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Expansions of quadratic numbers in a p-adic continued fraction.

Umberto Zannier

Abstract It goes back to Lagrange that a real quadratic irrational always has a periodic continued fraction. Starting from several decades ago, some authors have proposed expansion in p-adic continued fractions. Here the expansion depends on the chosen system of residues mod p, and results are different. We shall adopt the simplest definition, due to Ruban. It turns out that not all expansions of quadratic numbers are periodic; but it was not known how to decide whether the expansion for a given quadratic number is or is not periodic. In recent work with L. Capuano and F. Veneziano, we have achieved a completely general algorithm in this sense. This algorithm depends on deep theorems in transcendence and diophantine analysis, and, somewhat surprisingly, depends on the "real" value of the "p-adic" continued fraction.

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Around discretized rotation

Shigeki Akiyama

Abstract Discretized rotation is modeled by a simple integer sequence (a_n) defined by

$$0 \le a_{n+2} + \lambda a_{n+1} + a_n < 1$$

with a fixed constant $\lambda \in (-2, 2)$. By an analogy with rotation, we expect that all orbits are bounded, i.e., periodic. This is a notorious conjecture on which we still know very little. From dynamical point of view, this problem is difficult by eigenvalues of modulus one, which makes the problem not at all expanding nor hyperbolic. We met this problem in the context of shift radix system, the parameter space for finiteness property of number systems.

With many colleagues for many years, we investigated this problem and found interesting directions to explore: beta expansion, self-inducing system, domain exchange, substitution, and nearly recursive sequences. In this talk, I wish to summarize several ideas and developments on this topic and present interesting remaining questions that you can join us from today.

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A Numeration System and a Gray Code Given by a Variant of the Tower of Hanoi

Benoît Rittaud

The tower of Hanoi game was introduced by Édouard Lucas ([Lucas1892], p. 55-59). In this game, n disks of increasing diameters from 1 to n are stacked onto the left post A of a set of three (the other ones being the central one, B and the right one, C). Each step consists in moving a disk (only one at a time) from one post to another, until the initial tower at the left post has been rebuilt on the right post. Each movement has to satisfy the *golden rule*: a disk cannot be stacked onto a disk of smaller diameter.

It is a classical exercise to show that the successive states are naturally coded in the binary numeration system as well as in the standard binary Gray code. Several variants of this game exist, and one may wonder whether these variants are also linked to other numeration systems. We show here in which way the answer is yes for a specific variant. What is interesting and makes the question more than a simple exercise is that the involved numeration systems are of a particular kind, different from the usual ones derived from a greedy algorithm and a fixed increasing sequence of integers. The usefulness of these numeration systems may outweight the single study of the tower of Hanoi game.

1 Recursive and iterative solutions

The easiest way to present an optimal solution to the game is probably the recursive approach, summed up by the following figure. Roughly speaking, to solve the game with n disks, an optimal recursive algorithm solves first the game with $(n-1)$ disks, then moves once the biggest disk, then solves again the game with $(n-1)$ disks.

An elementary induction shows that the number of states seen by this algorithm is 2^n. Let us label them by the set of increasing integers from 0 to $2^n - 1$. Again, an

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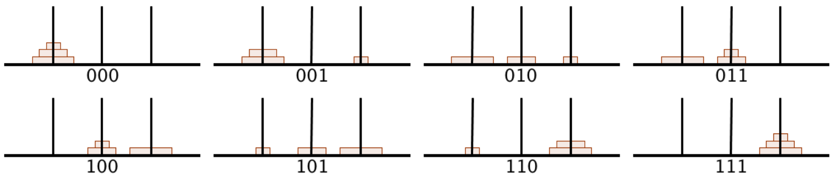
induction shows that, at the k-th state, the disk to be moved is given by the index of the leftmost changing digit when computing the successor of k in base 2. We will call this the *digit property*.



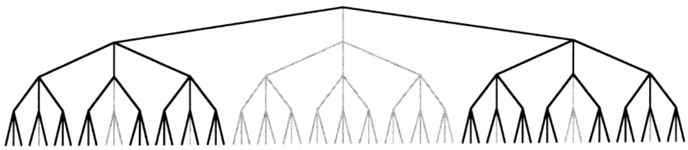
Knowing the disk which is to be moved is not equivalent to know where to put it, a question that needs some more investigation and would be of no real interest for us, so we will not consider it here.

Now, consider the *linear* variant (the terminology is from [Hinz2014]), in which direct moves between the left and the right posts are forbidden. Again, elementary considerations provides a recursive optimal solution, by which we can prove that the number of states is 3^n, and that the base 3 numeration system also satisfies the digit property. Hence, since in this case knowing the disk to be moved is enough to know on which post it has to go, we get an iterative algorithm for the linear variant.

Since the number of all possible states allowed by the golden rule is also 3^n (the number of possible partition of $\{1, \ldots, n\}$ in three subsets), we get that the linear game visits the set of all allowed states. Therefore, the standard and linear games are extremal: the first one provides the shortest path from the initial to the final state, the second one the (or *a*) longest path (without redundancy). Therefore, to no other kind of restriction in the moves can correspond a numeration system in some other integer base.

2 The clockwise-cyclic case

The *clockwise-cyclic* variant, introduced in [Atkinson1981], consists in allowing only moves from A to B, from B to C and from C to A. Atkinson wrote his paper in the context of teaching. In the standard game, he wrote, it is quite easy to find an iterative solution. The clockwise-cyclic variant was designed to provide a situation easily solved by a recursive procedure, but not by iteration. He computed the explicit number of steps needed by the optimal recursive algorithm for this cyclic variant, and found that its complexity is $O((1+\sqrt{3})^n)$. Some years after, [Gedeon1996] made a step further by presenting an iterative algorithm and by mentioning a ternary tree structure associated to it (see figure).



Our aim is to go further in the study, by providing an explicit numeration system that corresponds to the clockwise-cyclic variant and satisfying the digit property.

Denote by Δ_n the set $\{1,\dots,n\}$, where each integer stands for the disk of equivalent diameter. Any state of the game with n disks can be written as an ordered partition of Δ_n into three subsets. Hence, the recursive algorithm to solve the clockwise-cyclic game with n disks can be presented in the following way, in which each arrow corresponds either to a single move or to the recursive application of the algorithm:

$$(\Delta_n,\varnothing,\varnothing) \longrightarrow (n,\varnothing,\Delta_{n-1}) \longrightarrow (\varnothing,n,\Delta_{n-1}) \longrightarrow (\varnothing,\{\Delta_{n-2},n\},n-1) \longrightarrow$$

$$(n-1,\{\Delta_{n-2},n\},\varnothing) \longrightarrow (\Delta_{n-1},n,\varnothing) \longrightarrow (\Delta_{n-1},\varnothing,n) \longrightarrow (\varnothing,\varnothing,\Delta_n).$$

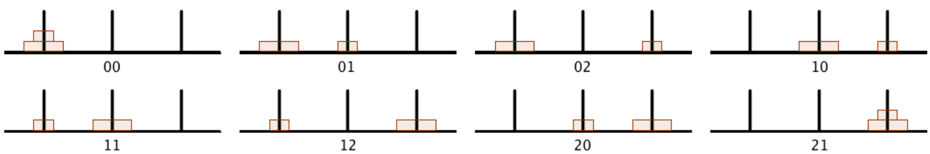
The number r_n of moves therefore satisfies $r_n = 2r_{n-1}+2r_{n-2}+3$ with $r_1 = 2$ and $r_2 = 7$, so the number of states $s_n = r_n + 1$ is given by:

$$s_1 = 3 \qquad s_2 = 8 \qquad s_n = 2s_{n-1}+2s_{n-2} \text{ (for } n \geq 2).$$

The roots of the polynomial X^2-2X-2 being $1\pm\sqrt{3}$, we recover the result of [Atkinson1981] about the run-time $O((1+\sqrt{3})^n)$ of the algorithm.

The sequence $(s_n)_n$ defines naturally a numeration system of integers, and we could expect that it satisfies the digit property, as the sequence $(2^n)_n$ does for the standard game and the sequence $(3^n)_n$ does for the linear variant.

The greedy algorithm for the sequence $(s_n)_n$ defines a codage of integers given by the set of finite words on the alphabet $\{0,1,2\}$ recognized by the language that excludes the factor 22. The set of states of the clockwise-cyclic game is in one-to-one correspondence with the set of integers whose expansion in this numeration system is made of n letters (allowing leading 0s). The point is that such a numeration system does not satisfy the digit property (see the states 12 and 20 in the next figure).



Hence we need something different. First, to avoid ambiguity in the notation, we choose $\mathscr{A} := \{a,b,c\}$ for the alphabet instead of $\{0,1,2\}$, and we consider only the words that do not contain bb as a factor. With the help of the alphabetic order on words of length n, we define a unique codage of integers between 0 and $s_n - 1$ by words of length exactly n, whose tree structure is the one given by [Gedeon1996]. Therefore we get an alternative way of labelling the states of the cyclic game. Let us call it the CCH numeration system (for *clockwise-cyclic Hanoi*).

Theorem 1. *The CCH numeration system satisfies the digit property.*

Proof. The partition of the set of states in the clockwise-cyclic game given by the recursive presentation above corresponds inductively to the following partition of the set $\mathscr{A}_n$ of words of length n without bb as a factor (NB: here, $\mathscr{A}_n^*$ stands for $\mathscr{A}_n \backslash \{a^n\}$):

$$\begin{aligned}\mathscr{A}_n = \{a^n\} \cup \{aW,\ W \in \mathscr{A}_{n-1}^*\} \cup \{ba^{n-1}\} \cup \{baW,\ W \in \mathscr{A}_{n-2}^*\} \cup \\ \{bca^{n-2}\} \cup \{bcW,\ W \in \mathscr{A}_{n-2}^*\} \cup \{ca^{n-1}\} \cup \{cW,\ W \in \mathscr{A}_{n-1}^*\}.\end{aligned}$$

The result follows by induction.

Since the forbidden factor is bb instead of cc (i.e. 11 instead of 22), there is a big difference between the classical greedy algorithm and the CCH numeration system, as it appears in the way we get the value of an integer knowing its CCH expansion.

Theorem 2. *Let $(b_n)_n$ and $(c_n)_n$ be the sequences defined by:*

$$b_0 = 1 \qquad b_1 = 3 \qquad b_n = 2b_{n-1} + 2b_{n-2} \text{ for } n \geq 2;$$

$$c_0 = 2 \qquad c_1 = 5 \qquad c_n = 2c_{n-1} + 2c_{n-2} \text{ for } n \geq 2.$$

Let $m_{n-1} \dots m_0$ be a finite word on the alphabet $\mathscr{A} = \{a,b,c\}$ with no factor equal to bb. For any $i \in \{0, \dots, n-1\}$, put

$$j_i := \begin{cases} 0 \ \text{ if } m_i = a; \\ b_i \ \text{ if } m_i = b \text{ and } m_{i+1} \neq b; \\ c_i \ \text{ if } m_i = c \text{ and } m_{i+1} \neq b; \\ b_i \ \text{ if } m_i = c \text{ and } m_{i+1} = b. \end{cases}$$

The integer that corresponds to the word $m_{n-1} \dots m_0$ is $\sum_i j_i$.

Theorem 2 is easily proved by induction and the partition $\mathscr{A}_n$ in the proof of Theorem 1. It is also easy to derive from Theorem 2 an algorithm that expands any nonnegative integer in the CCH numeration system.

In a sense, the cyclic tower of Hanoi game is the simplest nontrivial case among a new kind of numeration systems that could be investigated. With only two letters, we may consider a kind of "negative-Fibonacci-Zeckendorf" numeration system in which the forbidden factor is 00 instead of 11, but the impossibility of leading zeroes makes it not very natural, at least for the usual purpose of numeration systems.

3 Gray codes

The standard *Gray code* is a binary codage such that the codage of any two consecutive integers differ by exactly one digit. Let $w_n \dots w_0 \in \{0,1\}^n$ the binary expansion of length $n+1$ of some integer k. The corresponding Gray code of k, $g(w_n \dots w_0)$, can be recursively defined by $g(0) = 0$, $g(1) = 1$ and, for $n \geq 1$:

$$g(w_n \dots w_0) = g(w_n \dots w_1)w, \text{ where } w = \begin{cases} w_0 & \text{if } w_2 w_1 \in \{00, 11\}; \\ 1 - w_0 & \text{if } w_2 w_1 \in \{01, 10\}. \end{cases}$$

When $n = 1$, a leading 0 is added to make this definition sensible.

Such a Gray code is linked to the tower of Hanoi by the same digit property as the usual binary numeration system is. To prove it, it is easier to start from a more classical definition of the binary Gray code: the code with n letters being given, write the sequence twice: first in increasing order then in decreasing order. Add a leading 0 (resp. a leading 1) to each element of the first part (resp. of the second part): here is the Gray code for words of length $n+1$. This latter definition (equivalent to the first one, as can be easily proved) is closer to the recursive solution we met in section 1. It provides therefore a simpler way to prove that such a Gray code satisfies the digit property. Nevertheless, it appears that the one we gave first is easier to extend to the CCH case, so we will stick to it. From a tower of Hanoi standpoint, looking at $w_n \dots w_0$ as $w_n(w_{n-1} \dots w_0)$ consists in forgetting the largest disk n to work with the others; looking at $w_n \dots w_0$ as $(w_n \dots w_1)w_0$ instead corresponds to the idea that we are interested in the "lower tower" $\{2, \dots, n\}$, the smallest disk of radius 1 being moved apart each time it is necessary to allow a move of one of the other disks.

For the linear variant, a Gray code (in base 3) $g(w_n \dots w_0)$ can be defined by $g(0) = 0$, $g(1) = 1$, $g(2) = 2$ and, for $n \geq 1$, by:

$$g(w_n \dots w_1)w, \text{ where } w = \begin{cases} w_0 & \text{if } w_1 = 0; \\ w_0 - 1 & \text{if } w_1 = 1; \\ w_0 + 1 & \text{if } w_1 = 2, \end{cases}$$

where $w_0 \pm 1$ is to be understood mod 3. It is easy to see that such a Gray code satisfies the digit property for the linear variant.

Now, it is natural to ask for a Gray code with the digit property in the case of the clockwise-cyclic game. Here is the result.

Theorem 3. *In the CCH numeration system, define $g(0) = 0$, $g(1) = 1$, $g(2) = 2$ and, for $n \geq 1$, let $g(w_n \dots w_0)$ be*

$$g(w_n \dots w_1)w, \textit{ where } w = \begin{cases} w_0 & \textit{if } w_2 w_1 \in \{00, 02, 12, 21\}; \\ 2 - w_0 & \textit{if } w_2 w_1 \in \{01, 10, 20, 22\}, \end{cases}$$

where a leading 0 is added if $n = 2$. Then, g is a Gray code that satisfies the digit property for the clockwise-cyclic variant.

It is easy to see that the condition $w_2w_1 \in \{00, 02, 12, 21\}$ is satisfied iff the integer that corresponds to $w_n \dots w_2w_1$ is even.

Also, it is a trivial consequence of Theorem 3 that all the digits 1 of $w_n \dots w_0$ are at the same place as those of $g(w_n \dots w_0)$.

4 Other restrictions, and the Abu Dhabi variant

There are other possible variants of the game based on constraints on the moves from post to post. In particular, we could have considered the *counterclockwise-cyclic* game as well (which is not isomorphic to the clockwise-cyclic game, since we asked the tower to go from A to C), and also some others. It is proved in [Hinz2014] Theorem 8.4 that the set of all constraints for which the game is solvable correspond to the set of digraphs of vertices A, B and C which are strongly connected. What precedes could be reproduced in these other cases, the point being that the order of the corresponding linear recurring sequences $(s_n)_n$ is bigger — up to 6. Hence, a general study of numeration systems with the same kind of "Markov property" as in Theorem 2 would be probably of better interest than the study of these specific variants of the game.

Incidentally, let us notice another kind of variant, found by some pupils during a workshop in Abu Dhabi in March 2017 ("MATh.en.JEANS"). In this variant, the successive moves of any disk has to follow the periodic pattern

$$A \longrightarrow C \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow \cdots.$$

It is an interesting exercise to show that the number of steps in this variant is $(3^n - 1)/2$, and that the corresponding numeration system is made of all words of length n in $\{0, 1, 2\}$ upper-bounded (in the lexicographical order) by 1^n. (The question of a "nice" Gray code seems a little bit more difficult.) One may wonder if other patterns of the same kind could give rise to more complex situations.

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Substitutions, coding prescriptions and Numeration

Paul Surer

Denote by $\mathscr{A}$ a finite set (alphabet) and by $\mathscr{A}^*$ the set of finite words over $\mathscr{A}$. For a word $X \in \mathscr{A}^*$ we write $|X|$ for the length of X. Consider a substitution σ, *i.e.* a morphism $\sigma : \mathscr{A} \to \mathscr{A}^*$ and its extension to $\mathscr{A}^*$ via concatenation.

A *coding prescription (with respect to σ)* is a function c with domain $\mathscr{A}^2$ that assigns to each pair of letters a finite set of integers with the following properties:

1. for all $k \in c(xx)$ we have $|k| < |\sigma(x)|$ (for all $x \in \mathscr{A}$);
2. $c(xx)$ is a complete set of representatives modulo $|\sigma(x)|$ (for all $x \in \mathscr{A}$);
3. $c(ab) = \{k \in c(aa) : k \leq 0\} \cup \{k \in c(bb) : k \geq 0\}$ (for all $ab \in \mathscr{A}^2$).

The notion of coding prescription was introduced in [4] in order to code substitution dynamical systems as shifts of finite type. In the actual presentation we want to concentrate on combinatorial aspects of coding prescriptions.

In particular, we will show how to compose coding prescriptions with respect to given substitutions σ and σ' over the same alphabet in order to obtain a coding prescription for the composition $\sigma' \circ \sigma$ or for powers σ^n. This will yield a way to represent integers quite analogously as the Dumont-Thomas numeration for natural integers (see [2, 3]). The set of representable numbers may consist of all (positive and negative) integers, it may have gaps, and it can consist of 0 only. This depends on the actual coding prescription. For a special one, that assigns to each $ab \in \mathscr{A}^2$ a set of non-negative integers, we retrieve exactly the results from [3].

For primitive substitutions we can also base a numeration system for real numbers on our setting where the domain can have various characteristics. Again, this depends on the coding prescription. For the special choice from above we obtain the Dumont-Thomas numeration for real numbers. Other choices of σ and c yield, for example, symmetric beta-expansions (*cf* [1]). We will outline several effects by examples.

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On the sum of digits of the factorial

Carlo Sanna

Abstract For integers $b \geq 2$ and m, let $s_b(m)$ be the sum of the digits of m when written in base b. We prove that $s_b(n!) > C_b \log n \log\log\log n$ for all integers $n > e^e$, where C_b is a positive constant depending only on b. This improves by a factor $\log\log\log n$ a previous lower bound for $s_b(n!)$ given by Luca. We also prove that the same inequality holds with $n!$ replaced by the least common multiple of $1, 2, \dots, n$.

1 Introduction

For integers $b \geq 2$ and m, let $s_b(m)$ be the sum of the digits of m when written in base b. Given a sequence of integers $(a_n)_{n \geq 1}$ with some combinatorial or number-theoretic meaning, proving a lower bound for $s_b(a_n)$ is usually a challenging task, which has attracted the attention of several researchers.

For example, if $(F_n)_{n \geq 1}$ is the sequence of Fibonacci numbers, defined as usual by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n, a result of Stewart [11] implies that

$$s_b(F_n) \gg \frac{\log n}{\log\log n}, \tag{1}$$

for all integers $n \geq 3$, where the implied constant depends on b (see also [4] for a more general result).

Moreover, if for each positive integer n we write

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{2}$$

for the nth Catalan number, then Luca and Shparlinski [8] showed that

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$$s_b(C_n) > \varepsilon(n)\sqrt{\log n}, \tag{3}$$

for almost all positive integers n (in the sense of the natural density), where $\varepsilon(n)$ is any function tending to zero as $n \to +\infty$. Furthermore, Luca and Young [9] proved that

$$s_2(C_n) \gg \log\log n, \tag{4}$$

holds for all integers $n \geq 3$ such that C_n is odd.

If $p(n)$ denotes the partition function of n, Luca [6] gave the lower bound

$$s_b(p(n)) > \frac{\log n}{7\log\log n}, \tag{5}$$

for almost all positive integers n.

Cilleruelo, Luca, Rué, and Zumalacárregui [2] proved a general result, which in particular implies that

$$s_b(B_n) > \frac{\log n}{60\log b}, \tag{6}$$

for almost all positive integers n, where B_n is the nth Bell number.

Also, lower bounds for the sum of digits of the Apéry numbers [7], the numerators of Bernoulli numbers [1], and a wide class of binomial sums [3], have been proved.

It is worth mentioning that all the previous results are much weaker than what is expected to be the order of s_b over such sequences of integers. Indeed, unless the terms of the sequence $(a_n)_{n\geq 1}$ have small $s_b(a_n)$ for trivial reasons (e.g., because the a_n's are powers of b), then it seems plausible to conjecture that the digits of a_n in base b behave like uniformly distributed random variables in $\{0, 1, \dots, b-1\}$. Therefore, it should be true that

$$s_b(a_n) \gg \log a_n, \tag{7}$$

for almost all positive integers n, or at least for infinitely many n, where the implied constant depends on b. However, it can the easily checked that all the previous lower bounds are instead of the form

$$s_b(a_n) \gg \log\log a_n, \tag{8}$$

or worse.

Regarding the sum of digits of the factorial, Luca [5] showed that

$$s_b(n!) \gg \log n, \tag{9}$$

for all positive integers n, with an implied constant depending on b.

In a recent paper [10], we improved Luca's result by proving that

$$s_b(n!) \gg \log n \log\log\log n, \tag{10}$$

for all integers $n > e^e$, where the implied constant depends on b.

The main ingredients of our proof are the following: First, a folklore result, which says that if a positive integer N is divisible by $b^m - 1$, for some positive integer m, then $s_b(N) \geq m$. Second, an asymptotic formula, as $x \to +\infty$, for the largest positive integer m such that $\varphi(m) \leq x$, where φ is Euler's totient function. Third, some important divisibility properties of the cyclotomic polynomials $\Phi_d(X)$ evaluated at b.

