neutrality definition, $|e^o| \equiv v'$, which is contrary to the hypothesis on $|e^o|$. Thus we conclude $|e| \not\equiv v$.

$\flat$-link: if π has a terminal $\flat$-link at depth 0, the case follows straightforwardly by induction hypotheses.

Of course: if π has a terminal ! link o. Let π^o be the o box, $a : !A$ (resp. $a' : A$) be the o conclusion (resp. premise), $b_1 : \flat B_1, \ldots b_h : \flat B_h$ be the π^o auxiliary doors, $c_1 : C_1, \ldots, c_t : C_t$ be the π conclusions which are not doors of π^o, i.e. $\Pi = !A, \flat B_1, \ldots \flat B_h, C_1, \ldots, C_t$.

Define π' from π by substituting the link o with its box π^o. Of course π' has conclusions $\Pi' = A, \flat B_1, \ldots, \flat B_h, C_1, \ldots, C_t$.

π' is not weakly correct, since no visible cycle of π passes through the o box, being o terminal. By induction there is an experiment $e' : \pi'$, s.t. for any (x, n)-simple element v' in $\wp\Pi'$ with degree less or equal m, $|e'| \smile v'$.

We define $e : \pi'$ be the extension of e' taking value $n\,[e']$ on the ! link o, i.e. for any π edge d at depth 0:

$$e(d) = \begin{cases} e'(d) & \text{if } d \text{ is not a } \pi^o \text{ door} \\ n\,[e'(a')] & \text{if } d = a \\ ne'(b_i) & \text{if } d = b_i \end{cases}$$

Remark that:

$$|e'| = < e'(a'), e'(b_1), \ldots, e'(b_h), e'(c_1), \ldots, e'(c_t) >$$

$$|e| = < n\,[e'(a')]\,, ne'(b_1), \ldots, ne'(b_h), e'(c_1), \ldots, e'(c_t) >$$

Let now v be a (x, n)-simple element in $\wp\Pi$ with degree less or equal m. We may write:

$$v =< n\,[v_0]\,, v_1, \ldots, v_h, w_1, \ldots, w_t >$$

where v_0, v_i for each $i \leq h$ and w_j for each $j \leq t$ are (x,n)-simple elements resp. in $\mathcal{A}$, $?\mathcal{B}_i$ and $\mathcal{C}_j$ with degree less or equal to m.

Firstly, let us prove $|e| \overset{\smile}{\frown} v\,[\parr\Pi]$. Define $v' =< v_0, v_1, \ldots, v_h, w_1, \ldots, w_t >$ and remark that v' is a (x,n)-simple element in $\parr\Pi'$ with degree less or equal to m. Thus by hypothesis $|e'| \overset{\smile}{\frown} v'\,[\parr\Pi']$, which implies $|e| \overset{\smile}{\frown} v\,[\parr\Pi]$.

Secondly, let us prove $|e| \not\equiv v$. Suppose $|e| \equiv v$ and let us prove a contradiction. Under such a supposition, $e'(a) \equiv v_0$, for each $j \leq t$, $e'(c_j) \equiv w_j$, and for each $i \leq h$, $\exists v'_i \subseteq v_i$, $e^o(b_i) \equiv v'_i$. Define $v' =< v_0, v'_1, \ldots, v'_h, w_1, \ldots, w_t >$ and remark that v' is a (x,n)-simple element in $\parr\Pi'$ with degree less or equal m. By the $\parr$ neutrality definition, $|e'| \equiv v'$, which is contrary to the hypothesis on $|e'|$. Thus we conclude $|e| \not\equiv v$.

□

Proof of theorem 104. Recall the statement of theorem 104:

> Let π be a cut-free **MELL** proof structure, $\mathcal{X}$ be a non-uniform coherent space with $x, y, z \in |\mathcal{X}|$ such that $x \equiv x\,[\mathcal{X}]$, $x \frown y\,[\mathcal{X}]$ and $x \smile z\,[\mathcal{X}]$.
>
> If $[\![\pi]\!]_{\mathcal{X}}$ is a clique, then π is weakly correct.

Proof. Let π be a cut-free proof structure with conclusion Π. Let us suppose that π is not weakly correct. We prove that $[\![\pi]\!]_{\mathcal{X}}$ is not a clique in $\parr\Pi$.

Let d be the π exponential depth, w be the maximal arity of the π ? links, k be the maximal number of doors of a box of π. Let us set $n = k$, $m = wn^d$.

Since π is not weakly correct then by lemma 111 there is an experiment $e : \pi$ such that for any (x,n)-simple element v in $\parr\Pi$ with degree less or equal to m, $|e| \smile v\,[\parr\Pi]$.

Let $e^{\pi}_{(x,n)}$ be the (x,n)-simple experiment on π. By proposition 109, $|e^{\pi}_{(x,n)}|$ is a (x,n)-simple element in $\parr\Pi$ with degree less or equal to m. So $|e| \smile |e^{\pi}_{(x,n)}|\,[\parr\Pi]$, i.e. $[\![\pi]\!]_{\mathcal{X}}$ is not a clique in $\parr\Pi$. □

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