Employing those tools, we constructed an integer $m \gg \log n \log\log\log n$, with a particular factorization, such that $b^m - 1$ can be written as a product of many "almost" pairwise coprime positive integers not exceeding n. This guarantees that $b^m - 1$ divides $n!$ and yields the desired conclusion. Our proof also works with $n!$ replaced by the least common multiple of $1, 2, \ldots, n$.

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On the semi-random Lüroth map

Marta Maggioni

Abstract We present the semi-random Lüroth map T as an open random system given by different ensembles of Lüroth maps. The state space is the unit interval with a fixed hole, namely the subinterval $[0,1/a)$ is removed from $[0,1]$, for an integer $a \geq 2$. We translate this open random system $[1/a,1] \to [0,1]$ to a closed one $[1/a,1] \to [1/a,1]$ by choosing either the standard Lüroth map or the alternating one, in such a way that, for each point of the domain, each of its iterates still lies in $[1/a,1]$. We then prove the existence of an absolutely continuous invariant measure on $[1/a,1]$ for the closed semi-random system. For $a = 2^m$ we show the semi-random Lüroth map T has a Markov partition, which allows us to use the theory of the Perron-Frobenius operator in matrix form to get an explicit formula for the density.

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k-regular Sequences & Mellin–Perron Summation for Analyzing Fluctuations in Pascal's Rhombus

Daniel Krenn (based on a joint work with C. Heuberger and H. Prodinger)

Abstract We focus on the asymptotic behavior of k-regular sequences, in particular on the example of the number of ones in Pascal's rhombus. A combination of Dirchlet series together with Mellin–Perron summation of orders one and two, and asymptotic results on k-regular sequences is used to analyze the fluctuation of the main term in the asymptotics of the sequence.

1 Introduction, Result and More

A k-regular sequence $f(n)$, $n \in \mathbb{N}_0$, can be represented as follows: Suppose, for some $d \in \mathbb{N}$, we have $d \times d$-matrices $M_0, \ldots, M_{k-1}$ and two d-dimensional vectors u and v. Then the sequence satisfies the k-linear representation

$$f(n) = u^T M_{n_0} M_{n_1} \ldots M_{n_{\ell-1}} v,$$

where $n = (n_{\ell-1} \ldots n_1 n_0)_k$, the standard k-ary expansion of n.

Pascal's rhombus is the array with entries $r_{i,j}$, where $r_{0,j} = 0$ for all j, $r_{1,0} = 1$ and $r_{1,j} = 0$ for all $j \neq 0$, and

$$r_{i,j} = r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} + r_{i-2,j}$$

for $i > 1$, see [3]. In the proposed talk, we will focus on the odd entries of this rhombus. A visualization is given in Figure 1.

The number of odd entries in a row of Pascal's rhombus equals one component of a certain system of recurrence relations, and it can be modelled by a 2-regular sequence.

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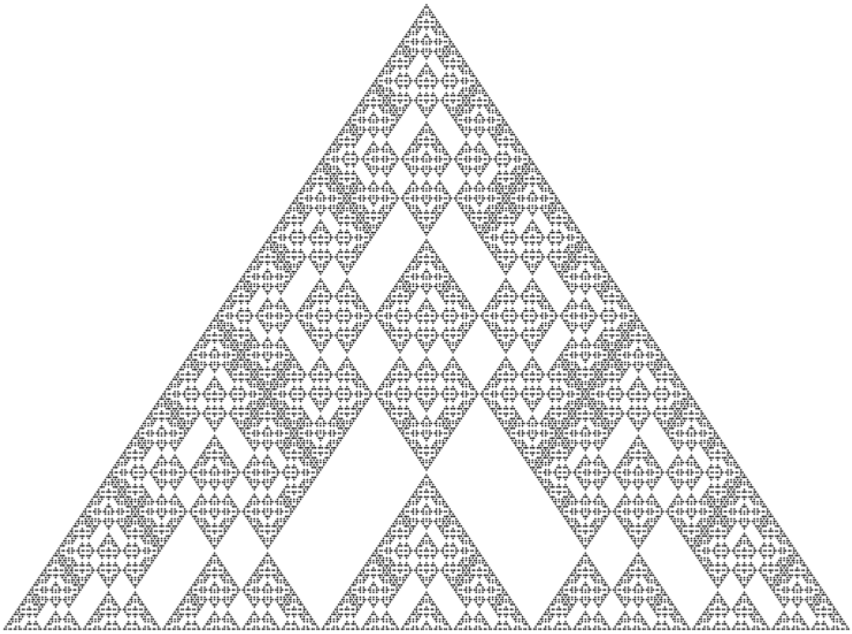

Fig. 1: Pascal's rhombus modulo 2.

We have the following asymptotic result.

Theorem 1 (Heuberger–K–Prodinger 2016). *Let a_n be the number of odd entries in the first n rows. Then we get*

$$a_n = n^{\kappa} \Phi_a(\log_2 n) + O(n \log_2 n)$$

with $\kappa = 2 - \log_2(\sqrt{17} - 3) = 1.8325063835804\ldots$ *and a continuous and* 1*-periodic function* $\Phi_a(u)$.

The Fourier coefficients of the fluctuation $\Phi_a(u)$ can be computed as well.

The proof of this result is based on the following ideas: As mentioned, the number of ones in the a certain row, i.e., the first order differences of the a_n, form a 2-regular sequence. This 2-linear representation is translated to a system of functional equations of Dirichlet series. Experiments show that the Mellin–Perron summation formula of first order gives the asymptotic behavior and the Fourier coefficients, however, due to (frequently recognized) convergence issues of this first order formula, it is not possible to use it. Instead we use an approach based on the second order Mellin–Perron summation formula, cf. [4] and [5], and show that we still get the results on the a_n despite using the same system of Dirichlet series as with the first order approach.

In order to make this work, a pseudo-Tauberian argument [2] and, to avoid a circular argumentation, an asymptotic statement based on [1] are needed.

Although the focus lies on this particular example of Pascal's rhombus, the method seems to cover many other questions that can be modelled by k-regular sequences; in the talk we draw our attention to these generalizations.

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High precision computing for continued fractions and mod 2 normal numbers

Geon Ho Choe

Abstract Optimal number of significant digits in numerical simulations for some number theoretical examples including continued fractions and mod 2 normal numbers is investigated in terms of divergence speed and the Lyapunov exponent of the associated discrete time dynamical system.

1 Discrete dynamical system associated with number theoretic examples

How can we be sure that the compute continued fraction expansion of a given irrational number obtained numerically is correct up to a given level with reasonable amount of theoretical justification? This question belongs to a more broad and fundamental problem of executing rigorous simulation of discrete time dynamical systems arising from number theoretic problems on a computer.

A discrete dynamical system is a repeated iteration T^n, $n \geq 1$, of a mapping (or a transformation) $T : X \to X$ defined on a probability measure space (X, μ) equipped with a T-invariant probability measure μ, i.e., $\mu(T^{-1}A) = \mu(A)$. In this talk we consider the case when X is the unit interval and T is piecewise differentiable. For example, $Tx = 2x \pmod 1$ or $Tx = 1/x \pmod 1$ with corresponding invariant measures $d\mu = \rho(x)dx$. A loop in a computer code is an iteration of the identical instruction. When a discrete time dynamical system is modeled as a *loop* in a numerical code, there is loss of accuracy in each application of the numerically defined instruction contained in a computer code.

Some theoretical concepts cannot be rigorously simulated numerically except when there is no need for high precision arithmetic. One such example is approximation of invariant measures including strange attractors in chaos theory based on the

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Birkhoff Ergodic Theorem which states that the time average $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} f(T^k(x))$ equals the space average $\int_X f(x)d\mu$ for an ergodic transformation T, i.e., if $f(Tx) = f(x)$ almost everywhere then f is constant almost everywhere. For more information consult [Bos],[P],[W]. Some software allows us to do practically unlimited precise computation that is indispensable for high precision numerical simulations without much technical coding requirements. Some previously unthinkable numerical experiments can be done on a personal computer.

When we try to iterate a mapping $T : [0,1] \to [0,1]$ with very many significant digits, it takes a lot of time to execute a program. In this talk we present a method to determine the number of digits sufficient for a rigorous simulation. If we take too many significant digits, then the computation would take too long. With small number of digits, the truncation error will ruin everything very soon. The optimal number of significant digits will be given in terms of Lyapunov exponent of T.

Definition 1. If $T : [0,1] \to [0,1]$ is piecewise continuously differentiable and if $\rho(x)dx$ is an ergodic T-invariant probability measure, then we define the *Lyapunov exponent* λ by

$$\lambda = \int_0^1 \log|T'(x)|\,\rho(x)\,dx$$

where the logarithmic base equals 10.

Suppose that we take D significant decimal digits in floating point computation in specifying a point x_0. For example, we may let $x_0 = 0.414$ with $D = 3$ or $x_0 = 0.4142$ with $D = 4$ in specifying $x_0 = \sqrt{2}-1$ as an input. (In fact, D is large in the numerical simulation.) The number D is the magnification level of a microscope that focuses on the point x_0.) After k iterations of T, we lose about $k \times \lambda$ decimal digits from original D digits in x_0.

Specify D significant decimal digits for x_0: $|x_0 - \tilde{x}_0| \approx 10^{-D}$. If λ is the Lyapunov exponent, then the Birkhoff Ergodic Theorem implies that, for sufficiently large k,

$$\lambda \approx \frac{1}{k}\sum_{j=0}^{k-1} \log|T'(T^j x_0)| \approx \frac{1}{k}\log\prod_{j=0}^{k-1}|T'(T^j x_0)|$$

and

$$10^{k\lambda} \approx \prod_{j=0}^{k-1}|T'(T^j x_0)| = |(T^k)'(x_0)|\,.$$

By the chain rule, $|T^k x_0 - T^k \tilde{x}_0| \approx 10^{k\lambda}|x_0 - \tilde{x}_0| \approx 10^{k\lambda - D}$.

Having some precision after k iterations in an orbit of $\tilde{x}_0$ is equivalent to $k\lambda - D \leq 0$, and hence we should not iterate T numerically more than about D/λ times.

Definition 2. Define the *divergence speed*

$$V_n(x) = \min\{j \geq 1 : |T^j x - T^j \tilde{x}| \geq 10^{-1}\}$$

where $|\tilde{x} - x| = 10^{-n}$. Since

$$|T^{V_n}x - T^{V_n}\tilde{x}| \approx 10^{V_n\lambda}\,|x-\tilde{x}| = 10^{V_n\lambda - n}\ ,$$

we have $V_n\lambda - n \approx 0$, so

$$\lim_{n\to\infty} \frac{n}{V_n(x)} = \lambda\,.$$

Example 1. For $Tx = 10x \pmod 1$ with $\lambda = \log_{10} 10 = 1$, we have $V_n(x) = n-1$ for $0 \le x \le 1 - 10^{-n}$.

Example 2. An irrational translation mod 1 has the Lyapunov exponent equal to 0, and hence $\lambda = 0$, and $V_n = +\infty$ for $n \ge 2$. See [CS],[KS] for related results.

2 Optimal number of significant digits

Each time when T is applied, we lose λ significant decimal digits on average. If we start with D significant decimal digits for x, then we may iterate T up to approximately $V_D \approx D/\lambda$ times with at least one decimal significant digit remaining in $T^{V_D}x$. Take a starting point of an orbit, say x_0, and let $\tilde{x}_0$ denote its approximation such that $|x_0 - \tilde{x}_0| \approx 10^{-D}$. For every $j \in \mathbb{N}$, $1 \le j \lesssim D/\lambda$, there corresponds an integer $k \approx D - j \times h$ such that

$$T^n\tilde{x}_0 = 0.\underbrace{a_1\,.........\,a_k}_{\text{significant}}\,\underbrace{a_{k+1}\,.........\,a_D}_{\text{meaningless}}$$

where the last $j \times \lambda$ digits have no information for the true theoretical orbit point T^jx_0. So, if $j \approx D/\lambda$, then $T^j\tilde{x}_0$ loses any amount of precision. We don't have to worry about the truncation errors of the size 10^{-D}, which are negligible in comparison with 10^{-k}. In simulations with irrational translations mod 1 we don't need many significant digits. For the details, consult [C3]. For a related result, see [C2],[CKb].

3 Accuracy in continued fraction expansions

Consider the continued fraction map $Tx = \{1/x\}$. Since $x_0 = [k,k,k,\ldots]$ is a fixed point of T, it is obvious that the average of $\log|T'|$ along the orbit of x_0 is equal to $\log|T'(x_0)|$. Take

$$x_0 = [k,k,k,\ldots] = \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cdots}}}\ .$$

Then $x_0 = (-k + \sqrt{k^2+4})/2$. Since $\log|T'(x_0)| = -2\log x_0$, if we are given D decimal significant digits we lose all the significant digits after approximately $D/(-2\log x_0)$ iterations of T. The number of correct partial quotients in the continued fractions of x_0 obtained from numerical computation is denoted by N_k. Let C_k be the maximal number of iterations of T that produces the correct partial quotients of x_0. If $|\widetilde{x}_0 - x_0| \approx 10^{-D}$, then $T^{C_k-1}\tilde{x}_0 \in I_k = (1/(k+1), 1/k)$ and $T^{C_k}\tilde{x}_0 \notin I_k$. Since $T^j x_0 = x_0$,

$$\frac{|T^n\tilde{x}_0 - x_0|}{|\tilde{x}_0 - x_0|} \approx \frac{|T^n\tilde{x}_0 - T^n x_0|}{|\tilde{x}_0 - x_0|} \approx \prod_{j=0}^{n-1} |T'(T^j x_0)| \approx |T'(x_0)|^n ,$$

hence

$$\log|T^n\tilde{x}_0 - x_0| + D \approx n\log|T'(x_0)| = n(-2\log x_0) .$$

Since $T^{C_k-1}\widetilde{x}_0 \in I_k$,

$$|T^{C_k-1}\widetilde{x}_0 - x_0| \le \max\{\,|x_0 - \frac{1}{k}|, |x_0 - \frac{1}{k+1}|\} .$$

An approximate upper bound U_k for N_k is given by

$$U_k = \frac{D + \log\max\{\,|x_0 - \frac{1}{k}|, |x_0 - \frac{1}{k+1}|\}}{-2\log x_0} + 1 .$$

Similarly, an approximate lower bound L_k for N_k is given by

$$L_k = \frac{D + \log\min\{\,|x_0 - \frac{1}{k}|, |x_0 - \frac{1}{k+1}|\}}{-2\log x_0} .$$

For more details and related results, see [A],[C1],[C3],[CKc]. For a general introduction to continued fractions, consult [DK],[KN],[RS].

4 Mod 2 normal numbers

Let $X = [0,1)$ with Lebesgue measure μ. Define $Tx = 2x \pmod 1$. Let $x = \sum_{k=1}^{\infty} a_k 2^{-k}$, $a_k \in \{0,1\}$, be the binary representation of x. Then $a_k = 1_E(T^{k-1}x)$ where $E = [\frac{1}{2}, 1)$. The normal number theorem states that almost every x is normal, i.e., $(1/n)\sum_{k=1}^{n} a_k$ converges to $\frac{1}{2}$.

Definition 3. Define $d_n(x) \in \{0,1\}$ by $d_n(x) \equiv \sum_{k=1}^{n} a_k \pmod 2$. Then x is called a mod 2 normal number if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n(x) = \frac{1}{2} . \tag{1}$$

Later we will see that almost every $x \in [0,1]$ is mod 2 normal. In general, if $T : X \to X$ is a transformation and 1_E is the characteristic function of $E \subset X$. Define $d_n(x) \in \{0,1\}$ by

$$d_n(x) \equiv \sum_{k=1}^{n} 1_E(T^{k-1}x) \qquad (\bmod\ 2)\ .$$

Then x is called a *mod 2 normal number* with respect to E if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n(x) = \frac{1}{2}\ . \tag{2}$$

(This type of problem was first studied by [W] for irrational translations modulo 1.) Put $e_n(x) = \exp(\pi i d_n(x)) = 1 - 2\,d_n(x) \in \{\pm 1\}$. Then Eq. (2) is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_n(x) = 0 \tag{3}$$

and Eq. (1) is a special case of Eq. (2) for $Tx = 2x \pmod 1$ and $E = [\frac{1}{2}, 1)$.

Theorem 1. *(i) For any interval $E \neq (\frac{1}{6}, \frac{5}{6})$ a.e. $x \in [0,1)$ is mod 2 normal with respect to E.*

(ii) For $E = (\frac{1}{6}, \frac{5}{6})$ a.e. $x \in [0,1)$ is not mod 2 normal with respect to E. More precisely, for $E = (\frac{1}{6}, \frac{5}{6})$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n = \frac{2}{3} \qquad a.e.\ x \in (\tfrac{1}{3}, \tfrac{2}{3})\ ,$$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n = \frac{1}{3}\ \ a.e.\ x \in (0, \tfrac{1}{3}) \cup (\tfrac{2}{3}, 1)\ .$$

For the proof and numerical simulations see [C3],[CHN].

Example 3 (Mod 2 uniform distribution). For θ irrational, $Tx = x + \theta \pmod 1$ with respect to $E = (0, \theta)$.

Example 4 (Mod 2 uniform distribution). $Tx = 2x \pmod 1$ with respect to $E = (0, \frac{1}{2})$

Example 5 (Failure of mod 2 uniformity). For $0 < \theta < 1/2$ irrational, $Tx = x + \theta \pmod 1$ with respect to $E = (0, 2\theta)$. Then $\exp(\pi i 1_E(x)) = q(x)q(Tx)$ is satisfied by $q = \exp(\pi i 1_F)$, $F = (\theta, 2\theta)$. Note that $\int q\, d\mu = 1 - 2\theta$.

Example 6 (Failure of mod 2 uniformity). $Tx = 2x \pmod 1$ with respect to $E = (\frac{1}{6}, \frac{5}{6})$ Then $\exp(\pi i 1_E(x)) = q(x)q(Tx)$ is satisfied by $q = \exp(\pi i 1_F)$, $F = (\frac{1}{3}, \frac{2}{3})$. Note that $\int q\, d\mu = \frac{1}{3}$.

Example 7 (Failure of mod 2 uniformity). Take $\beta = \frac{\sqrt{5}+1}{2}$, $Tx = \beta x \pmod 1$, $E = (1 - \frac{1}{\beta}, 1)$. Then $\exp(\pi i 1_E(x)) = q(x)q(Tx)$ with $q = \exp(\pi i 1_F)$, $F = (0, \frac{1}{\beta})$. Note that $\int q\, d\mu = -\frac{1}{\sqrt{5}}$.

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Limits of families of canonical number systems

Dávid Bóka and Péter Burcsi

Abstract We consider expansion structure and expansion length in numeration systems obtained from polynomial families with canonical digits sets. We examine how the length of the expansion of constant polynomials changes as we increase the degree. In some special cases we prove that the expansion length is quadratic in the degree, and introduce a limiting numeration system which captures some aspects of the complete family. We also pose some open questions about the period structure of some families of polynomials.

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An upper bound on prolongation of periods of continued fractions by Möbius transformation

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A number x has an eventually periodic continued fraction if and only if x is quadratic irrational. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2,2}$. Möbius transformation (also called fractional linear transformation) determined by the matrix M is the function h_M given by $h_M(x) = \frac{ax+b}{cx+d}$. We study the relation of the periods of the continued fraction of a quadratic number x and of $h_M(x)$ for a given M.

Raney in [?] constructed transducers representing Möbius transformations. Using these transducers, we construct an upper bound on the period of the continued fraction of $h_M(x)$ that depends only on the period of the continued fraction of the number x and on the determinant of the matrix M. More precisely, let $n = |\det M|$ and $\pi_n(x)$ be the period of the continued fraction of $h_M(x)$ divided by the period of the continued fraction of x. Furthermore, set $\xi(e,f)$ to be the number of steps of Euclidean algorithm of coprime positive integers e and f (until we reach the remainder 1). If n is prime, we show for all quadratic irrational x the following estimate:

$$\pi_n(x) \leq \begin{cases} 5 & \text{if } n = 2, \\ 2 + 2\sum_{i=1}^{\frac{n-1}{2}} (\xi(i,n) + 3) & \text{if } n \equiv 3 \pmod 4, \\ 1 + 2\sum_{i=1}^{\frac{n-1}{2}} (\xi(i,n) + 3) & \text{if } n \equiv 1 \pmod 4. \end{cases} \tag{1}$$

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For n composite, we show a generalization of this formula. We exhibit computational evidence that this upper bound is sharp for some small prime determinants and for other small determinants it is not greater than $2\max_x(\pi_n(x))$.

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The Thompson groups, graph polynomials, and knot theory.

Valeriano Aiello and Roberto Conti

Abstract We describe some results and problems arising from the graphical description of the elements of the Thompson groups in terms of planar rooted binary trees and a recent connection with knot theory discovered by Jones.

1 Introduction

In the 1960s Richard Thompson introduced the groups $F \subset T \subset V$, which ever since their debut have received a great deal of attention. Among other things, we remark that it is still unknown whether F is amenable or not. For our purposes, it is important to stress that their elements admit a nice pictorial description in terms of pair of planar rooted binary trees. By using this description and Vaughan Jones's recent work, many invariants of geometric nature, either coming from graph or knot theory, give rise to positive-definite functions on some of these groups. One thus gets many unitary representations that may shed further light on some analytical aspects. Two notable examples coming from graph theory are given by the Tutte polynomial and the chromatic polynomial. In the process of giving an elementary proof of the positive definiteness of the chromatic polynomial, we came across a family of matrices satisfying a somewhat curious set of conditions, namely symmetric real matrices $A = (a_{ij})$ of arbitrary size, with entries $0 \leq a_{ij} \leq 1$, and such that for every i, j, k it holds $a_{ij} + a_{ik} - 1 \leq a_{jk}$. Many of these matrices can be shown to be positive-semidefinite and counterexamples are not known. Along similar lines, it can also be shown that several specializations of

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the HOMFLY polynomial, a celebrated link invariant, yields a positive-definite function on the oriented Thompson group $\vec{F}$, a notable subgroup of the Thompson group F.

2 Preliminaries

For brevity, we will focus only on the Thompson group F, however, many of the issues make sense for the other groups too. The Thompson group F can be described in many different ways. Here we will recall just one of them. Let $\mathscr{T}$ be the set of rooted planar binary trees. Given a tree $T \in \mathscr{T}$ we will denote by ∂T the set of its leaves. The Thompson group F can be defined as

$$F \doteq \{(T_+, T_-) \in \mathscr{T} \times \mathscr{T} \mid |\partial T_+| = |\partial T_-|\}/\sim$$

where $\sim$ is the equivalence relation generated by the addition of opposite carets, the interested reader is referred to [5, 4]. Given two elements $(T_+^1, T_-^1), (T_+^2, T_-^2) \in F$, in order to compute their product we pick two representatives $(T'^1_+, T'^1_-) \sim (T_+^1, T_-^1), (T'^2_+, T'^2_-) \sim (T_+^2, T_-^2)$ such that $T'^1_- = T'^2_+$ and then

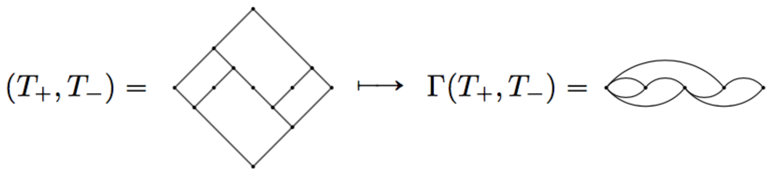
$$(T_+^1, T_-^1) \cdot (T_+^2, T_-^2) \doteq (T'^1_+, T'^2_-) \ .$$

It can be shown that (T, T) is the unit for any $T \in \mathscr{T}$ and $(T_+, T_-)^{-1} = (T_-, T_+)$. We mention that F is finitely presented as

$$F = \langle x_0, x_1 \mid x_2 x_1 = x_1 x_3, x_3 x_1 = x_1 x_4 \rangle,$$

where $x_n \doteq x_0^{1-n} x_1 x_0^{n-1}$ for $n \geq 2$.

Jones introduced in [7] a graph $\Gamma(T_+, T_-)$ associated with a representative (T_+, T_-) of an element of F, e.g.



The chromatic polynomial of G is the polynomial $\mathrm{Chr}(G,t)$ such that, for any $Q \in \mathbb{N}$, $\mathrm{Chr}(G, Q)$ is the number of proper colourings of the vertices of G with Q colours. Moreover, if $e \in E(G)$, it satisfies the following condition

$$\mathrm{Chr}(G, Q) = \begin{cases} 0 & \text{if } e \text{ is a loop} \\ \mathrm{Chr}(G - e, Q) - \mathrm{Chr}(G/e, Q) & \text{otherwise.} \end{cases}$$

Jones defined a function on F by $Q^{-1}(Q-1)^{-n+1}\mathrm{Chr}(\Gamma(T_+,T_-),Q)$ for $T_\pm$ with n-leaves, as this expression is independent of the choice of the representative of an element of F. Actually, this function is of positive type on F [7, Proposition 5.2.1] according to the following definition.

Definition 1. Let G be a discrete group. A function $\varphi : G \to \mathbb{C}$ is said *of positive type* (or *positive definite*) if the matrix $(\varphi(g_i g_j^{-1}))_{i,j}$ is positive semi-definite for any $r \in \mathbb{N}$, $g_1,\dots,g_r \in G$, and $a_1,\dots,a_r \in \mathbb{C}$.

The chromatic polynomial allows us to introduce the oriented Thompson group as

$$\vec{F} = \{g \in F \mid \mathrm{Chr}_{\Gamma(g)}(2) = 2\} = \{g \in F \mid \Gamma(g) \text{ is bipartite}\}$$

For $g \in F$, the graph $\Gamma(T_+,T_-)$ admits a preferred 2-colouring, with colours $+$ and $-$, starting with $+$ on the leftmost vertex.

3 Colourings and a positive semi-definiteness problem

We examine some further combinatorial aspects of the chromatic polynomial in the above setting.

Proposition 1. *Let $T_i, T_j, T_k \in \mathscr{T}_n$. Consider the Q-colourings of $\Gamma(T_i,T_j)$ and $\Gamma(T_i,T_k)$. We denote their cardinality by Chr_{ij} and Chr_{ik}, respectively. The number of colourings that are compatible for both $\Gamma(T_i,T_j)$ and $\Gamma(T_i,T_k)$ is denoted by Chr_{ijk}.Then the following inquality holds*

$$Chr_{ij} + Chr_{ik} - Chr_{ijk} \le Chr_{ii} \, .$$

In particular, it holds

$$Chr_{ij} + Chr_{ik} - Chr_{ii} \le Chr_{jk} \, .$$

Proof. Taking into account double counting, we have the following inequality

$$Chr_{ij} + Chr_{ik} - Chr_{ijk} \le Q(Q-1)^{n-1}$$

(as colourings of $\Gamma_+(T_i)$), which implies

$$Chr_{ij} + Chr_{ik} - Q(Q-1)^{n-1} \le Chr_{ijk} \le Chr_{jk}.$$

One can recover the fact that $\vec{F}$ is a group from the above lemma. Indeed, for $Q=2$, the Chromatic polynomial takes only two values 0, 2 and, thus, given three trees $T_i, T_j, T_k \in \mathscr{T}$ with n-leaves such that $Chr(\Gamma(T_i,T_j),Q) = 2$, $Chr(\Gamma(T_i,T_k),Q) = 2$, the above inequality tells us that $Chr(\Gamma(T_k,T_j),Q) = 2$.

By Jones' results the matrices $(Q^{-1}(Q-1)^{-n+1}\mathrm{Chr}(\Gamma(T_+^i,T_+^j),Q))_{i,j}$ are positive semi-definite for all choices of trees $\{T_+^i\}_i$ with n-leaves. One might wonder whether the same conclusion holds only assuming the inequalities in Proposition 1 (together with few other natural assumptions). In particular, consider a matrix $A \in M_r([0,1])$ such that

1. $A_{ii} = 1$, for all i;
2. $A_{ij} = A_{ji}$, for all i, j;
3. let $(i, j, k) \in \{1, \cdots, r\}^3$ be a triple with mutually distinct indices, then $a_{ij} + a_{ik} - 1 \leq a_{jk}$.

We tend to believe that for any $r \in \mathbb{N}$ a matrix A as above is always positive semi-definite. This is obviously true for $r = 2$ and if $r = 3$ it can be proved without much troubles. However, already for $r = 4$ the situation becomes much more involved and the general argument remains elusive. We mention however that the claim can be proved in many special cases. Over the years, several conditions ensuring the positive semi-definiteness have been examined in the literature, although we are not aware of any result implying a complete solution to the above problem.

If we make the further assuption that $A \in M_r(\{0,1\})$, then the matrix satisfies the following stronger condition

- let $(i, j, k) \in \{1, \cdots, r\}^3$ be a triple with mutually distinct indices. If $A_{ij} \neq 0$ and $A_{ik} \neq 0$, then $A_{jk} \neq 0$.

Moreover, the above properties imply that

- for any i, $[A^2]_{ii} = \sum_j (A_{ij})^2$ is equal to the number of 1 on the i^{th} row (and column).
- for any $i \neq j$, $A_{ij} = 0$ implies $[A^2]_{ij} = 0$. In fact, $[A^2]_{ij} = \sum_k A_{ik}A_{kj}$. If $[A^2]_{ij} \neq 0$, then there exists $k_0 \notin \{i, j\}$ such that $A_{i,k_0} \neq 0$, $A_{k_0,j} \neq 0$. However, this implies that $A_{ij} \neq 0$.

This case corresponds precisely to 2-colourings and there are a handful of proofs of the claim in this setting. We give a quick sketch of one of them. Our aim is to show that A can be expressed as a linear combination of orthogonal projections (with positive coefficients). This establishes the positive semi-definiteness of A.

Consider A^2. We define the following matrices $A^{(1)}, \cdots, A^{(r)} \in M_r(\{0,1\})$ as follows. For any $l \in 1, \ldots, r$, the entries of $A^{(l)}$ are all zero but those such that the corresponding entries in A^2 are equal to l, that is

$$A^{(l)}_{ij} = \begin{cases} 1 & \text{if } (A^2)_{ij} = l \\ 0 & \text{otherwise} \end{cases}$$

.

Lemma 1. *If $A^{(l)}_{ij} \neq 0$, for some $1 \leq l \leq r$, then for any $k \in 1, \ldots, r$,*

$$A_{ik} \neq 0 \Leftrightarrow A_{jk} \neq 0.$$

Therefore, the rows i and j of A contain the same number l of non-zero entries (in the same positions).

Proof. Suppose instead that $A_{ik} = 1$ and $A_{jk} = 0$. Then i, j, k are necessarily distinct and A_{ij} must be 0. However, this is impossible by the assumption on A.

Next lemma shows that if a row of $A^{(l)}$ is nonzero then it must coincide with the corresponding row of A.

Lemma 2. *If $A^{(l)}_{ij} \neq 0$ and $A_{ik} \neq 0$, then $A^{(l)}_{ik} \neq 0$.*

Proof. Suppose instead $A^{(l)}_{ik} = 0$. Then by the definition of $A^{(l)}$ we have that $[A^2]_{ik} = \sum_h A_{ih}A_{kh} \neq l$. As $A^{(l)}_{ij} \neq 0$, we have by the Lemma 1 that the rows i, j of A have exactly l entries equal to 1 (in the same positions). The hypothesis $A^{(l)}_{ik} = 0$ means that the entries equal to 1 in the row k of A are in different positions with respect to those in the row i. Therefore, there exists an index m such that $A_{im} = 1$, $A_{km} = 0$. As i, k, m can be easily seen to be distinct, one has $A_{ik} = 0$, contradicting the assumption.

As a consequence, if for some i, j one has $A^{(l)}_{ij} \neq 0$ then the rows i and j of $A^{(l)}$ coincide and contain the same number l of non-zero entries.

Lemma 3. *The matrices $A^{(k)}/k$ are orthogonal projections.*

Proof. By definition, all the matrices $A^{(k)}/k$ are clearly symmetric. They are also idempotent. In fact, by the above lemma, a row (or a column) of $A^{(k)}$ has either all the entries equal to 0 or has the same entries of the corresponding row in A. The conclusion now follows by a straightforward combination of the previous results.

Theorem 1. *Under the above hypotheses we have that*

$$A = \sum_{k=0}^{n} A^{(k)}$$

In particular, the matrix A is positive semi-definite.

Proof. The first equation follows from the definition of $A^{(l)}$. The second claim follows from the above lemma.

Recently, Vilmos Komornik [9] gave another elegant proof of this result.

4 Further remarks

In the previous section we saw that the chromatic polynomial gives rise to functions of positive type on the Thompson groups. We mention that many other functions (coming from graph and knot theory) admit a similar treatment. The following invariants have been studied [7, 1, 2, 3]: Chromatic polynomial, Tutte polynomial, Kauffman bracket, Jones polynomial, HOMFLY polynomial, number of Fox-colourings, 2-variable Kauffman polynomial. We briefly explain the connection with knot theory. Jones [7] showed that links can be obtained as the *closure* of elements of the Thompson group F and the result does not depend on the representative up to distant unknots. Moreover, for elements in $\vec{F}$ one can naturally orient the obtained link by using the data corresponding the 2-colouring of the Γ-graph. Therefore, in the same spirit as for the chromatic polynomial, one can use the HOMFLY polynomial $P_{\vec{L}}(\alpha,z)$ to produce functions of positive type on $\vec{F}$, the corresponding unitary representation being associated to the category of oriented forests [3]. There are actually two possible ways to deal with the HOMFLY, according to whether one choses the trivial *-structure or not. In the former case one defines the HOMFLY function on $\vec{F}$ as

$$P_{(T_+,T_-)}(\alpha,z) = \left[\alpha^{-1}\left(\frac{1-\alpha^{-2}}{z}+z\right)\right]^{-n+1} P_{\vec{L}(T_+,T_-)}(\alpha,z)$$

and then shows that for $q\in\mathbb{R}\setminus\{\pm1,0\}$ and k a positive integer, $P_{(T'_+,T'_-)}(q^k,q-q^{-1})$ is of positive type. The proof relies upon the statistical mechanical interpretation of the HOMFLY polynomial discovered in [6] and the following facts

- the quantity $c(L(T_+,T_-))-n$ is always even, where $(T_+,T_-)\in\vec{F}$, (T_+,T_-) is a pair of trees with n leaves and $c(L(T_+,T_-))$ is the number of connected components of the associated link;
- the rotation number $\mathrm{rot}(\vec{L}\,(T_+,T_-)) = n_+ - n_-$, where n_+ (resp. n_-) is the number of vertices of $\Gamma(T_+,T_-)$ labeled with colour $+$ (resp. $-$) and by $\mathrm{rot}(\vec{L})$ the rotation number of the link $\vec{L}$.

The latter case is dealt with in [3] by using an approach developed in [8].

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Finiteness in real cubic fields

Zuzana Masáková and Magdaléna Tinková

Positional representation of numbers in systems with non-integer base $\beta > 1$ as defined by Rényi [11] is a suitable alternative to classical decimal or binary notation, when encountered with the need of finite representation of numbers in an algebraic field. A particular question is what is the most suitable choice of the base in a real number field. One of the natural requirements is that elements with eventually periodic β-expansion form the field $\mathbb{Q}(\beta)$. By a result of Schmidt [13], this happens exactly for Pisot bases, i.e. such algebraic integers $\beta > 1$ that have all conjugates inside the unit circle. It is also natural to ask that the set $\mathrm{Fin}(\beta)$ of numbers with finite β-expansion is closed under addition, subtraction and multiplication. This so-called finiteness property or property (F) can be reformulated by $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta, \beta^{-1}]$, and is satisfied only by Pisot numbers [6]. Moreover, not all of them. A complete algebraic characterization of Pisot numbers with (F) is known only for the quadratic case and cubic unit case [1]. For general degree, some sufficient conditions have been given [6, 7]. If the base is chosen to be a unit, then $\mathbb{Z}[\beta, \beta^{-1}] = \mathbb{Z}[\beta]$ and the (F) property is equivalent to a certain property of the corresponding Rauzy tiling. Such geometric interpretation is useful in describing some other arithmetic features of the numeration system, such number of fractional digits arising in arithmetic operations.

Salem in [12] proved that every real number field can be generated by a Pisot number β. Bertin et al. [4] provide a non-constructive proof for the fact that β can be chosen to be a Pisot unit. Requiring moreover the finiteness property, it is no longer true that every real number field contains such Pisot unit generator. This can be seen on the case of real quadratic fields. It turns out that existence of a Pisot unit generator with (F) in $\mathbb{Q}(\sqrt{m})$ depends on the period-length of the continued fraction expansion

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of the square-free integer $m \in \mathbb{N}$. However, every quadratic field contains a generator with property (PF), i.e. the set of finite expansions is closed at least under addition.

We focus on the question whether every cubic number field contains a generator which is a Pisot unit with property (F). We show that this is true in case of all real cubic fields which are not totally real. We also illustrate that in case of totally real cubic fields the situation is more complicated. We give classes of fields which contain unit with (F), but we also give examples of cubic totally real field where no Pisot unit satisfies the finiteness property.

As a compensation for the number fields without a generator possessing property (F), we suggest to consider representation in negative base [8]. Analogous property ($-$F) can be defined by requiring $\mathrm{Fin}(-\beta) = \mathbb{Z}[\beta, \beta^{-1}]$. Based on the necessary conditions on β for ($-$F) provided in [10, 5] and the description of cubic Pisot units with ($-$F) given recently in [9], we show that in real quadratic and cubic number fields not generated by a Pisot unit with (F), one always finds a Pisot unit generator with property ($-$F).

We also provide bounds $L_{\oplus}(\beta)$ on the number of fractional digits appearing when performing addition and subtraction of β-expansions. Such bounds were sofar given only for a few cubic examples [3, 2].

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Nearly linear recursive sequences, especially SRS

Attila Pethő

My talk is based on join works with Shigeki Akiyama and Jan-Hendrik Evertse, as well as with Jörg Thuswaldner and Mario Weitzer.

Let $\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{R}^d$, then the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ such that

$$\tau_{\mathbf{r}}(a_0, \dots, a_{d-1}) = (a_1, \dots, a_{d-1}, -[\mathbf{ra}])$$

is called SRS. Equivalently we define the sequence of integers (a_n) by the nearly linear recurrence relation, in the sequel *nlrs*,

$$r_0 a_n + \cdots + r_{d-1} a_{n+d-1} + a_{n+d} = e_n,\ 0 \le e_n < 1.$$

SRS was extensively studied recently, but only if $\tau_{\mathbf{r}}$ is contractive or indifferent, i.e. if the roots of the polynomial $X^d + p_{d-1}X^{d-1} + \cdots + p_0$ lie inside or on the unit circle. Then the orbits of $\tau_{\mathbf{r}}$ are either periodic or grow polynomially. If, however, $\tau_{\mathbf{r}}$ is expansive no systematic investigation is known for me.

SRS is a special instance of nlrs (b_n) of complex numbers for which there exist $A_0, \dots, A_{d-1} \in \mathbb{C}$ such that

$$A_0 b_n + \cdots + A_{d-1} b_{n+d-1} + b_{n+d} = e_n$$

and the sequence (e_n) is bounded.

With Akiyama and Evertse [1] we investigated nlrs. We proved a Binet-type formula for such sequences, justifying that they are growing usually exponentially. This does not mean monotone increase, nlrs may have surprisingly large fluctuation. We show that there exist lrs (b_n) such that $\limsup b_n = \infty, \liminf |b_n| = 0$ and $b_n \neq 0$ for all n. Using this result we construct nlrs of integers, for which the Skolem-Mahler-

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Lech theorem does not hold. This means that infinitely many terms of the sequence is zero, but their indices do not fit a line.

If (a_n) is a nlrs and (b_n) a bounded sequence then (a_n+b_n) is also a nlrs. Having stronger restrictions to nlrs, for example concentrating to expansive SRS, we can observe similar patterns as for the contractive SRS. There are expansive SRS, which have some bounded, thus periodic, orbits, others have only unbounded orbits. We prove that for expanding SRS the existence of bounded orbits is algorithmically decidable.

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Order statistics of the values of words with respect to the generalised multinomial measure

Ligia L. Cristea and Helmut Prodinger

Okada, Sekiguchi and Shiota [1] introduce the multinomial measure on the unit interval, that is defined with the help of the digital expansions of the numbers in the unit interval in a certain integer base q in the following way.

Let $q \geq 2$ be a positive integer. Denote $I = I_{0,0} = [0,1]$ and $I_{n,j} = \left[\frac{j}{q^n}, \frac{j+1}{q^n}\right)$, for $j = 0,1,\ldots,q^n-2$, $I_{n,q^n-1} = \left[\frac{q^n-1}{q^n}, 1\right]$, for $n = 1,2,3,\ldots$. Let $\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$ with $0 \leq r_i \leq 1$ and $\sum_{k=0}^{q-1} r_k = 1$. The *multinomial measure* $\mu_{q,\mathbf{r}}$ is the probability measure on I defined by

$$\mu_{q,\mathbf{r}}(I_{n+1,qj+k}) = r_k \cdot \mu_{q,\mathbf{r}}(I_{n,j}) \tag{1}$$

for $n = 0,1,2,\ldots$, $j = 0,1,\ldots,q^n-1$, $k = 0,1,\ldots,q-1$.

We recall that the *binomial measure* (and generalisations thereof) is used in a method designed by the authors Kobayashi, Muramoto, Okada, Sekiguchi and Shiota in order to study moments of higher order $\sum_{n<N} S_q^d(n)$ of the sum-of-digits function, which for an integer $q \geq 2$ and any natural number n with the q-adic expansion $n = \sum_{i\geq 0} a_i q^i$, with $a_i \in \{0,1,\ldots,q-1\}$, is defined by $S_q(n) := \sum_{i\geq 0} a_i$.

In our paper [2] we introduced the *generalised multinomial measure*. Here a generalisation consists, roughly speaking, in the fact that instead of dividing the unit interval into a finite number of subintervals of equal length, we divide it into infinitely (and denumerably) many intervals, such that the j-th subinterval has length pq^{j-1}, where $p,q > 0$, and $p+q = 1$.

One way to define the generalised multinomial measure is the following. We consider the set $\mathscr{W}$ of all (finite and infinite) words over the infinite alphabet $\{0,1,\ldots\}$

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and a probability measure $\mathbb{P}_{\mathbf{r}}$ defined on the set of all words. A function value associates to every word $\omega \in \mathscr{W}$ a real number $\mathsf{value}(\omega) \in [0,1)$, such that the closure of the set of all such values $\mathsf{value}(\omega)$, is the interval $[0,1]$. Then the measure of an interval $\mu_{\mathbf{r},q}([0,a))$, with $0 \le a \le 1$ can be defined in a natural way with the help of the probability $\mathbb{P}_{\mathbf{r}}$, where $\mathbf{r} = (r_0, r_1, \dots)$ is a sequence of real numbers (parameters) with $0 \le r_i \le 1$ and $\sum_{k=0}^{q-1} r_k = 1$.

In my talk I present results on the behaviour of the average minimum value a_n among n words of $\mathscr{W}$ chosen independently at random with respect to the generalised multinomial measure $\mu_{\mathbf{r},q}$ for certain values of the parameters r_j, $j = 0,1,\dots$.

Furthermore, I show results for the average maximum value among n words. We note that the final formulae obtained for the asymptotics show a certain duality.

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Discrepancy bounds for β-adic Halton sequences

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Van der Corput and Halton sequences are well-known low-discrepancy sequences. In the 1990ies Ninomiya defined analogues of van der Corput sequences for β-numeration and proved that they also form low-discrepancy sequences provided that β is a Pisot number. Hofer, Iacó, and Tichy define β-adic Halton sequences and show that they are equidistributed for certain parameters $\beta = (\beta_1, \ldots, \beta_s)$.

In this talk we give discrepancy estimates for β-adic Halton sequences for which the components β_i are m-bonacci numbers. Our methods include dynamical and geometric properties of Rauzy fractals that allow to relate β-adic Halton sequences to rotations on high dimensional tori. The discrepancies of these rotations can then be estimated by classical methods relying on W. M. Schmidt's Subspace Theorem.

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Algebraic structure and numeration systems for circular words

Isabelle Dubois

We investigate some properties of finite abelian groups defined by equivalence relations on circular words. In particular, we show how the formalism of circular words gives rise naturally to the notion of numeration systems for some finite abelian groups, and to periodic expansions of real numbers in $[0;1[$.

Our basic structure is the notion of circular words, introduced in the algebraic context in [2] by B. Rittaud and L. Vivier.

A circular word of length ℓ is a finite word made of ℓ letters on the alphabet $\mathbb{Z}$, indexed by $\mathbb{Z}/\ell\mathbb{Z}$. The set of circular words of length ℓ is then an abelian group and we consider an equivalence relation that defines a "carry" given by an integer polynomial. More precisely, given a polynomial $P(X) = \sum_{0 \le i \le d} a_i X^i \in \mathbb{Z}[X]$, the carry equivalence $\approx$ defined by P on circular words $W = (w_0 \dots w_{\ell-1})$ (where the indices are modulo ℓ) is based on the relations:

$$\forall i \mod \ell, W \approx (w_0 \dots (w_{i-d} + a_0) \dots (w_{i-1} + a_{d-1})(w_i + a_d) w_{i+1} \dots w_{\ell-1}).$$

When P has no roots of unity, the quotient group of circular words of length ℓ by this carry equivalence is a finite abelian group whose structure is determined by algebraic and arithmetic tools. We can then define a numeration system on these groups, which leads to a numerical system of the classes of finite circular words and to periodic expansions of some real numbers.

Some examples are given, in particular for a carry equivalence from a linear polynomial $aX - b$, or from a quadratic polynomial $X^2 - pX - q$ generalizing the results of [1]. Algebraic rather than combinatorial tools are used, which allow a more systematic and general study.

This is a joint work with Benoît Rittaud.

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A constrained Diophantine equation on symmetric numbers in different bases

Stefano Arnone, Corrado Falcolini, Francesco Moauro, and Matteo Siccardi

Abstract Any number may be written in many different ways, using different strings in different bases. In few, very special cases, a symmetry emerges which is usually hidden beneath the surface: the strings “230” in base 164 and “164” in base 230 are both equal to 54284 in base 10. This talk analyzes the infinite solution set of the (constrained) Diophantine equation that implements such symmetry, and shows interesting patterns emerging both from a geometrical and analytic point of view. We will discuss these patterns and formulate a conjecture on the number of solutions of the equation.

The relationship between numbers in different bases will be investigated.

In order to avoid ambiguities, our symbols will be numerals in base ten, *e.g.* in the following $\{1,15\}_{16}$ will be used in place of what is commonly written as $1\mathrm{F}_{16}$. As far as notation is concerned, N with no lower indices will be used to denote the list, $N = \{n_P, n_{P-1}, \cdots, n_2, n_1, n_0\}$, while N with a lower index (representing a base) will indicate the number, evaluated in that base, $N_\mathrm{b} = n_P \mathrm{b}^P + n_{P-1}\mathrm{b}^{P-1} + \cdots + n_1 \mathrm{b} + n_0$. For instance, $230 = \{2,3,0\}$ whereas $230_{164} = 2\cdot(164)^2 + 3\cdot(164)^1 + 0\cdot(164)^0$.

Now, “a kind of magic” may take place: $\{2,3,0\}_{\{1,6,4\}} = \{1,6,4\}_{\{2,3,0\}}$, where the equal sign refers to the equality of the two numbers represented by the different lists in different bases. The above is a shorthand for

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$$
\begin{aligned}
&\{2,3,0\}\_\{1,6,4\} \\
&= 2\cdot(1\cdot 10^2+6\cdot 10^1+4\cdot 10^0)^2+3\cdot(1\cdot 10^2+6\cdot 10^1+4\cdot 10^0)^1+0\cdot(1\cdot 10^2+6\cdot 10^1+4\cdot 10^0)^0 \\
&\qquad = \{5,4,2,8,4\}\_\{1,0\} = \{1,6,4\}\_\{2,3,0\}
\end{aligned}
$$

This phenomenon is not restricted to or peculiar of base ten. For instance,

$$\{2,1,1\}\_\{1,2,1\}_{\text{three}} = \{1,2,1\}\_\{2,1,1\}_{\text{three}},$$

where the reference base has been written in letters.

Generalizing the simple observation that

$$N\_\{1,0\} = \{1,0\}\_N, \tag{1}$$

N being any list except $\{0\}$ and $\{1\}$, we will speculate on the existence of solutions to the following equation:

$$N\_M_{\mathrm{b}} = M\_N_{\mathrm{b}}. \tag{2}$$

For the sake of clarity and effectiveness, we expand eq. (2)

$$\sum_{k=0}^{P} n_k \left(\sum_{i=0}^{Q} \mathrm{b}^i m_i\right)^k = \sum_{i=0}^{Q} m_i \left(\sum_{k=0}^{P} \mathrm{b}^k n_k\right)^i, \tag{3}$$

where b is the base and the lists N, M contain $P+1$ and $Q+1$ elements respectively. All elements — m_i, n_k — are constrained to the range $[0, \mathrm{b}-1]$ and, in order to avoid ill-defined expressions, if one list contains one element only, the other cannot — that is $(P,Q) \neq (0,0)$. Such restrictions upon the form of possible solutions will be seen to play an important role in the following. Solving our problem amounts to finding all of the solutions to eq. (3).

Meaningful solutions of eq. (2) are lists of integers satisfying a number of constraints. Therefore, we are bound to solve a Diophantine equation with additional constraints.

As eq. (2) is symmetric under the exchange $M \leftrightarrow N$, we can and will search for solutions in M, at fixed N. We can also impose $M \geq N$ as a non-restrictive constraint. An immediate consequence of the above is that the solution $M = N$, hereafter referred to as the *symmetric solution*, satisfies eq. (2).

Another solution that is easy to find is given by the following

Theorem 1 *Any M satisfies eq.* (2) *if $N = \{1,0\}$*

which generalizes eq. (1) to any base b. Such a solution will be hereafter referred to as the *trivial solution*.

Regular patterns are seen to emerge when considering simplifying assumptions. One such well-defined set of solutions is given in the following

Theorem 2 *Solutions of eq.* (2) *exist of the form $n_0 = 0$, $n_k = \binom{P}{k}$ for $k > 0$, $m_0 = 0$, $m_i = \binom{Q}{i}$ for $i > 0$, $\forall P > Q > 0$ and $\forall \mathrm{b} > \max_k \binom{P}{k}$.*

Due to their structure, for a given P we can read all lists M which are solutions of eq. (3) from the first rows of a slightly modified Pascal's triangle.

We next show that solutions exist for lists having only one non-zero element.

Theorem 3 *Solutions of eq.* (2) *exist of the form* $M = \{a^h, 0, \ldots, 0\}$, $N = \{a^k, 0, \ldots, 0\}$, $Q = nh + 1$, *and* $P = nk + 1 \quad \forall n \in \mathbb{N}_0, \forall a \in \{1, \ldots, \mathrm{b} - 1\}$ *such that* $a^h, a^k < \mathrm{b}$.

The solutions so far presented are independent of the reference base b, although Theorems 2 and 3 do have a base-dependent constraint.

Eq. (3) can be interpreted as a constraint on two polynomials in the reference base, b. More specifically, finding a base-independent solution is tantamount to constructing two polynomials in b which commute under composition. Block and Thielman [Block and Thielman, 1951], building on previous work by Ritt [Ritt, 1922, Ritt, 1923], have shown what the properties of commuting polynomials are in the case they do not have the same degree, and were able to organize them in two different classes. Translating their results into our "language" allowed us to conclude that the only base-independent solutions with different number of elements are those listed above [Pakovich, 2013, Arnone et al., 2017].

1 Solutions with equal number of elements

1.1 Two elements

When M, N contain two elements each, that is $P = Q = 1$, it is possible to rewrite the Diophantine eq. (3) as

$$(n_1 - 1)m_0 = (m_1 - 1)n_0. \tag{4}$$

As before, the only dependence upon the base b comes from the constraint that the elements cannot exceed it, which implies an increase in the number of solutions as the base grows large. The exact result is

$$S_2(\mathrm{b}) = 4(\mathrm{b}-1)^2 + \sum_{m_1=1}^{\mathrm{b}-2} \left[\varphi(m_1) \left\lfloor \frac{\mathrm{b}-2}{m_1} \right\rfloor^2 + \sum_{\substack{m_0=m_1+1 \\ \gcd(m_0,m_1)=1}}^{\mathrm{b}-1} \left\lfloor \frac{\mathrm{b}-1}{m_0} \right\rfloor^2 \right], \tag{5}$$

where $\varphi(x)$ is the Euler totient function.

1.2 Three elements

When M, N contain three elements each, that is $P = Q = 2$, the Diophantine eq. (2) takes the form

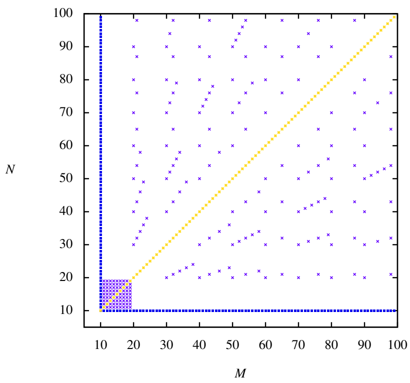

Fig. 1 Solutions to eq. (4) for any base greater than or equal to ten.

$$
\begin{aligned}
N\_M_{\mathrm{b}} &= n_2(m_2\mathrm{b}^2+m_1\mathrm{b}+m_0)^2+n_1(m_2\mathrm{b}^2+m_1\mathrm{b}+m_0)+n_0 \\
&= m_2(n_2\mathrm{b}^2+n_1\mathrm{b}+n_0)^2+m_1(n_2\mathrm{b}^2+n_1\mathrm{b}+n_0)+m_0 = M\_N_{\mathrm{b}}, \quad (6)
\end{aligned}
$$

that is a polynomial equation of degree 7 in $3+3+1$ variables, $\{m_2,m_1,m_0;n_2,n_1,n_0;\mathrm{b}\}$, where the dependence upon b no longer disappears.

Solutions to eq. (6), other than the symmetric ones, have been obtained, up to $\mathrm{b}=400$, via a numerical algorithm [Arnone et al., 2017].

Preliminary analysis shows that infinite families of solutions exist, which may be cast into two different categories. The former contains solutions where the base can be expressed as a (linear or quadratic) function of an integer, k, an example of which is given by $M=\{k+1,2k+1,13k-5\}$, $N=\{k,10k+1,5k\}$, $\mathrm{b}=16k+5$. The latter is exemplified by $M=\{4,\frac{1}{2}\left(1+8i-8p+\sqrt{1+16p+48p^2}\right),0\}$, $N=\{3,3i+1,0\}$, $\mathrm{b}=p-i$, for an infinite sequence of ps, coming from the Pell-type structure of one of the constraints, and a finite range for i for each p. Further investigation reveals that another, geometric perspective may be adopted, which sheds more light on the structure of the solution set.

Figure 2 shows the graphs of two second-degree polynomials which commute under composition at a specific, positive integer value of the independent variable b, together with two secants to such parabolae. The secant lines go through the points $(\mathrm{b},N(\mathrm{b}))$, $(M(\mathrm{b}),N(M(\mathrm{b})))$ and $(\mathrm{b},M(\mathrm{b}))$, $(N(\mathrm{b}),M(N(\mathrm{b})))$ respectively. Owing to the fact that the two polynomials commute at b, $N(M(\mathrm{b}))$ equals $M(N(\mathrm{b}))$, so that b is a solution to eq. (6).

The point of intersection of the secants, I, can be shown to lie on the bisector of the first quadrant angle; its coordinates are bounded between $\mathrm{b}-1$ and b. Computing the coordinates of I for each of the known solutions holds the key to a better representation of the whole set. As a matter of fact, re-expressing the known families in terms of the base b — as opposed to k or p — and other two parameters, and plotting the pairs[1] $(\mathrm{b},1-\{x_I(\mathrm{b})\})$ reveals the patterns shown in fig. 3.

[1] Here $\{x_I\}$ stands for the fractional part of x_I, $\{x_I\}=1-\lfloor x_I \rfloor$.

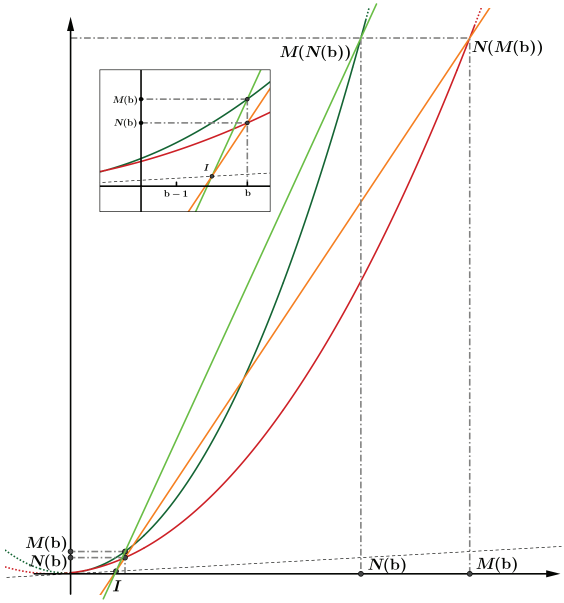

Fig. 2 A new, geometric perspective elucidating the structure of solutions. The red curve is $N(x)$, whereas the green one is $M(x)$.

The grey curves, corresponding to equal values of the two additional parameters, represent the fine spectral lines of our "Diophantine atom": for a given value of n_2 these are seen to form a band, which gets smaller and smaller the larger n_2 grows. The curves in different colours correspond to equal values of the other additional parameter: when they intersect a spectral line at a solution, they always cut through the mid-point of the patch.

Several solutions have been found that have not been classified yet, and the question remains as to whether a base $\bar{\mathrm{b}}$ exist such that solutions can be found for all bases greater than $\bar{\mathrm{b}}$.

1.3 Four or more elements

When $P = Q = 3$, that is in the case of four elements, eq. (2) is of degree 13 in $4+4+1$ variables.

No explicit solutions, except for the symmetric ones, have been found numerically. This appears to be consistent with a series of results that have been obtained on bounds for feasible solutions. Indeed, the region in the $M-N$ plane where solutions might be found shrinks down to a smaller and smaller size as the number of elements increases.

The lack of non-symmetric solutions when N, M have at least four elements each motivated us to formulate the following

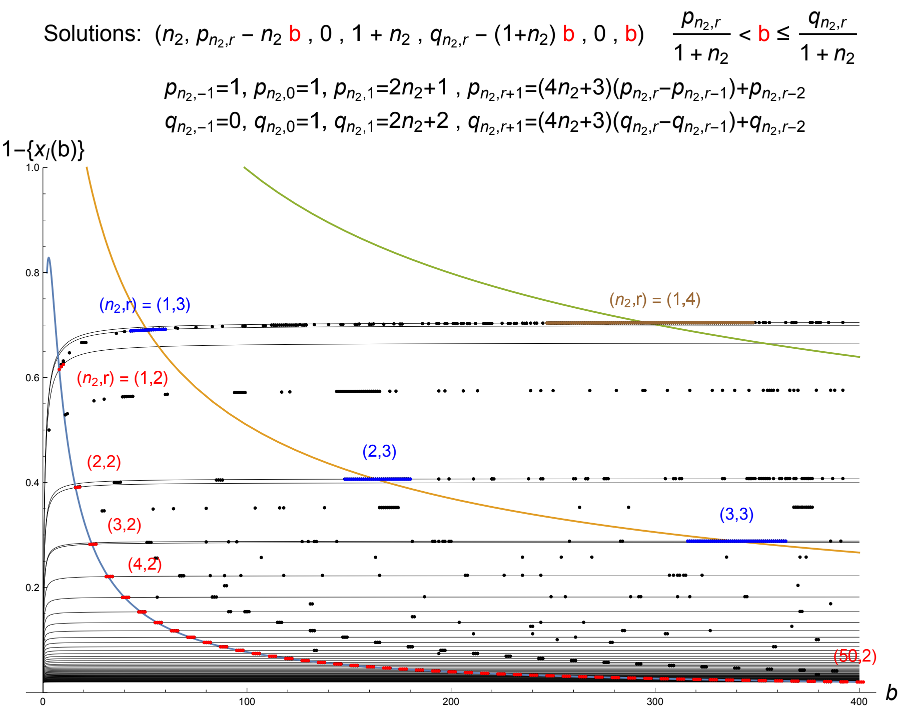

Fig. 3: Our “Diophantine atomic” structure.

Conjecture 1. Let n_i, m_i be non-negative integers,

$$N = \{n_P, n_{P-1}, \dots, n_0\}, \quad \textit{and} \quad M = \{m_P, m_{P-1}, \dots, m_0\},$$

with $m_P n_P \neq 0$. Then, $\forall \mathrm{b} \in \mathbb{N}\ \forall P > 2$,

$$N \neq M \quad \Rightarrow \quad N\_M_{\mathrm{b}} \neq M\_N_{\mathrm{b}}.$$

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On the structure of periodic elements of simultaneous systems

Gabor Nagy

Abstract It is well-known (see [1]) that $(-N_1, -N_2, A_c)$ is a simultaneous number system if and only if $2 \leq N_1, N_2$ and $|N_1 - N_2| = 1$, where N_1, N_2 are rational integers and $A_c = \{0, 1, \ldots, |N_1||N_2| - 1\}$. The aim of this talk is to determine the structure of periodic elements when N_1, N_2 do not fulfil the aforementioned assumption.
The case of simultaneous systems of Gaussian integers is also considered (see [2]).

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Numbers, systems, applications

Attila Kovács

Abstract The lecture presents the newest results achieved in (1) constructing number systems in the ring of integers of real quadratic fields, (2) showing how the number system concept can be applied to kleptographing the RSA public key scheme.

Let Λ be a lattice in $\mathbb{R}^n$ and let $M : \Lambda \to \Lambda$ be a linear operator such that $\det(M) \neq 0$. Let furthermore $0 \in D \subseteq \Lambda$ be a finite subset.

Definition 1 *The triple* (Λ, M, D) *is called a* number system *(GNS) if every element* x *of* Λ *has a unique, finite representation of the form*

$$x = \sum_{i=0}^{L} M^i d_i \ ,$$

where $d_i \in D$ *and* $L \in \mathbb{N}$. *L is the* length of the expansion.

Here M is called the *base* and D is the *digit set*.

Theorem 2 *If* (Λ, M, D) *is a number system then*

1. *D must be a full residue system modulo M,*
2. *M must be expansive,*
3. $\det(I_n - M) \neq \pm 1$. *(*unit condition*)*

If a system fulfills the first two conditions then it is called a radix system.

Let $\varphi : \Lambda \to \Lambda, x \overset{\varphi}{\mapsto} M^{-1}(x-d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a norm $\|.\|$ on Λ and a constant C such that the orbit of every $x \in \Lambda$ eventually enters the finite set $S = \{x \in \Lambda \mid \|x\| < C\}$ for the repeated application of φ. This means that the sequence $x, \varphi(x), \varphi^2(x), \ldots$ is eventually periodic for all $x \in \Lambda$. Clearly, (Λ, M, D) is a number system iff for every $x \in \Lambda$ the orbit of x eventually reaches 0.

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A point p is called *periodic* if $\varphi^k(p) = p$ for some $k > 0$. The orbit of a periodic point p is a *cycle*. The set of all periodic points is denoted by $\mathscr{P}$. The *signature* $(l_1, l_2, \ldots, l_\omega)$ of a radix system (Λ, M, D) is a finite sequence of non-negative integers in which the periodic structure $\mathscr{P}$ consists of $\#l_i$ cycles with period lenght i $(1 \leq i \leq \omega)$. Clearly, the signature of a number system is (1).

The following problem classes are in the mainstream of the research: for a given (Λ, M, D)

- the *decision problem* asks if the triple form a number system or not,
- the *classification problem* means finding all cycles (*witnesses*),
- the *parametrization problem* means finding parametrized families of number systems,
- the *construction problem* aims at constructing a digit set D to M for which (Λ, M, D) is a number system. In general, construct a digit set D to M such that (Λ, M, D) satisfies a given signature.

Regarding the decision and classification problem the algorithmic complexity is unknown. For the practical analysis software packages were designed and implemented (please, visit the talk of Tamás Krutki of this conference), see also [1, 22, 27, 3, 33, 37].

Families of number systems were analysed by different authors and research groups, especially for canonical digit sets in the $\{1, \theta, \ldots, \theta^{n-1}\}$ basis, where M is the companion of the polynomial $p(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + x^n \in \mathbb{Z}[x]$ and $f(\theta) = 0$, since this is the most natural generalization of the usual binary or decimal number system concept (see [1, 2, 3, 4, 5, 7, 8, 12, 18, 19, 20, 21, 23, 24, 25, 26, 28, 32]).

The construction problem was analysed in the quadratic fields by ([6, 11, 31]), in two dimensional simultaneous systems by [34, 35], and in general by [14, 29, 36]. There are only a few results regarding the signature analysis [13, 16, 17]. The first part of the talk is about number system constructions with integers in real quadratic fields.

There are only a few application regarding the concept of generalized number systems [9, 10].

Kleptography is the study of stealing information securely and subliminally. A kleptographic attack is an attack which uses asymmetric cryptography to implement a cryptographic backdoor. Kleptographic attacks have been designed for RSA key generation, the Diffie–Hellman key exchange, the Digital Signature Algorithm, and other cryptographic algorithms and protocols.

The other part of the talk is about how generalized binary number systems can be applied in cryptography, especially in kleptography.

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Interactions between digits in Fibonacci Numeration

Anne Bertrand Mathis

A positive integer m can be written in "Fibonacci base" in a unique way: let $(G_n)_{n\geq 0}$ be the numeration sequence $G_1 = 1$, $G_2 = 2$, and $G_n = G_{n-1} + G_{n-2}$ for $n \geq 3$; then $m = \varepsilon_r G_r + ... + \varepsilon_1 G_1$ where the digits ε_i belongs to $\{0,1\}$ and where $\varepsilon_{i+1}\varepsilon_i = 0$ for all i; we say that $\varepsilon_r...\varepsilon_1$ is the expansion of m. We set $\Phi = \frac{1+\sqrt{5}}{2}$ so the algebraic conjugate of Φ is $\frac{1-\sqrt{5}}{2} = \frac{-1}{\Phi}$, and $G_n = \frac{1}{\sqrt{5}}\left(\Phi^{n+1} - \frac{(-1)^{n+1}}{\Phi^n}\right)$; G_n is equivalent to $\frac{1}{\sqrt{5}}(\Phi)^{n+1}$. We say that ε_n is the n^{th} digit on the left and we say also that ε_n is "the digit of the Φ^{n-1}" (ε_1 is the digit of unities, ε_2 the digit of the Φ, ε_3 the digits of the $\Phi^2$and so on) as in base ten in 645 the first digit 5 is the digit of unities, the second digit 4 is the digit of the dozens, the third digit 6 is the digit of the hundreds.

We prove that at the time where the n^{th} digit increases by 1 it was late with respect to all $(n-i)^{th}$ digits and becomes (just after growing) early with respect to all $(n-i)^{th}$ digits. So the digits are interlinked with an extraordinary precision, their motion being orchestrated by a substitution. You can imagine an helical gears with an infinite numbers of pulleys, the letters of the substitution playing the role of the pulleys.

An alphabet is a set of letters, a word (finite or infinite) is a sequence on this alphabet; the length of a finite word w is the number of letters that it contains and is denoted by $\mid w \mid$. The classical Fibonacci word is defined as follows [6] [7] : we set $\sigma'(a) = ab$ and $\sigma'(b) = a$; the map σ' extends to all word on the alphabet $\{a,b\}$ by concatenation; as $\sigma'(a)$ begin with a, $\sigma'^k(a)$ is the beginning of $\sigma'^{k+1}(a)$ and iterating the processus we obtain a $\sigma'-$invariant infinite word $\sigma^\infty(a) = abaababa...$: this is the Fibonacci word. We shall use another very similar Fibonacci word adapted to our problem: let $B = \{b_1, b_2, b_3, ...\}$ be an enumerable set of letters, and let σ be the map $b_1 \longmapsto b_1b_2$, $b_2 \longmapsto b_3$, $b_3 \longmapsto b_1b_4$, $b_4 \longmapsto b_5$ and more generally $b_{2k} \longmapsto b_{2k+1}$, $b_{2k+1} \longmapsto b_1b_{2k+2}$; the map σ extend to all finite word by concatenation and as b_1 is the beginning of $\sigma(b_1)$, for all n $\sigma^n(b_1)$ is the beginning of $\sigma^{n+1}(b_1)$; thus

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iterating the process we obtain a $\sigma-$invariant infinite word $\sigma^{\infty}(b_1) = u_0u_1...$ also called Fibonacci substitutive word. The following Lemma is clear:

Lemma 1. *If we replace all the b_{2k+1} by a and all the b_{2k} by b, σ becomes the classical substitution σ': $a \longmapsto ab$ and $b \longmapsto a$; for all $n \geq 0$, $\mid \sigma^n(b_1) \mid = \mid \sigma'^n(a) \mid$ and $\sigma^{\infty}(b_1)$ becomes the classical Fibonacci word $\sigma'^{\infty}(a)$.*

Transferring classical results from σ' *to σ we obtain the:*

Lemma 2. *For all $n \geq 0$, $\mid \sigma^n(b_1) \mid = G_{n+1}$ and $G_n = \frac{1}{\sqrt{5}}\left(\Phi^{n+1} - \frac{(-1)^{n+1}}{\Phi^{n+1}}\right)$.*

For all $n \geq 1$ the expansions of the $\mid \sigma^n(b_1) \mid = G_{n+1}$ first numbers (starting from 0) are exactly those admitting at most n digits.

More, $m = \varepsilon_r G_r + ... + \varepsilon_1 G_1$ if and only if the beginning of the substitutive word $\sigma^{\infty}(b_1) = u_0u_1....$ is $u_0...u_m = (\sigma^r(b_1))^{\varepsilon_r}(\sigma^{r-1}(b_1))^{\varepsilon_{r-1}}...(\sigma^1(b_1))^{\varepsilon_2}(\sigma^0(b_1))^{\varepsilon_1}$.

Going from m to $m+1$, the h^{th} digit either increases by 1, or vanishes, or remains fixed.

Theorem 1. *Suppose that the expansion of an integer m is $\varepsilon_r...\varepsilon_1$ and that the term u_m in the sequence $\sigma^{\infty}(b_1) = u_0u_1...$ is b_k; then the expansion of $m+1$ is obtained from the expansion of m as follows: the k^{th}digit on the left ε_k increased from $+1$ (or appears if they were no k^{th}digit), the digits $\varepsilon_{k-1}, ..., \varepsilon_1$ becomes 0 and the digits $\varepsilon_{k+1}, \varepsilon_{k+2}, ...$ do not change.*

Thus we obtain a method allowing to write the sequence of integers in Fibonacci base; the term u_m point out what to do in order to obtain the expansion of $m+1$ from the expansion of m.

Remark: it is possible to expand positive integers in all base $\beta > 1$ [3] and this substitutif phenomen can be generalised to all the bases; one even can expand numbers of $\mathbb{Z}$ in a base $-\beta < -1$ [4], define an associated substitution and study his effect on the expansion of successive integers.

Look for example at the first expansions: $G_1, G_2, G_3, G_4, G_5, G_6 = 1, 2, 3, 5, 8, 13$; the beginning $u_0...u_{12} = u_0...u_{G_6-1}$ of $\sigma^{\infty}(b_1) = u_0u_1...u_{12}$ is

$$b_1b_2b_3b_1b_4b_1b_2b_5b_1b_2b_3b_1b_6;$$

the following data (expansions smaller than $G_6 = 13 = \sigma^5(b_1)$, just below their position in the sequence $(u_n)_{n\geq 0}$ and the corresponding b_i) explain how to go from m to $m+1$:

0	1	10	100	101	1000	1001	1010	10000	10001	10010	10100	10101
u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
b_1	b_2	b_3	b_1	b_4	b_1	b_2	b_5	b_1	b_2	b_3	b_1	b_6

You can see that till $m = 12$ if the letter u_m is b_k, this letter says: to go from the expansion of the number m to the expansion of $m+1$, add 1 to the k^{th} digit on the left, replace the h^{th} digit by zero if $h < k$ and do not change the other digits.

And the term u_{G_n-1} is always equal to b_n and point out that $m+1 = G_n = 10^{n-1}$ owns n digits.

The k^{th} digit grows Φ^h= $\left(\frac{1+\sqrt{5}}{2}\right)^h$faster than the $(h+k)^{th}$ digit. So

Theorem and definition 2. *Suppose that until reaching a number m the k^{th} digit increased x_k times and the $(h+k)^{th}$ digit increased x_{h+k} times; we say that the $(h+k)^{th}$ is late with respect to the k^{th} digit if $x_{h+k} < \frac{x_k}{\Phi^h}$ and is early if $x_{h+k} > \frac{x_k}{\Phi^h}$.($\Phi$ is an irrational number so there is never equality).*

At the time where the n^{th} digit increase by 1 it was late with respect to all the $(n-i)^{th}$ digits and becomes (just after growing) early with respect to all $(n-i)^{th}$ digits.

For all $k \geq 1$ and $h \geq 1$, $\left| x_{h+k} < \frac{x_k}{\Phi^h} \right| < 1$.

Let d_n the number of times where the first digit (the digit of unities) has increased before reaching the number $G_n = 10^{n-1}$and set $G_0 = G_{-1} = 1$; then for all $n \geq 1$, $d_n = G_{n-2}$ (we shall consider that $d_0 = 0$) and for $n \geq 2$ $d_n = \frac{1}{\sqrt{5}} \left(\Phi^{n-1} - \left(\frac{-1}{\Phi^{n-1}} \right) \right)$.

Let $n \geq k$. The number of times where the $(k+1)^{th}$ digit increased in the $\mid \sigma^n (b_0) \mid$ first numbers (owning at most n digits) is d_{n-k}.

Let $m = \varepsilon_r G_r + ... + \varepsilon_1 G_1$ an integer ≥ 1.

Then the number $x_1(m)$ of times where the right digit increased between 0 and m is $x_1(m) = \varepsilon_r d_r + ... + \varepsilon_{k+1} d_{k+1} + ... + \varepsilon_1 d_1 = \sum_{i=1}^{r} \varepsilon_i G_{i-2}$

The number of times $x_{k+1}(m)$ where the $(k+1)^{th}$ digit increased between zero and m is $x_{k+1}(m) = \varepsilon_r d_{r-k} + ... + \varepsilon_{k+1} d_1 = \sum_{i=k+1}^{r} \varepsilon_i G_{r-k}$.

A calculus allows to compare $x_1(m)$ and $x_{k+1}(m)$ and to prove the Theorem 2. Another proof can be given using the simplicity of the Rauzy tiling and remains valid in the quadratic unimodular Pisot case.

Remark: for every base β there is a similar substitution who controls the passing from $m = 1$ to $m+1$; the k^{th} digit grows β^h faster than the $(h+k)^{th}$ digit, this allows by example to recover Christiane Frougny's results [1] [2] on carries.

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Bernoulli convolutions, Garsia entropy and local dimension

Kevin G. Hare

This talk will discuss joint work with Kathryn E. Hare, Kevin R. Matthews, Michael Ka Shing Ng, Nikita Sidorov and Grant Simms. Much of this work can be found in [10, 11, 12, 13, 14].

Let $\beta \in (1,2)$. Consider the expansion

$$x = \sum_{j=1}^{\infty} a_j \beta^{-j}$$

where $a_j \in \{0,1\}$. Then $a_1 a_2 a_3 \cdots$ is a *beta-expansion* for x. It is easy to see that x will have a beta-expansion if and only if $x \in I_\beta := [0, \frac{1}{\beta-1}]$.

One interesting property of non-integer expansions is that it is possible to have multiple different representations for the same value. For example, consider β the golden ratio, $\beta \approx 1.618$, the larger root of $x^2 - x - 1 = 0$. Notice that

$$\begin{aligned} 1 &= \tfrac{1}{\beta} + \tfrac{1}{\beta^2} && \text{or} \\ &= \tfrac{1}{\beta} + \tfrac{1}{\beta^3} + \tfrac{1}{\beta^4} && \text{or} \\ &= \tfrac{1}{\beta} + \tfrac{1}{\beta^3} + \tfrac{1}{\beta^5} + \tfrac{1}{\beta^7} + \tfrac{1}{\beta^9} + \cdots . \end{aligned}$$

In the case of the golden ratio, some x have only one expansion (such at $x = 0$), some have countably many expansions, and some even have uncountably many expansions.

Intimately connected with beta-expansion is the idea of *Bernoulli convolutions*, denoted by μ_β. We can think of this as a measure which has weight focused where there are lots of beta-expansions. These were first studied by by Rényi and by Parry [20, 23]. Let us recall the basic definitions.

Let μ_β denote the *Bernoulli convolution* parameterized by β on I_β by

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$$\mu_\beta(E) = \mathbb{P}\left\{(a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k \beta^{-k} \in E\right\}$$

for any Borel set $E \subseteq I_\beta$, where $\mathbb{P}$ is the product measure on $\{0,1\}^{\mathbb{N}}$ with $\mathbb{P}(a_1 = 0) = \mathbb{P}(a_1 = 1) = 1/2$.

An alternate way of defining μ_β is as the unique measure such that:

$$\mu_\beta = \frac{1}{2}\mu_\beta \circ S_0^{-1} + \frac{1}{2}\mu_\beta \circ S_1^{-1},$$

where $S_j = x/\beta + j$ for $j = 0, 1$.

Bernoulli convolutions have been studied since the 1930s (see, e.g., [25] and references therein). An important property of μ_β is the fact that it is either absolutely continuous or purely singular.

Recall that a number $\beta > 1$ is called a *Pisot number* if it is an algebraic integer whose other Galois conjugates are less than 1 in modulus. Erdős [2] showed that if β is a Pisot number, then μ_β is singular. Garsia [7] introduced a new class of algebraic integers – now referred to as *Garsia numbers* – and proved that μ_β is absolutely continuous if β is a Garsia number. Solomyak [24] proved that for Lebesgue-a.e. $\beta \in (1,2)$ the Bernoulli convolution is absolutely continuous.

Definition 1. Given a Bernoulli convolution μ_β, by the *upper local dimension* of μ_β at $x \in \operatorname{supp}\mu_\beta$, we mean the number

$$\overline{\dim}_{loc}\mu_\beta(x) = \limsup_{r\to 0^+} \frac{\log \mu_\beta([x-r, x+r])}{\log r}.$$

Replacing the lim sup by lim inf gives the *lower local dimension*, denoted $\underline{\dim}_{loc}\mu_\beta(x)$. If the limit exists, we call the number the *local dimension* of μ_β at x and denote this by $\dim_{loc}\mu_\beta(x)$.

It is also known that μ_β is *exact-dimensional* (which is a special case of a general result in [15]). Namely, there is a number α such that

$$\dim_{loc}(\mu) = \lim_{r\to 0} \frac{\log \mu_\beta(x-r, x+r)}{\log r} = \alpha$$

for μ_β-almost every x. We will call this number α *the dimension of* μ_β and denote by $\dim(\mu_\beta)$. In particular, the exact-dimensionality implies $\dim_H(\mu_\beta) = \dim(\mu_\beta)$. Clearly, if μ_β is absolutely continuous, then $\dim(\mu_\beta) = 1$. Whether the converse is true for this family, remains unknown.

Garsia [8] introduced the following useful quantity H_β, called the *Garsia entropy*, associated with a Bernoulli convolution. If β is transcendental or algebraic but not satisfying an algebraic equation with coefficients $\{-1, 0, 1\}$ (i.e., it is not of *height one*), then $H_\beta = \log 2/\log\beta > 1$.

However, if β is Pisot, then it was shown in [8, 21] that $H_\beta = \dim(\mu_\beta) < 1$. Furthermore, Garsia also proved that if $H_\beta < 1$, then μ_β is singular.

Alexander and Zagier in [1] managed to evaluate H_β for the golden ratio $\beta = \tau$ with an astonishing accuracy. It turned out that H_τ is close to 1 – in fact $H_\tau \approx 0.9957$. Grabner, Kirschenhofer and Tichy [9] extended this method to the multinacci numbers, which are the positive real roots of $x^m = x^{m-1} + x^{m-2} + \cdots + x + 1$. No other values of H_β correct up to at least two decimal places are known to date.

The Garsia entropy and the dimension of a Bernoulli convolution are connected by the following result:

Theorem 1 (Hochman, 2014 [15]). *If $\beta \in (1,2)$ is algebraic, then*

$$\dim(\mu_\beta) = \min\{H_\beta, 1\}.$$

Essentially, this remarkable theorem says that the topological quantity, $\dim(\mu_\beta)$, coincides with the combinatorial one, H_β. If β is Pisot, then this is relatively straightforward (and known since [21]); for all other algebraic β this fact is highly non-trivial.

Using this, we prove that

Theorem 2. *For all algebraic $\beta \in (1,2)$ we have* $\dim(\mu_\beta) > 0.82$.

We also show that:

Proposition 1. *If $[\mathbb{Q}(\beta^{1/k}) : \mathbb{Q}(\beta)] = k$ for some $k \geq 2$, then $H_{\beta^{1/k}} = kH_\beta$.*

Although it is possible for $[\mathbb{Q}(\beta^{1/k}) : \mathbb{Q}(\beta)] \neq k$, this does not happen often.

Corollary 0.1. *Let $k \geq 2$ and $\beta \in (1,2)$ a Pisot number. Assume further that $\beta^{1/k}$ is not a Pisot number. Then $H_{\beta^{1/k}} \geq 1$ and* $\dim(\mu_{\beta^{1/k}}) = 1$.

Using a similar technique we also show

Corollary 0.2. *Let $k \geq 2$, $\beta \in (1,2)$ and* $\deg(\beta) = r$. *Assume further that* $\deg(\beta^{1/k}) > r$. *Then $H_{\beta^{1/k}} \geq 1$ and* $\dim(\mu_{\beta^{1/k}}) = 1$.

Further, when β is a Pisot number, these Bernoullie convolutions are know to be of finite type, (first introduced by Ngai and Wang [NW]). This allows us to study in more detail the possible local dimensions arising from a Bernoulli convolution.

Building on earlier work (c.f., [6, 16, 18, 22]), Feng undertook a study of equicontractive, self-similar measures of finite type in [3, 4, 5]. His main results were for Bernoulli convolutions. In particular, he proved that despite the failure of the open set condition, the multifractal formalism still holds for the Bernoulli convolutions whose contraction factor was the reciprocal of a simple Pisot number (meaning, a Pisot number whose minimal polynomial is of the form $x^n - x^{n-1} - \cdots - x - 1$). A particularly interesting example is when the contraction factor is the golden ratio with minimal polynomial $x^2 - x - 1$ (also called the golden mean).

We first give a simple formula for the value of the local dimension of μ at any "periodic" point of its support. As a corollary we get that the local dimension exists at "periodic" points. The finite type condition leads naturally to a combinatorial notion we call a "loop class". For a "positive" loop class we prove that the set of attainable local dimensions of the measure is a closed interval and that the set of local dimensions at periodic points in the loop class is a dense subset of this interval. Similar results are also given for upper and lower local dimensions. Given two values $\ell \leq u$ within this interval, we can find an x in this positive loop class with lower local dimension equal to ℓ and upper local dimension equal to u.

A consequence of our result is that the set of attainable local dimensions is the union of a closed interval together with the local dimensions at points in finitely many loop classes external to the essential class. We will say that a point is an essential point if it is in the essential class. The set of essential points has full Lebesgue measure on the support of the measure and in many interesting examples the set of essential points is the interior of the support of the measure.

When the essential set is the interior of the support of the measure μ, then μ has no isolated point in its set of attainable local dimensions if and only if $\dim_{loc} \mu(0)$ coincides with the local dimension of μ at an essential point. In that case, the set of attainable local dimensions of μ is a closed interval. The Bernoulli convolution μ_ρ, with ρ^{-1} a simple Pisot number has this property.

However, we construct other examples of Bernoulli convolutions (with contraction factor a Pisot inverse) which do have an isolated point in their set of attainable local dimensions. As far as we are aware, these are the first examples of Bernoulli convolutions known to admit an isolated point. We also construct a Cantor-like measure of finite type, whose set of local dimensions consists of (precisely) two distinct points. In all of these examples, the essential set is the interior of the support of the measure.

The computer was used to help obtain some of these results. In principle, the techniques could be applied to other convolutions of Bernoulli convolutions, however even with the simple examples given here, the problem can become computationally difficult.

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Bases with two expansions

Vilmos Komornik and Derong Kong

Abstract In this paper we answer several questions raised by Sidorov on the set $\mathscr{B}_2$ of bases in which there exist numbers with exactly two expansions. In particular, we prove that the set $\mathscr{B}_2$ is closed, and it contains both infinitely many isolated and accumulation points in $(1, q_{KL})$, where $q_{KL} \approx 1.78723$ is the Komornik-Loreti constant. Consequently we show that the second smallest element of $\mathscr{B}_2$ is the smallest accumulation point of $\mathscr{B}_2$. We also investigate the higher order derived sets of $\mathscr{B}_2$. Finally, we prove that there exists a $\delta > 0$ such that

$$\dim_H(\mathscr{B}_2 \cap (q_{KL}, q_{KL} + \delta)) < 1,$$

where $\dim_H$ denotes the Hausdorff dimension. For more details see arXiv:1705.00473.

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A shrinking hole for β-transformations

Niels Langeveld

Abstract For $\beta \in (1,2]$ let T_β be the expanding map on the interval $[0,1)$ defined by $T_\beta(x) = \beta x \pmod 1$. In this talk we consider the set E_β of all $t \in [0,1)$ for which the forward orbit $\left\{T_\beta^n(t) : n \geq 0\right\}$ avoids the hole $[0,t)$. The set E_β is a Lebesgue null set of full Hausdorff dimension for all $\beta \in (1,2)$. Different from $\beta = 2$, we prove that for Lebesgue almost every $\beta \in (1,2)$ the set E_β contains both infinitely many isolated and accumulation points. Furthermore, we characterise the set of β such that $E_\beta \cap [0,\delta)$ has isolated points for all $\delta > 0$, the set of β for which there exists a $\delta > 0$ such that $E_\beta \cap [0,\delta)$ has no isolated points and the set of β for which E_β has no isolated points. In the last case we show that $\overline{E_\beta}$ is a Cantor set satisfying

$$\lim_{\delta \to 0} \dim_H \left(E_\beta \cap (t-\delta, t+\delta)\right) = \dim_H K_\beta(t) \quad \text{for all} \quad t \in E_\beta, \tag{1}$$

where $K_\beta(t)$ is the survivor set defined by $K_\beta(t) = \left\{x \in [0,1) : T_\beta^n(x) \geq t \text{ for any } n \geq 0\right\}$. This is joint work with Charlene Kalle, Derong Kong and Wenxia Li.

1 The main theorems

Let us also define the function $\eta_\beta : [0,1) \to [0,1]$ by

$$\eta_\beta(t) := \dim_H K_\beta(t). \tag{2}$$

Then by the results from [4] the function η_β is a continuous, monotone function with $\eta_\beta(0) = 1$ and $\eta_\beta(1) = 0$ and such that $\eta_\beta'(t) = 0$ for Lebesgue almost every $t \in [0,1)$. Denote by B_β the *bifurcation set* of η_β, i.e., the set of $t \in [0,1)$ where the function η_β is not locally constant. Since η_β is continuous, B_β has no isolated

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points. For the doubling map Nilsson proved in [2] that the bifurcation set B_2 of this dimension function is a Cantor set of full Hausdorff dimension. In [1] Carminati and Tiozzo proved for the maps $x \mapsto kx \pmod 1$ ($k \in \mathbb{N}_{\geq 2}$) that at every bifurcation parameter t of the function $t \mapsto \dim_H K(t)$ the local Hölder exponent equals the value of the function itself. Both [2] and [1] used the fact that B_2 equals the set $E_2 := \{t \in [0,1) : T_2^n(t) \geq t \text{ for any } n \geq 0\}$, proved in [2]. Urbański proved in [4] that for any $t \in E_2$,

$$\lim_{\delta \to 0} \dim_H \big(E_2 \cap (t-\delta, t+\delta)\big) = \dim_H K(t). \tag{3}$$

Similar to $\beta = 2$ we define the set E_β for $\beta \in (1,2)$ as

$$E_\beta := \Big\{t \in [0,1) : T_\beta^n(t) \geq t \text{ for any } n \geq 0\Big\}. \tag{4}$$

One might expect that the behaviour of E_β is similar to that of $E_2 = B_2$. Some of our results indicate similarities between the sets E_2 and E_β. Namely, we have

Theorem 1. *For any $\beta \in (1,2)$ the set E_β is a Lebesgue null set of full Hausdorff dimension.*

This result is unexpected, since the symbolic dynamical system associated to the β-transformation becomes more and more restricted as $\beta \to 1$. Apparently this has no significant influence on the size of E_β. For other properties we identify a set of $\beta \in (1,2)$ which have more similarities as $\beta = 2$. To be able to do so we define cyclically balanced words and Sturmian words.

For any word $u \in \{0,1\}^*$, use $|u|_1$ to denote the number of ones. A word w is called *balanced* if for every two factors u, v of w of the same length we have $\big||u|_1 - |v|_1\big| \leq 1$ and cyclically balanced if w^2 is balanced. Let $w \in \{0,1\}^\infty$ be a sequence. If it is non-periodic and $\big||u|_1 - |v|_1\big| \leq 1$ for any two factors u and v of w, then the sequence is called *Sturmian*. Let $\alpha : (1,2) \to \{0,1\}^{\mathbb{N}}$ denote the function assigning to β the corresponding quasi-greedy β-expansion of 1, i.e.,

$$\alpha(\beta) = \alpha_1(\beta)\alpha_2(\beta)\cdots,$$

where $1 = \sum_{n \geq 1} \frac{\alpha_n(\beta)}{\beta^n}$ and $\alpha(\beta)$ does not end in an infinite string of zeros. We define the set

$$R := \{\beta \in (1,2) : \alpha(\beta) \text{ is cyclically balanced or Sturmian}\}.$$

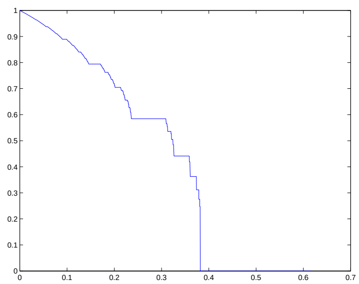
As the following theorem will show this will be the set that has similarities with $\beta = 2$.

Theorem 2. *For $\beta \in R$ the following statements hold.*

(i) *$\overline{E}_\beta$ is a Cantor set. Furthermore, for any $t \in E_\beta$ we have*

$$\lim_{\delta \to 0} \dim_H \big(E_\beta \cap (t-\delta, t+\delta)\big) = \eta_\beta(t). \tag{5}$$

(ii) *The survivor set $K_\beta(t)$ has the following properties.*



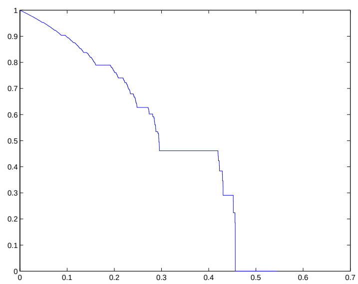


Fig. 1: Left: the graph of η_β with β the golden mean. Right: the graph of η_β with β the tribonacci constant.

- *If* $t \in E_\beta$, *then* $\dim_H \left(K_\beta(t) \cap [t, t+\delta)\right) = \eta_\beta(t)$ *for all* $\delta > 0$.
- *If* $t \notin E_\beta$, *then* $K_\beta(t) \cap [t, t+\delta) = \emptyset$ *for some* $\delta > 0$.

On the other hand, many of our results show that there are fundamental differences between E_2 and E_β as well. For $\beta \in (1,2)$ the set E_β has no isolated points if and only if $\beta \in R$. Moreover, the Hausdorff dimension of R is zero. In other words, in most cases $E_\beta \neq B_\beta$ since η_β is continuous.

Let us now define

$$C_3 = \{\beta \in (1,2) : \text{ the orbit of 1 under the map } T_\beta \text{ avoids } [0,\delta) \text{ for some } \delta > 0\}. \tag{6}$$

We have the following theorem

Theorem 3. *Let $\beta \in (1,2)$. The set $E_\beta \cap [0,\delta)$ has isolated points for all $\delta > 0$ if and only if $\beta \notin C_3$.*

Note that $R \subset C_3$. In [3] it is shown that C_3 is a Lebesgue null set with $\dim_H(C_3) = 1$. What we find is that for almost all $\beta \in (1,2)$ any neighbourhood of zero contains both isolated points and non-isolated points.

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Expansions in non-integer bases in control problems

Paola Loreti

In this talk we discuss some classes of expansions in non integer-bases describing how bases can represent physical properties of the system and the alphabet can be associated to a control acting on the system. Then we discuss a result obtained in collaboration with C. Baiocchi and V. Komornik (work in progress).

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Periodic representations in algebraic non-integer base

Tomáš Vávra

Abstract We consider $(\beta,\mathscr{A})$-representations of the form $\sum_{i=-\infty}^{k} a_i\beta^i$ with algebraic base β, $|\beta| > 1$, and a finite alphabet $\mathscr{A} \subset \mathbb{Q}(\beta)$.

We study the following problem: "For which bases β does there exist an alphabet $\mathscr{A}$ such that each element of $\mathbb{Q}(\beta)$ has eventually periodic $(\beta,\mathscr{A})$-representation?"

We show that this question can be answered in the affirmative if β has no Galois conjugates on the unit circle. This is an extension of the result of of Baker, Masáková, Pelantová, and Vávra.

It remains to solve the problem for the bases that have a conjugate on the unit circle. Such a class of numbers contains for example the Salem numbers for which is the positive answer to our question conjectured by K. Schmidt. We will discuss the possible application of our method to these cases.

This is a joint work with V. Kala.

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Self-similar manipulators, Fibonacci sequence and number systems

Anna Chiara Lai

Abstract We introduce a model for planar manipulators characterised by a self-similar morphology. We show that related control problems can be set in the framework of non-standard number systems and theory of Iterated Function Systems. We then focus on models for snake-like manipulators based on Fibonacci sequences: reachability and local controllability are investigated. This presentation is based on joint works with Paola Loreti and Pierluigi Vellucci.

1 Introduction

A manipulator is a robotic device composed by several rigid, possibly extensible, links joint by rotating junctions. Robot fingers, snake-like manipulators and robot octopus-inspired tentacles are examples of manipulators. The motion of *planar* manipulators is constrained on the plane, while *redundant* manipulators are characterised by a number of links which is gretar than the degrees of freedom of the device. We study a model for planar manipulators with an arbitrarily large number of links, the so-called *hyper-redundant planar manipulators*. This class of robotic arms is knowning an increasing interests among researchers due to their good performances in constrained enviroments [1]. The peculiarity of the model presented here is a self-similarity assumption on the links, that gives access to classical methods of the theory of Iterated Function Systems for the investigation of the reachable workspace. We then focus on the particular case of links scaling according to Fibonacci sequence.

In order to build our model, we assume links and junctions composing the manipulator to be thin, so to be respectively approximated by their middle axes and barycentres. We also assume axes and barycentres to be coplanar and, by employing

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the isometry between $\mathbb{R}^2$ and $\mathbb{C}$, we use the symbols $x_0, x_1, ..., x_n \in \mathbb{C}$ to denote the position of the barycentres of the junctions. We define l_k as the maximal length of the k-th link. A binary control $u_k \in \{0,1\}$ rules the extension of the k-link in particular we set $|x_k - x_{k-1}| = u_k l_k$, so that if $u_k = 1$ then the k-th link has length l_k and if $u_k = 0$ then the k-th link is unextended and the k-th and the $k-1$-th joints coincide. Moreover a binary control v_k rules the rotation of the k-th link: if $v_k = 0$ then the $k-1$-th link and the k-th link are collinear, if $v_k = 1$ they form an angle $\pi - \omega$, $\omega \in (0,\pi)$ is fixed. In view of these assumptions, the position of k-th junction x_k depends on the $k-1$-th junctions according to the discrete controlled dynamical system:

$$x_k = x_{k-1} + u_k l_k e^{-i\omega \sum_{j=1}^{k} v_j}. \tag{1}$$

A closed formula for x_k is given by

$$x_k = \sum_{k=0}^{n} u_k l_k e^{-i\omega \sum_{j=1}^{n} v_j}. \tag{2}$$

We say that a sequence (l_k) in $\mathbb{R}$ is *Linear-Contractive-Recursive* (LCR) if there exists an integer $n \in \mathbb{N}$ and a linear map $F : \mathbb{R}^n \to \mathbb{R}$ satisfying $l_k = F(l_{k-1}, \dots, l_{k-n})$ for every $k > n$ and there exists $L < 1$ such that $|F(x) - F(y)| < L||x-y||_\infty$, where $||(x_1, \dots, x_n)||_\infty := \max\{|x_k|, k = 1, \dots, n\}$ denotes the L^∞-norm of $x \in \mathbb{R}^n$. We are interested in the *asymptotic reachable workspace*

$$\mathscr{W} := \{\lim_{n\to\infty} x_n(u,v) \mid u,v \in \{0,1\}^\infty\} \qquad = \left\{ \sum_{k=0}^{\infty} u_k l_k e^{-i\omega \sum_{j=1}^{n} v_j} \mid u,v \in \{0,1\}^\infty \right\}.$$

Also we investigate a subset of $\mathscr{W}$ corresponding to the so called *full-extension configurations*, characterised by extension controls constantly equal to 1:

$$\mathscr{E} := \{\lim_{n\to\infty} x_n(\mathbf{1},v) \mid v \in \{0,1\}^\infty\} = \left\{ \sum_{k=0}^{\infty} l_k e^{-i\omega \sum_{j=1}^{n} v_j} \mid u,v \in \{0,1\}^\infty \right\}.$$

Example 1. The geometric sequence $l_k = q^{-k}$ for some $q > 1$ is LCR. Indeed $l_k = l_{k-1} q^{-1}$ for every $k \in \mathbb{N}$. In this case the asymptotic reachable workspace $\mathscr{W}$ is a self-similar set [3]. Moreover note that the *full-rotation configurations* $x(u,\mathbf{1}) = \sum_{k=0}^{\infty} u_k \frac{e^{-i\omega k}}{\rho^k}$ are expansions in the complex base $q := \rho e^{-i\omega}$ with digits (u_k).

Example 2. The *scaled Fibonacci sequence* $l_k = f_k q^{-k}$ is LCR, where f_k is the k-th Fibonacci number, $q > \varphi$ and φ is the Golden Mean. We discuss this case in Section 3 below.

2 Self-similar manipulators

In [4], the properties of LCR sequences are applied to the study of a simplified version of (1) corresponding to case $u_n \equiv 1$, i.e., to the full-extension configurations. In particular, a relation with fractal geometry is established: the reachable set is showed to be a suitable projection of a self-similar set. The core idea is to build an Iterated Function System (IFS) naturally associated to (l_k). To this end, recall that the dual of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ itself, namely every linear map $F:\mathbb{R}^n \to \mathbb{R}$ can be identified with an element $a \in \mathbb{R}^n$. In other words, there exists $a = (a_1,\dots,a_n) \in \mathbb{R}^n$ such that for every $k > n$

$$l_k = \langle a, (l_{k-1},\dots,l_{k-n})\rangle.$$

Now, we define the $n \times n$ matrix

$$A := \begin{pmatrix} a_1 \dots a_{n-1} & a_n \\ \mathbf{I}_{n-1} & 0_{n-1} \end{pmatrix},$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1)$-dimensional identity matrix. By construction, for every $k > n$

$$A(l_{k-1}, l_{k-2}, \dots, l_{k-n})^T = (l_k, l_{k-1}, \dots, l_{k-n+1})^T, \tag{3}$$

that is A acts on (l_k) as a shift operator with window of length n. We say that a map F is *eventually contractive* if some of its iterations F^k, with k sufficiently large, is a contractive map. We call a finite set of eventually contractive maps an *eventually contractive* IFS. We now consider the affine eventually contractive IFS $(F_u)_{u\in\{0,1\}}$ defined on $\mathbb{C}^n$

$$F_u(x) := e^{-i\omega u}(Ax+B) \qquad u \in \{0,1\}, \tag{4}$$

where $B = (l_n, l_{n-1}, \dots, l_1)^T$. Note that by (3)

$$A^k B = (l_{k+n}, \dots, l_{k+1})^T \qquad \forall k > 0. \tag{5}$$

Moreover by Hölder inequality the spectral radius of A is lower than 1 and it can be proved that this a sufficient condition to have an unique compact attractor for the IFS $(F_u)_{u\in\{0,1\}}$.

Theorem 1 ([4]). *Let $\mathscr{F}(S) := \cup_{u\in\{0,1\}} F_u(S)$ be the Hutchinson operator associated to $(F_u)_{u\in\{0,1\}}$ and denote by π_1 the projection of a vector of $\mathbb{C}^n$ on its first component. Let Q_∞ be the (unique) invariant set of $(F_u)_{u\in U}$. Then asymptotic reachable workspace corresponding to full extension configurations $\mathscr{E}$ satisfies*

$$\mathscr{E} = \left\{ \sum_{k=0}^{\infty} l_k e^{-i\omega \sum_{n=0}^{k} v_n} \mid v_n \in \{0,1\} \right\} = \pi_1(Q_\infty)$$

and for every bounded set $S \subset \mathbb{C}^n$ we have $\mathscr{E} = \pi_1(\lim_{k\to\infty} \mathscr{F}^k(S))$.

3 Fibonacci manipulators

We assume now that the length of the links is governed by the recursive relation

$$l_{k+1} = \frac{1}{q} l_k + \frac{1}{q^2} l_{k-1}$$

with $l_0 = 1$ and $l_1 = \frac{1}{q}$ for some $q > \varphi$ – recall that φ denotes the Golden Mean. The assumption $q > \varphi$ ensures the convergence of the series $\sum_{k=0}^{\infty} l_k$, representing the maximal total length of the manipulator: a closed formula for (l_k) is given by

$$l_k = \frac{f_k}{q^k}.$$

where (f_k) is Fibonacci sequence, namely $f_0 = f_1 := 1$ and $f_{k+2} = f_{k+1} + f_k$ for all $k \geq 0$ and one has that the radius of convergence of the power series $\sum_{k=0} f_k y^k$ is $\frac{1}{\varphi}$. The following local controllability holds[1]

Theorem 2 ([6]). *Let $p \in \mathbb{N}$ and $q(p)$ be the greatest real solution of the equation*

$$\sum_{k=0}^{\infty} \frac{f_{pk}}{q^{pk}} = 2.$$

If $\omega = 2\pi \frac{d}{p}$ for some coprime integers $d, p \in \mathbb{N}$ and if $q \in (\varphi, q(p)]$, then the asymptotic reachable workspace

$$\mathscr{W} = \left\{ \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k} e^{-i\omega \sum_{n=0}^{k} v_n} \mid u, v \in \{0,1\}^{\infty} \right\}$$

contains a neighborhood of the origin.

We present also a result on the quantity

$$L(u) = \sum_{k=0}^{\infty} u_k \frac{l_k}{q^k}$$

representing the total length of manipulator corresponding to the extension control sequence $u \in \{0,1\}^{\infty}$. We consider the set

$$\mathscr{L} := \{L(u) \mid u \in \{0,1\}^{\infty}\} = \left\{ \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k} \mid u \in \{0,1\}^{\infty} \right\}.$$

In order to give a complete characterisation of $\mathscr{L}$ we introduce the quantities:

[1] We recall that a manipulator is locally controllable if its reachable workspace contains a neighborhood of the origin

$$
\begin{aligned}
S(q,j) &:= \sum_{k=0}^{\infty} \frac{f_{j+k}}{q^k} \\
Q(-1) &:= \varphi; \\
Q(j) &:= \text{greatest solution of the equation } S(q,j+1) = qf_j \\
&= \frac{1}{2f_j}\left(f_{j+2} + \sqrt{f_{j+2}^2 + 8f_j^2}\right).
\end{aligned}
$$

Note that $Q(0) = 1+\sqrt{3}$.

Theorem 3 ([6, 2]). *For all $j \geq 0$ if $q \in (Q(j-1), Q(j)]$, then $\mathscr{L}$ is composed by the disjoint union of 2^j intervals:*

$$
\mathscr{L} = \bigcup_{u_0,\ldots,u_{j-1}\in\{0,1\}} \left[\sum_{k=0}^{j-1} \frac{f_k}{q^k} u_k, \sum_{k=0}^{j-1} \frac{f_k}{q^k} u_k + S(q,j)\right]. \tag{6}
$$

In particular, if $q \in (\varphi, 1+\sqrt{3}]$, then $\mathscr{L}$ is the interval $[0, \sum_{k=0}^{\infty} \frac{f_k}{q^k}]$.

Moreover if $q \geq \frac{1}{2}(\varphi^2 + \sqrt{\varphi^2+1})$ then the map $u \mapsto x_u = \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k$ is increasing with respect to the lexicographic order and $\mathscr{L}$ is a totally disconnected set.

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Digit frequencies and self-affine sets with non-empty interior

Simon Baker

Abstract In this talk I will discuss recent results on digit frequencies in the setting of expansions in non-integer bases, and self-affine sets with non-empty interior. Within expansions in non-integer bases I will give details of a recent result which states that if $\beta \in (1, 1.787\ldots)$ then every $x \in (0, \frac{1}{\beta-1})$ has a simply normal β-expansion. Employing similar ideas I will show that if $\beta \in (1, \frac{1+\sqrt{5}}{2})$ then every $x \in (0, \frac{1}{\beta-1})$ has a β-expansion for which the digit frequency does not exist, and a β-expansion with limiting frequency of zeros p, where p is any real number sufficiently close to $1/2$. For a class of planar self-affine sets I will show that if the horizontal contraction lies in a certain parameter space and the vertical contractions are sufficiently close to 1, then every nontrivial vertical fibre contains an interval. This approach lends itself to explicit calculation and give rise to new examples of self-affine sets with non-empty interior. One particular strength of this approach is that it allows for different rates of contraction in the vertical direction.

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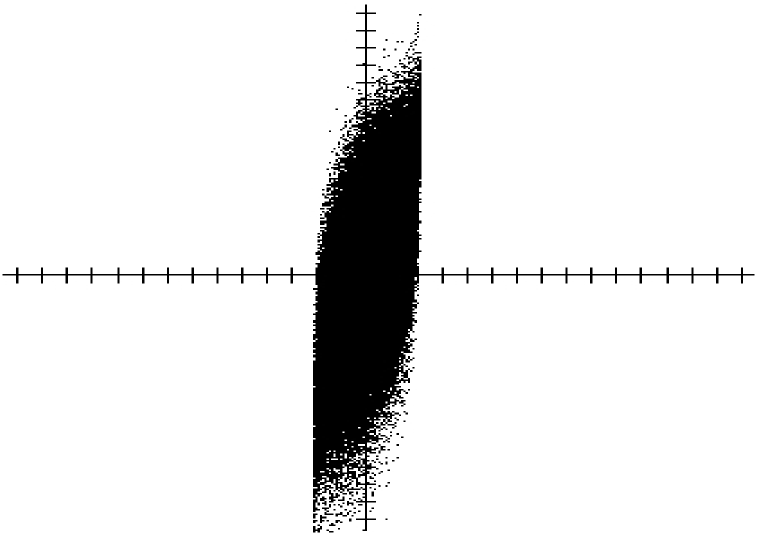

Fig. 1: An example of a self-affine set where it can be shown that each vertical fibre contains an interval.

Essentially nonnormal numbers for random Cantor series expansions

Bill Mance and Roman Nikiforov

The set of *essentially nonnormal numbers in base b*, L_b, consists of those numbers where no digital frequencies exists in their b-ary expansion. It was proven by S. Albeverio, M. Pratsiovytyi, and G. Torbin in [2] that L_b has full Hausdorff dimension. We extend this result to a large class of Cantor series expansions and considering numbers not only whose digital frequencies do not exist but whose block frequencies also do not exist.

A *basic sequence* is a sequence of integers greater than or equal to 2. Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the *Q-Cantor series expansion* of a real number x is the (unique) expansion of the form

$$x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \tag{1}$$

where $E_0 = \lfloor x \rfloor$ and E_n is in $\{0, 1, \cdots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. For a basic sequence $Q = (q_n)$, a block $B = (b_1, b_2, \cdots, b_\ell)$, and a natural number j, define

$$\mathscr{I}_{Q,j}(B) = \begin{cases} 1 & \text{if } b_1 < q_j, b_2 < q_{j+1}, \cdots, b_\ell < q_{j+\ell-1} \\ 0 & \text{otherwise} \end{cases}.$$

and let

$$Q_n(B) = \sum_{j=1}^{n} \frac{\mathscr{I}_{Q,j}(B)}{q_j q_{j+1} \cdots q_{j+\ell-1}}.$$

A real number x is *Q-normal* if for all blocks B such that $\lim_{n\to\infty} Q_n(B) = \infty$

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$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n(B)} = 1.$$

where $N_n^Q(B,x)$ is the number of occurences of the block B in the sequence $(E_i)_{i=1}^n$ of the first n digits in the Q-Cantor series expansion of x. Motivated by this definition we say that x is *Q-essentially nonnormal* if the limit $\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n(B)}$ does not exist for all B such that $\lim_{n\to\infty} Q_n(B) = \infty$ and call this set L_Q^*. Clearly, when $q_n = b$ for all n, we have $L_b \subsetneq L_Q^*$. We improve the result in [2] by not only showing that L_Q^* has full Hausdorff dimension in this special case, but that its Hausdorff dimension is 1 for other classes of basic sequences. Furthermore, we will show that the Hausdorff dimension of L_Q^* will be 1 almost surely when the bases q_n are i.i.d. random variables where the expecation of $E[\log q_1] < \infty$. Furthermore, our result will even hold if we consider the set L_Q^{AP} of real numbers where a similar limit doesn't exist for numbers whose Cantor series digits are sampled along all nontrivial arithmetic progressions.

Our construction surprisingly will use a recent result of Joseph Vandehey on numbers normal with respect to the regular continued fraction expansion. In [1], he proved that if $x = [a_0; a_1, a_2, \cdots]$ is the continued fraction of x and if x is continued fraction normal, then for every pair of integers m, r with $m \geq 2$ and $r \geq 0$, the real number $[0; a_r, a_{m+r}, a_{2m+r}, \cdots]$ is not normal with respect to the regular continued fraction expansion. We will use this property to construct members of L_Q^{AP}.

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Computing with generalized number systems using the computer algebra system SYGNM

Tamás Krutki, Bence Németh and Attila Kovács

Abstract As computational methods become increasingly important in scientific research and education there is a high demand for all kinds of mathematical software. Computer algebra systems aim to provide both symbolic and numerical capabilities covering a wide range of mathematical topics. The development of some of the most widely used computer algebra systems has began over 30 years ago, which means that these systems are outdated in some aspects and do not fully take advantage of new technologies and software engineering methods that appeared since then. A new, general purpose computer algebra system architecture and its implementation is presented which aims to fix several problems of current systems. Part of this new system is a built-in package for computations with generalized number systems, containing functionality which is not found in other computer algebra systems. The number systems package is efficient and well integrated into the general purpose computer algebra environment, however it is also available as a separate C++ library. The current capabilities of the number systems package and plans for future number system related developments are presented. Both the new computer algebra system and the generalized number systems library will be released as open source software.

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Examining the number system property with probabilistic algorithms

Péter Hudoba and Attila Kovács

Abstract The world of generalized number systems contains many challenging areas. In some cases the complexity of the arising problems is unknown. Researching these problems are important since they can be applied e.g. in cryptography. In this talk we focus on experimenting with probabilistic methods by which we can support the analysis of the "decision problem". To be more precise we present two Las Vegas type probabilistic algorithms and measurements regarding the attraction domain distributions. We implemented our solutions in the computer algebra system Sage (some parts in C++).

1 Extended abstract

Let Λ be a lattice in R^n and let $M : \Lambda \to \Lambda$ be a linear operator such that $\det(M) \neq 0$. Let furthermore $0 \in D \subseteq \Lambda$ be a finite subset. Lattices can be seen as finitely generated free Abelian groups.

Lattices have many significant applications in pure mathematics (Lie algebras, number theory and group theory), in applied mathematics (coding theory, cryptography) because of conjectured computational hardness of several lattice problems, and are used in various ways in the physical sciences.

In this talk we consider number expansions in lattices.

Definition 1. The triple (Λ, M, D) is called a *number system* (GNS) if every element x of Λ has a unique, finite representation of the form

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$$x = \sum_{i=0}^{L} M^i d_i \ ,$$

where $d_i \in D$ and $L \in N$. L is the *length of the expansion*.

Here M is called the *base* and D is the *digit set*. It is easy to see that similarity preserves the number system property, i.e., if M_1 and M_2 are similar via the matrix Q then (Λ, M_1, D) is a number system if and only if $(Q\Lambda, M_2, QD)$ is a number system at the same time. If we change the basis in Λ a similar integer matrix can be obtained, hence, no loss of generality in assuming that M is integral acting on the lattice Z^n.

If two elements of Λ are in the same coset of the factor group $\Lambda/M\Lambda$ then they are said to be *congruent* modulo M. The following theorem is well-known.

Theorem 1. *If (Λ, M, D) is a number system then*

1. *D must be a full residue system modulo M,*
2. *M must be expansive,*
3. $\det(I_n - M) \neq \pm 1$. *(*unit condition*)*

If a system fulfills the first two conditions then it is called a radix system*.*

Let $\varphi : \Lambda \to \Lambda$, $x \overset{\varphi}{\mapsto} M^{-1}(x-d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a norm $\|.\|$ on Λ and a constant C such that the orbit of every $x \in \Lambda$ eventually enters the finite set $S = \{x \in \Lambda \mid \|x\| < C\}$ for the repeated application of φ. This means that the sequence $x, \varphi(x), \varphi^2(x), \ldots$ is eventually periodic for all $x \in \Lambda$. Clearly, (Λ, M, D) is a number system iff for every $x \in \Lambda$ the orbit of x eventually reaches 0.

A point p is called *periodic* if $\varphi^k(p) = p$ for some $k > 0$. The orbit of a periodic point p is a *cycle*. The set of all periodic points is denoted by $\mathscr{P}$.

The following problem classes are in the mainstream of the research: for a given (Λ, M, D)

- the *decision problem* asks if the triple form a number system or not.
- the *classification problem* means finding all cycles (*witnesses*).
- the *parametrization problem* means finding parametrized families of number systems.
- the *construction problem* aims at constructing a digit set D to M for which (Λ, M, D) is a number system.

The algorithmic complexity of the decision and classification problems is still unknown. In this talk we propose two new probabilistic methods improving the running time of determining the number system property.

- We analysed the Garsia operators (the companions of all expansive polynomials with constant term ± 2) and monitored the orbits of randomly chosen points. In case of non-number systems in order to find a witness it was enough to calculate the orbit of a very few number of randomly chosen points from S.

- According to the conjecture of the second author ([2]) we obtained that in a non-number system case there were always at least one point around zero (having coordinates from $\{0, \pm 1\}$) that converged to a non-zero periodic point. If we combine this observation with the previous algorithm we can find witnesses really fast with high probability.

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Normal Subsequences of Automatic Sequences

Clemens Müllner

In 2013 Drmota, Mauduit and Rivat observed that the subsequence along the squares $(t(n^2))_{n\geq 0}$ of the Thue-Morse sequence $(t(n))_{n\geq 0}$ (that can be defined by $t(n) = s_2(n) \bmod 2$, where $s_2(n)$ denotes the binary sum-of-digits function) is a normal sequence on the alphabet $\{0,1\}$.

The purpose of this talk is to discuss this result also from a more general point of view. The Thue-Morse sequence is a special case of an k-automatic sequence $(a(n))_{n\geq 0}$, that is, a sequence where the n-th element is the output of a deterministic finite state automaton, where the input is the base k expansion of n. Automatic sequence have a sub-linear subword complexity - so they are far from being normal. Furthermore, linear subsequences of automatic sequences are again automatic sequence and, therefore, have again sub-linear subword complexity. However, when we consider a subsequence $a(\phi(n))$, where $\phi(n)/n \to \infty$ the situation can change completely - as $(t(n^2))_{n\geq 0}$ shows. Thus, we discuss the question for which subsequences - and for which automatic sequences - we may expect a normal sequence.

Recent results in this direction, e.g., by Lukas Spiegelhofer and the author (who considered $\phi(n) = \lfloor n^c \rfloor$ for the Thue-Morse sequence, where $1 < c < 3/2$) and by the author (who considered block-additive functions like the Rudin-Shapiro sequence and $\phi(n) = n^2$) indicate that there might be a more general principle behind.

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Totally Real Algebraic Numbers, Bogomolov Property, and Dynamical Zeta Function of the β-shift

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Abstract Schinzel (1973) obtained the lower bound $\frac{1}{2}\log(1+\sqrt{5}/2) = 0.24\ldots$ for the Weil's height $h(\alpha)$ of any totally real algebraic integer $\alpha \neq 0, \neq \pm 1$, optimally. This problem of minoration of the height is related to the problem of Lehmero for Salem numbers with Mahler's measure $\mathrm{M}(\alpha)$. Bombieri and Zannier (2001) introduced the property of Bogomolov for any field $F \subset \overline{\mathbb{Q}}$, by analogy with Bogomolov's Conjecture: by definition F has the property of Bogomolov relative to h if and only if $h(\alpha) = 0$ or admits a lower bound > 0 for any $\alpha \in F$. Amoroso and Zannier (2000) proved it for K^{ab}, where K is a number field. Bombieri and Zannier (2001) for totally p-adic fields, Habegger (2011) for $\mathbb{Q}(E_{tors})$, where $E/\mathbb{Q}$ is an elliptic curve. Fili and Miner (2016), using limit equidistribution theorems of Favre and Rivera-Letelier, proved $\liminf h(\alpha) \geq 0.12\ldots$ for α in the field of totally real algebraic numbers $\mathbb{Q}^{tr}$; Pottmeyer (2016) obtained the limit infimum $\liminf h(\alpha) \geq \frac{7}{4\pi^2}\zeta(3)$ by other techniques. In this work we show that the dynamical zeta function of the β-shift relative to the arithmetical Rényi-Parry dynamical system, with $\beta = \lceil \alpha \rceil$, $\alpha \in \mathbb{Q}^{tr}$, allows to prove that the property of Bogomolov for $\mathbb{Q}^{tr}$ is true, with a global explicit minoration.

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On the Hausdorff dimension faithfulness of expansions with infinite alphabet and properties of non-normal numbers

Roman Nikiforov

The notion of the Hausdorff dimension is well known, but in many cases it is a rather non-trivial problem to give the exact dimension. One approach to the simplification of the calculation of the Hausdorff dimension consists in some restrictions of admissible coverings.

A fine covering family Φ is said to be a faithful family of coverings for the Hausdorff dimension calculation on $[0,1]$ if $\dim_H(E,\Phi) = \dim_H(E), \forall E \subseteq [0,1]$.

The first phenomenon we will talk about is connected with the problem of faithfulness of the family of cylinders generated by the expansions with infinite alphabet (continued fractions, Lüroth expansion, Sylvester expansion and other).

We will also discuss properties of sets of essentially non-normal numbers defined in different systems of numeration.

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On Weyl's theorem on Uniform Distribution and Ergodic Theorems

Radhakrishnan Nair and Entesar Nasr

We say sequence a $\{x_1,\ldots,x_N\}$ is uniformly distributed modulo one if

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\le n\le N : x_n\in I\}=|I|$$

for every interval $I\subseteq[0,1)$.

We also say a sequence of natural numbers $(k_n)_{n\ge1}$ is Hartman uniform distributed (on $\mathbb{Z}$) if it is uniformly distributed in residue classes modulo m, for each natural number $m>1$, and for each irrational number α, the sequence $(\{k_n\alpha\})_{n\ge1}$ is uniformly distributed modulo one. Here and henceforth, for a real number y we use $\{y\}$ to denote its fractional part. Note that if $(k_n)_{n\ge0}$ is Hartman uniformly distributed, and if we set

$$F(N,z):=\frac{1}{N}\sum_{n=0}^{N-1}z^{k_n}, \qquad (N=1,2,\cdots)$$

we have $F(N,1)=1$ for all $N\ge1$ and if $z\ne1$ we have $\lim_{N\to\infty}F(N,z)=0$. A list of examples is given in the next section .

Let $(X,\mathscr{B},\mu)$ be a probability space and let $T:X\to X$ be a measurable map, that is also measure-preserving. That is, given $A\in\mathscr{B}$, we have $\mu(T^{-1}A)=\mu(A)$, where $T^{-1}A$ denotes the set $\{x\in X : Tx\in A\}$. We call $(X,\mathscr{B},\mu,T)$ a dynamical system. We

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say the dynamical system is ergodic if $T^{-1}A = A$ for $A \in \mathscr{B}$ means that either $\mu(A)$ or $\mu(X\backslash A)$ is 0.

We say $(k_n)_{n\geq 0}$ is L^p good universal if for each dynamical system $(X, \mathscr{B}, \mu, T)$ and for each $f \in L^p(X, \beta, \mu)$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n}x),$$

exists μ almost everywhere.

Let $\psi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_k x^k$, with α_k irrational. In the sequel call $(k_j)_{j\geq 1}$ good if it is Hartman uniform distributed on $\mathbb{Z}$ and L^∞-good universal.

Theorem 1. *The sequence $(\psi(k_j))_{j\geq 1}$ is uniform distribution modulo one if $(k_j)_{j\geq 1}$ is good.*

This theorem has a number of implications.

We say a set of natural numbers S has positive Banach density $B(S)$ if there exists a sequence of intervals $(I_k)_{k=1}^\infty$ in $\mathbb{N}$ with $I_k = [a_k, b_k] \cap \mathbb{N}$ and $b_k - a_k$ tending to infinity as k tends to infinity such that

$$\lim_{k\to\infty} \frac{|S \cap I_k|}{|I_k|} = B(S)$$

and for any other sequence of intervals $(I'_k)_{k=1}^\infty$ in $\mathbb{N}$ such that $|I'_k|$ tends to infinity as k tends to infinity we have

$$\limsup_{k\to\infty} \frac{|\, S \cap I'_k \,|}{|\, I'_k \,|} \leq B(S).$$

We say a sequence $\mathbf{k}$ of positive integers is a set of intersectivity if given any set of natural numbers S of positive Banach density $B(S)$ there exists an integer k in $\mathbf{k}$ such that we can find s_1 and s_2 both in S satisfying

$$k = s_1 - s_2.$$

We say a sequence of natural numbers $\mathbf{k} = (k_j)_{j=1}^\infty$ is a sequence of Poincaré recurrence if given any measure preserving dynamical system on a probability space (X, β, μ, T) and any set A in β of positive measure there exists an element k of $\mathbf{k}$ such that

$$\mu(A \cap T^{-k}A) > 0.$$

Theorem 2. *Suppose ϕ is a polynomial mapping the natural numbers to themselves, such that $\phi(0) = 0$. Then the sequence $(\phi(k_j))_{j\geq 1}$ is Poincaré recurrent if $(k_j)_{j\geq 1}$ is good.*

Theorem 3. *Suppose ϕ is a polynomial mapping the natural numbers to themselves, such that $\phi(0) = 0$. Then the sequence $(\phi(k_j))_{j\geq 1}$ is intersective if $(k_j)_{j\geq 1}$ is good.*

For $f \in L^p(X, \mathscr{B}, \mu)$, with $p \geq 1$ set $\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$.

Theorem 4. *Suppose that ϕ is a polynomial mapping the natural numbers to themselves, such that $\phi(0) = 0$ and that $(k_j)_{j\geq 1}$ is good. Then for the dynamical system $(X, \mathscr{B}, \mu, T)$ and $f \in L^p(X, \mathscr{B}, \mu)$, for $p \geq 1$. Then there exists $\overline{f}$ such that*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^{\phi(k_n)}x) - \overline{f} \right\|_p = 0.$$

It is natural to ask if Theorem 1.5 is true almost everywhere.

We say a set S contained in $\mathbb{Z}$ is Glasner if for every infinite set A contained in $\mathbb{T}$ and $\varepsilon > 0$, some dilation $nA = \{nx : x \in A\}$ is ε dense (that is nA intersects every interval of length ε). This definition is motivated by the fact that in 1979 S. Glasner showed that given an infinite set $A \subset \mathbb{T}$ there exists a natural number n such that nA is ε dense in $\mathbb{T}$. We have

Theorem 0.6 : *Suppose $(k_n)_{n\geq 1}$ good. Let ψ be a polynomial of degree $k \geq 1$ mapping the natural numbers to themselves. Suppose $\delta > 0$. Then there exists $\varepsilon(\psi, \delta) > 0$ such that if $0 < \varepsilon < \varepsilon(\psi, \delta)$ given any set X contained in $\mathbb{T}$ of cardinality s with*

$$s > (\frac{1}{\varepsilon})^{2k+\delta}$$

has an ε dense dilation of the form $\psi(k_n)X$, for some integer n.

1 List of known good sequences

1. *The natural numbers:* The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.
2. *Condition* H*:* Sequences $(a_n)_{n=1}^{\infty}$ that are both L^p-good universal and Hartman uniformly distributed can be constructed as follows. Denote by $[x]$ the integer part of a real number x. Set $a_n = [\tau(n)]$ $(n = 1, 2, \ldots)$, where $\tau : [1, \infty) \to [1, \infty)$ is a differentiable function whose derivative increases with its argument. Let Ω_m denote the cardinality of the set $\{n \colon a_n \leq m\}$, and suppose, for some function $\varphi : [1, \infty) \to [1, \infty)$ increasing to infinity as its argument does, that we set

$$\rho(m) = \sup_{\{z\} \in \left[\frac{1}{\varphi(m)}, \frac{1}{2}\right)} \left| \sum_{n \colon a_n \leq m} e(za_n) \right|,$$

where $e(x) = e^{2\pi i x}$ for a real x. Suppose also, for some decreasing function ρ : $[1, \infty) \to [1, \infty)$ and some positive constant $\omega > 0$, that

$$\frac{\rho(m)+\Omega_{[\varphi(m)]}+\frac{m}{\varphi(m)}}{\Omega_m} \le \omega\rho(m).$$

Then if we have

$$\sum_{n=1}^{\infty} \rho(\theta^n) < \infty$$

for all $\theta > 0$, we say that $(a_n)_{n=1}^{\infty}$ satisfies condition H.
Sequences satisfying condition H are known to be both Hartman uniformly distributed and L^p-good universal. Specific sequences of integers that satisfy condition H include $a_n = [\tau(n)]$ $(n = 1,2,\dots)$ where:

I. $\tau(n) = n^{\gamma}$ if $\gamma > 1$ and $\gamma \notin \mathbb{N}$.
II. $\tau(n) = e^{\log^{\gamma} n}$ for $\gamma \in (1, \frac{3}{2})$.
III. $\tau(n) = b_k n^k + \cdots + b_1 n + b_0$ for $b_k, \dots, b_1$ not all rational multiples of the same real number.
IV. *Hardy fields:* By a Hardy field, we mean a closed subfield (under differentiation) of the ring of germs at $+\infty$ of continuous real-valued functions with addition and multiplication taken to be pointwise. Let $\mathscr{H}$ denote the union of all Hardy fields. Conditions for $(a_n)_{n=1}^{\infty} = ([\eta(n)])_{n=1}^{\infty}$, where $\eta \in \mathscr{H}$ to satisfy condition H are given by the hypotheses of Theorems 3.4, 3.5 and 3.8. in [**?**].

3. *A random example:* Suppose that $S = (b_n)_{n=1}^{\infty}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function χ_S, we may view it as a point in $\Lambda = \{0,1\}^{\mathbb{N}}$, the set of maps from $\mathbb{N}$ to $\{0,1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$, where, for each natural number n, we have $X_n = \{0,1\}$ and specify the probability ν_n on X_n by $\nu_n(\{1\}) = \omega_n$ with $0 \le \omega_n \le 1$ and $\nu_n(\{0\}) = 1 - \omega_n$ such that $\lim_{n\to\infty} \omega_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\nu = \prod_{n=1}^{\infty} \nu_n$. The underlying σ-algebra $\mathscr{A}$ is that generated by the cylinders
$$\{(\Delta_n)_{n=1}^{\infty} \in \Lambda : \Delta_{n_1} = \alpha_{n_1}, \dots, \Delta_{n_k} = \alpha_{n_k}\}$$
for all possible choices of $n_1, \dots, n_k$ and $\alpha_{n_1}, \dots, \alpha_{n_k}$. Then almost every point $(a_n)_{n=1}^{\infty}$ in Λ, with respect to the measure ν, is Hartman uniformly distributed. See Proposition 8.2 (i) in [**?**] for the details of this. Again Hartman uniformly distributed sequences are called ergodic sequences in this paper.
4. *Block sequences:* Suppose that $(a_n)_{n=1}^{\infty} = \bigcup_{n=1}^{\infty} [d_n, e_n]$ is ordered by absolute value for disjoint $([d_n, e_n])_{n=1}^{\infty}$ with $d_{n-1} = O(e_n)$ as n tends to infinity. Note that this allows the possibility that $(a_n)_{n=1}^{\infty}$ is zero density. This example is an immediate consequence of Tempelman's semigroup ergodic theorem. See page 218 of [T]. Being a group average ergodic theorem this pointwise limit must be invariant. which ensures that the block sequence must be Hartman uniformly distributed. The proof of this, which we don't need in this paper and is hence forgone is a simple exercise in spectral theory.
5. *Random perturbation of good sequences:* Suppose that $(a_n)_{n=1}^{\infty}$ is an L^p-good universal sequence which is also Hartman uniformly distributed. Let $\theta = (\theta_n)_{n=1}^{\infty}$ be

a sequence of $\mathbb{N}$-valued independent, identically distributed random variables with basic probability space $(Y, \mathscr{A}, \mathscr{P})$, and a $\mathscr{P}$-complete σ-field $\mathscr{A}$. Let $\mathbb{E}$ denote expectation with respect to the basic probability space $(Y, \mathscr{A}, \mathscr{P})$. Assume that there exist $0 < \gamma < 1$ and $\beta > 1/\gamma$ such that

$$a_n = O(e^{n^\gamma}) \qquad \text{and} \qquad \mathbb{E} \log_+^\beta |\theta_1| < \infty.$$

Then $(a_n + \theta_n(\omega))_{n=1}^\infty$ is both L^p-good universal and Hartman uniformly distributed [NW].

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Fractals and space-filling curves viewed by a new numerical computation system using infinities and infinitesimals

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Abstract In the first part of the talk we briefly introduce a new computational system based on infinite and infinitesimal quantities and we show how to work *numerically* with it in a very handy way. Then we describe the "state of the art" giving a quick overview of how this new computational methodology has already been successfully applied in a number of theoretical and computational research areas: if time allows we will choose and explain briefly some examples.

In the second part we focus on some space-filling curves and d-dimensional fractals, and we present a double level of treatment: at "finite" and at "infinite". In the first setting we analyze deeply the generating sequences of the considered fractals and, in particular, we study the characteristics of various d-dimensional generalizations; in the second setting we use the new computational machinery to investigate their behavior at "infinity" making a precise and detailed analysis of such geometric fractal objects, in a similar way as for ordinary finite shapes and familiar geometric entities.

Lastly, adopting the new point of view, we show as each fractal or space-filling curve considered previously, gives rise to a large family of infinitely many self-similar objects.

1 Introduction

In this paper we want to give, rather than a detailed treatment of the topics of the talk, some guidelines to introduce the reader to a new computational system, to show him some of the several possible applications in different fields and, above of all, to provide some references and a discussed bibliography for further, more complete readings.

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In particular the organization of the paper is as follows: in Section 2 we will give the *rudimenta* of the mentioned new numerical system recently introduced by Ya. D. Sergeyev; in its essence, it represents an alternative to the classical view of the set of natural numbers $\mathbb{N}$ (and of the larger sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$), and it is simple to use as we ordinarily do with $\mathbb{N}$. Sergeyev's system is based on infinite and infinitesimal quantities like Robinson's approach to non-standard analysis (see [1]), but it is different from the last, especially for the great versatility of use and its strong computational character. For detailed introduction surveys to Sergeyev's system, which show how one is able to work *numerically* with infinities and infinitesimals in a handy way as with finite numbers, the reader can see [2, 3, 4, 5] and the book [6], written in a popular way and easily understandable also by a high school student.
This computational methodology has already been successfully applied in a number of theoretical and computational research areas as optimization and numerical differentiation (see [7, 8, 9, 10]), cellular automata (see [11, 12]), Euclidean and hyperbolic geometry (see [13, 14]), percolation (see [15, 16, 17]), fractals (see [18, 19, 20, 21, 22, 17]), infinite series and the Riemann zeta function (see [23, 24, 25]), the first Hilbert problem, Turing machines and supertasks (see [26, 27, 28, 29]), numerical solution of ordinary differential equations (see [30, 31, 32, 33]), etc.

In the talk at the conference we will focus on some space-filling curves and d-dimensional fractals, referring to some works of the author ([34], [35], [18] and [36]), but here, for problem of space, we give some details only for one of them: the d-dimensional Sierpinski tetrahedron.
We want to point out that, for each of the mentioned objects of the talk and for the d-dimensional Sierpinski tetrahedron in the following, we present a double level of treatment: at "finite" and at "infinite". In the first setting we analyze deeply the generating sequences which approximate the considered fractals, first in dimension 2 and 3 to have viewable pictures of them, then we study accurately the characteristics of various d-dimensional generalizations. In the second setting, i.e. at infinity, almost all measures related to such fractals (or space-filling curves) approaches 0 or $+\infty$, and we can not say any more by traditional analysis. For instance, these are totally indistinguishable from classical mathematics, even though they have, per se, a completely different meaning. So, using the new computational machinery we referred above to investigate the behavior at "infinity", we obtain, in each case, a whole family of values expressed in the new system and, by defining equivalence relations between them, or looking at their ratios written in power series in the new system, or employing other methods, we are able to make a precise and detailed analysis of the behavior of such geometric fractal objects, in a similar way as for ordinary finite shapes and familiar geometric entities.
During the discussion of the topics we will suggest various further directions of research and how it is possible to generalize the results communicated in the talk. We in fact believe that papers using the new approach will be more and more numerous in the next years for two main reasons: the great versatility which allows to apply the new computational system to a really large variety of subjects, and the fact that it has been introduced so recently that many of the possible application areas are still totally unexplored.

2 The new computational system: a brief description

In this section we want to give some of the strictly indispensable notions and some examples that allow us to work with the new computational system. The main idea followed by Ya. D. Sergeyev in introducing his new system, is to find a way to measure infinite and infinitesimal quantities in an easy, handy and useful way for calculations and applications. Hence, in the early 2000s, he proposed a system based on two different fundamental units of measure: the familiar natural number 1 for finite quantities and another that had the same functions as 1, but for infinite quantities: it is the *grossone*, expressed by the new numeral $\textcircled{1}$ and formally introduced by the *Infinite Unit Axiom* (*IUA*)(see, for example, [6, 2, 37]). The new infinite number $\textcircled{1}$ has several different properties and characterizations: for instance, it represent the number of elements in the set of natural numbers $\mathbb{N}$, but it is necessary to emphasize immediately that $\textcircled{1}$ is not equal to the Cantor's cardinality ω, $\aleph_0$, $\aleph_1$, etc., or to other yet known entity or symbols. An important property we need for our employ of the new computational method in this paper, is that $\textcircled{1}$ is *the last element* in the set of natural numbers of the new system, and moreover, every sequential process can not have more than $\textcircled{1}$ steps.
As regards explicit computations, the numeral $\textcircled{1}$ allows one to express and to order a huge variety of numerals representing different infinities and infinitesimals; for instance, $\textcircled{1}$, $\frac{\textcircled{1}}{3}$, $2^{\textcircled{1}}$, $\textcircled{1}-10$, $\textcircled{1}^{5.2}$ are examples of infinities and $\textcircled{1}^{-1}$, $\frac{3}{\textcircled{1}}$, $\textcircled{1}^{-10}$, $-\textcircled{1}^{-5.2}$ are examples of infinitesimals.
The way to do computations inside the new system is easy and intuitive: for example, in many simple cases it coincides with the one in the field of rational functions $\mathbb{R}[x]$, where $\textcircled{1}$ takes the place of the unknown variable x. Even though the variety of possible operations, expressions and computations in the new system, is much larger than the one in the field $\mathbb{R}[x]$, we remark that all they are quite intuitive and easy to use; in the following we give some examples of computations and main relations:

- $0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1}^0 = 1, \quad 0^{\textcircled{1}} = 0, \quad 1^{\textcircled{1}} = 1, \quad \textcircled{1} - \textcircled{1} = 0, \quad \dfrac{\textcircled{1}}{\textcircled{1}} = 1;$
- $0 \cdot \textcircled{1}^{-1} = \textcircled{1}^{-1} \cdot 0 = 0, \quad \left(\textcircled{1}^{-1}\right)^0 = 1, \quad \textcircled{1}^{-1} - \textcircled{1}^{-1} = 0, \quad \dfrac{\textcircled{1}^{-1}}{\textcircled{1}^{-1}} = 1;$
- $\textcircled{1} \cdot \textcircled{1}^{-1} = 1, \quad \textcircled{1}^{-1} > 0, \quad \textcircled{1}^2 \cdot \textcircled{1}^{-3.4} = \textcircled{1}^{-1.4}, \quad \dfrac{\textcircled{1}^2}{\textcircled{1}^{-3.4}} = \textcircled{1}^{5.4};$
- $3 \cdot \left(\textcircled{1} - 4\right) = 3\textcircled{1} - 12, \quad \left(\dfrac{3}{2}\textcircled{1}^2 + 3\right) \cdot \left(1 - 2\textcircled{1}^{-2}\right) = \dfrac{3}{2}\textcircled{1}^2 - 6\textcircled{1}^{-2};$
- $\dfrac{\textcircled{1}^{2.1} + 2 - 3\,\textcircled{1}^{-3.4}}{\textcircled{1}^{1.1}} + 2\textcircled{1} = 3\textcircled{1} + 2\textcircled{1}^{-1.1} - 3\textcircled{1}^{-4.5};$
- $\left(2^{\textcircled{1}}\right)^3 = 2^{3\textcircled{1}}, \qquad \left(2 \cdot 3^{\textcircled{1}} - 5\,\textcircled{1}\right) \cdot \left(-4 \cdot 2^{\textcircled{1}}\right) = -8 \cdot 2^{6\textcircled{1}} + 20 \cdot \textcircled{1} \cdot 2^{\textcircled{1}}\ ;$

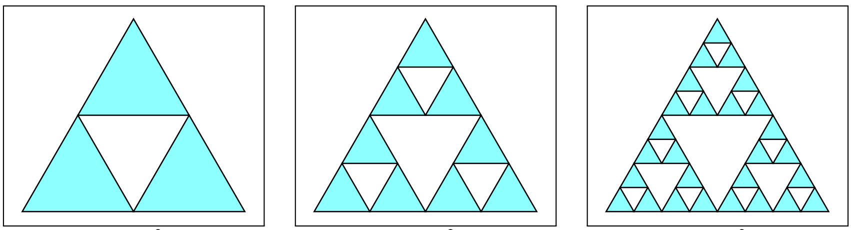
- $\left(2^{①} - 3 + 3 \cdot 2^{-2①}\right) \cdot \left(2^{2①} - 2 \cdot 2^{-①}\right) = 2^{3①} - 3 \cdot 2^{2①} + 1 + 6 \cdot 2^{-①} - 6 \cdot 2^{-3①}$;
- $0 < \frac{2\pi}{①^{3.5}} < \frac{1}{①^2} < 1 < \frac{①}{6} - 4 < \frac{①}{6} < ① < ① + 3 < ①^2 - 4 < 2^{①} - 3①$.

We think that the previous examples are enough to give an idea sufficiently clear of how to do basic computations in the new system. We point out that it is also possible to use an on-line calculator called the *Infinity Computer* and available at [38]. For more details about the system, the way to do computations, and some applications, the reader can see, for example, [2, 3, 4, 5, 6, 39, 40], or also several other papers and information available at the home page of Ya. D. Sergeyev, at `http://wwwinfo.deis.unical.it/yaro/`

Since the conference *Numeration 2017* focuses also on mathematical logic, we conclude this section by informing that, for a formal logic point of view on the theory of grossone, the reader can also see, for example, the paper of G. Lolli [41]. Moreover, in [42] it is possible to find a brief survey on the history of infinities and infinitesimals in mathematics.

3 The Sierpinski d-dimensional tetrahedron

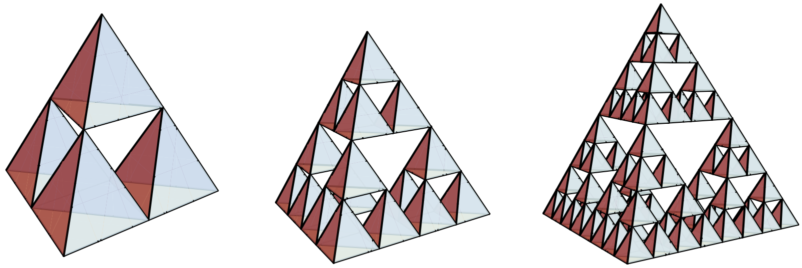
The Sierpinski gasket, also called the Sierpinski triangle and denoted by Δ^2, is one of the most known and very popular fractal set; it has the exterior shape of an equilateral triangle, and is subdivided recursively into smaller similar triangles, from which the central ones are removed at each step as in Fig. 1.



(a) Δ_1^2 (b) Δ_2^2 (c) Δ_3^2

Fig. 1: The first three steps in the construction of the Sierpinski gasket Δ^2, starting from Δ_0, the regular triangle of unitary side.

The equivalent 3-dimensional object of the Sierpinski gasket is called the *Sierpinski tetrahedron* and denoted by Δ^3: the first steps in its construction are shown in Fig. 2. But clearly, it is also possible to construct, in the same way, a d-dimensional fractal object in every dimension $d \geqslant 2$: this is what we will do, with some details, in the



(a) Δ_1^3 (b) Δ_2^3 (c) Δ_3^3

Fig. 2: The first three steps in the construction of the 3-dimensional Sierpinski tetrahedron, starting from the regular tetrahedron Δ_0^3 of side one.

following.
First of all we recall that the generalization of a triangle or tetrahedron to arbitrary dimension $d \geqslant 0$, is what is called a *d-simplex* or, less commonly, a *d-dimensional tetrahedron*. More precisely, a d-simplex is a d-dimensional polytope which is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^D$, $D \geqslant d+1$ and, of course, it is called *regular* if all its edges have the same length.
The use of simplexes is widespread in many areas of mathematics like algebraic geometry, algebraic topology and especially in singular homology; but we advise the reader that what in literature is called the *standard d-simplex* is different from our notion of *unitary d-simplex* or *unitary d-tetrahedron*, denoted by Δ_0^d and widely used in the following. In fact, whilst the first has edge length $\sqrt{2}$ because it is the convex hull of the standard basis $(1,0,\dots,1),\dots,(0,\dots,0,1)$ of $\mathbb{R}^{d+1}$, that is

$$\left\{ (x_0,\dots,x_d) \in \mathbb{R}^{d+1} \;\middle|\; \sum_{i=0}^{d} x_i = 1 \text{ and } x_i \geqslant 0 \text{ for all } i = 0,\dots,d \right\},$$

the second is a d-tetrahedron whose edges have all unitary length. We recall that the d-volume of a regular d-simplex is equal to

$$\frac{\sqrt{d+1}}{d!\sqrt{2^d}} \cdot l^d, \tag{1}$$

where l is the length of its edges. Moreover, if $0 \leqslant k \leqslant d$, every k-dimensional face (briefly k-face) of a d-simplex is a k-simplex itself, and since any $k+1$ points from the $d+1$ vertices of a d-simplex identify uniquely a k-face, then, the number of the k-faces of a d-simplex is given by the binomial coefficient

$$\binom{d+1}{k+1}. \tag{2}$$

Now let $d \geqslant 2$ be a fixed integer; to build a fractal Δ^d which generalizes the Sierpinski gasket Δ^2 and the Sierpinski tetrahedron Δ^3, we define a sequence $\{\Delta_n^d\}_n$. Note that Δ_0^d is yet defined and Δ_1^d is the union of $d+1$ regular tetrahedra of side $1/2$, each one built in a corner of Δ_0^d. Then we can adopt two equivalent inductive constructions at the generic step $n \geqslant 2$ to obtain Δ_n^d: the first one consists to repeat a copy of Δ_1^d, scaled by $l_{n-1} := (1/2)^{n-1}$, in each small tetrahedron of side l_{n-1} constituting Δ_{n-1}^d, instead the second construction consists to replicate a copy of Δ_{n-1}^d, scaled by $l_1 = 1/2$, in each of the $d+1$ tetrahedrons of side l_1 composing Δ_1^d.
Finally, for every $d \geqslant 2$, the *d-dimensional Sierpinski tetrahedron* Δ^d is the limit $\lim_{n\to+\infty} \Delta_n^d$, and note that it is also equal to $\bigcap_{n\in\mathbb{N}} \Delta_n^d$.
Now we want to attach to each of such fractal Δ^d, some sequences of real numbers $\{v_n^{d,k}\}_n$ which give a k-dimensional valuation of the elements of the generating sequence $\{\Delta_n^d\}_n$. More precisely we pose the following

Definition 1. For all integers $d \geqslant 2$ and $n \geqslant 0$, let $v_n^{d,d}$ be the d-volume of Δ_n^d. Moreover, if $0 \leqslant k < d$, let $v_n^{d,k}$ be the sum of the k-volumes of the *k-dimensional elements* (briefly *k-elements*) lying on the $(d-1)$-dimensional boundary surface of Δ_n^d.

The next proposition gives a general expression for $v_n^{d,k}$, but first note that, if $N_n^{(d)}$ is the number of tetrahedra of side $l_n = \left(\frac{1}{2}\right)^n$ which make up Δ_n^d, then

$$N_n^{(d)} = (d+1)^n, \tag{3}$$

for every $n \geqslant 0$.

Proposition 1. *For all $n \geqslant 0$ and $d \geqslant 2$, we have*

$$v_n^{d,k} = \begin{cases} \dfrac{\sqrt{k+1}}{k!\sqrt{2^k}} \cdot \dbinom{d+1}{k+1} \cdot \left(\dfrac{d+1}{2^k}\right)^n & \text{if } \ 1 \leqslant k \leqslant d, \\[2ex] \dfrac{(d+1)^{n+1} + d + 1}{2} & \text{if } \ k = 0. \end{cases} \tag{4}$$

For the proofs of the results in this and the next section we refer to [34].
For every $d \geqslant 2$ and $0 \leqslant k \leqslant d$, we pose

$$v_\infty^{d,k} := \lim_{n\to+\infty} v_n^{d,k} \tag{5}$$

and denote by $\dim(\Delta^d)$ the fractal dimension of Δ^d. For comprehensive references about the general theory of the dimension of a fractal, the reader can see [43] or [44]; in our particular case, it is simple to prove that

$$\dim(\Delta^d) = \frac{\ln(d+1)}{\ln 2} = \log_2(d+1), \tag{6}$$

Table 1: The values of the limit 5 in dependence from d and k, until the dimension $d = 8$ and for $d = 15, 32, 63, 127$. It is obvious that in the missing lines, for example $d \in \{16, \ldots, 30\}$ or $d \in \{32, \ldots, 62\}$, the values of $v_\infty^{d,k}$ until $k = 9$, are the same as for $d = 15$ and $d = 31$, respectively.

d	$\dim(\Delta^d)$	$v_\infty^{d,0}$	$v_\infty^{d,1}$	$v_\infty^{d,2}$	$v_\infty^{d,3}$	$v_\infty^{d,4}$	$v_\infty^{d,5}$	$v_\infty^{d,6}$	$v_\infty^{d,7}$	$v_\infty^{d,8}$	$\ldots$
2	$\log_2 3$	$+\infty$	$+\infty$	0	-	-	-	-	-	-	$\ldots$
3	2	$+\infty$	$+\infty$	$\sqrt{3}$	0	-	-	-	-	-	$\ldots$
4	$\log_2 5$	$+\infty$	$+\infty$	$+\infty$	0	0	-	-	-	-	$\ldots$
5	$\log_2 6$	$+\infty$	$+\infty$	$+\infty$	0	0	0	-	-	-	$\ldots$
6	$\log_2 7$	$+\infty$	$+\infty$	$+\infty$	0	0	0	0	-	-	$\ldots$
7	3	$+\infty$	$+\infty$	$+\infty$	$\frac{35}{3\sqrt{2}}$	0	0	0	0	-	$\ldots$
8	$\log_2 9$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	0	0	0	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
15	4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{91\sqrt{5}}{2}$	0	0	0	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
31	5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{18879\sqrt{3}}{10}$	0	0	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
63	6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{3235501\sqrt{7}}{30}$	0	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
127	7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{5957094385}{84}$	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

for every $d \geqslant 2$ (see, for example, [45]). In Table 1 there are collected some values of $v_\infty^{d,k}$ and $\dim(\Delta^d)$. From Proposition 1, it is trivial that the sequence $v_n^{d,k}$ converges if and only if $d \leqslant 2^k - 1$, and converges to a nonzero value if and only if $k \geqslant 2$ and $d = 2^k - 1$; in the last case the limit value is

$$v_\infty^{2^k-1,k} = \frac{\sqrt{k+1}}{k!\sqrt{2^k}} \cdot \binom{2^k}{k+1}. \tag{7}$$

It is also clear that the number of occurrences of $+\infty$ in the d-th row of Table 1 is given by $\lceil \log_2(1+d) \rceil$ for all $d \geqslant 2$ (cf. (6)), where $\lceil x \rceil$ is the ceiling of $x \in \mathbb{R}$.

4 The exact measures of the d-dimensional Sierpinski tetrahedron through infinite and infinitesimal computation

In the previous section we computed $v_n^{d,k}$ for finite values of n and we obtained that $v_\infty^{d,k}$ is zero, or a positive number equal to (7), or $+\infty$: that is all what we can say from classical analysis. It is also quite obvious that the various zeros and infinities appearing in Table 1 have not the same meaning, because they arise from a computation of a k-dimensional volume related to a d-dimensional object where k and d are different for each entry. For example, $v_\infty^{2,0}$, representing the infinite grow of the number of vertices of Δ_n^2, has a completely different meaning from $v_\infty^{2,1}$, representing the grow of its perimeter, or from $v_\infty^{3,1}$, $v_\infty^{4,2}$ which comes from a 2-dimensional area, or from $v_\infty^{8,3}$ which is related to a 3-volume in a 8-space, etc. But traditional analysis, having got no appropriate language and notations, is not able to see these differences and such infinite quantities are all equal, and denoted by the same symbol $+\infty$. We can also use the same argument for the zeros of Table 1, and hence it is clear that traditional computing systems are inadequate to treat many phenomena occurring in mathematics, like those we are speaking about.

In this section we show as, adopting the new computing system for infinities and infinitesimals quantities based on the grossone ①, we can give a rich description of the behavior at infinity of the previous constructive processes as, and maybe more than, the one at finite.
For example, if we execute ① steps in the construction of Δ^d, we obtain the following values for the related k-volumes

$$v_{①}^{d,k} = \frac{\sqrt{k+1}}{k!\sqrt{2^k}}\binom{d+1}{k+1}\cdot\left(\frac{d+1}{2^k}\right)^{①} \tag{8}$$

in the case $1 \leqslant k \leqslant d$, and

$$v_{①}^{d,0} = \frac{d+1}{2}\cdot(d+1)^{①} + \frac{d+1}{2} \tag{9}$$

for $k = 0$ (see Proposition 1).

Definition 2. Let α and β be any quantities of the new computational system.

(i) α and β are said *of the same order* (in symbols $\text{ord}(\alpha) = \text{ord}(\beta)$, or $\alpha \sim_{\text{ord}} \beta$) if their quotient is finite but not infinitesimal, that is, more precisely, if there exist two positive real number $m, M \in \mathbb{R}$ such that $0 < m < \alpha/\beta < M$.
In case α, β are both infinite or infinitesimal quantities, they are also called *infinities of the same order* or *infinitesimals of the same order*, respectively.
(ii) α and β are said *equivalent* (in symbols $\alpha \sim_{eq} \beta$, or simply $\alpha \sim \beta$) if their quotient is 1 up to infinitesimals. In other words, if for all $\varepsilon \in \mathbb{R}^+$ we have $1-\varepsilon < \alpha/\beta < 1+\varepsilon$.
As before, in case α, β are infinite or infinitesimal quantities, they are also called *equivalent infinities* or *equivalent infinitesimals*, respectively.

It is trivial that both the relations introduced in Definition 2 (i) and (ii) are equivalence relations, so for example, we can ask about the equivalence class of $v_{①}^{d,k}$, using the relation $\sim_{\text{ord}}$ or $\sim_{eq}$.
Recall that at the beginning of the section we said that the symbols $+\infty$ and 0 in Table 1 are not distinguishable for traditional mathematics; instead, Corollary 0.1, which is an immediate consequence of Theorem 1 (ii), states that, in the analogue new version

of Table 1, the elements $v_{①}^{d,k}$ are all distinct. But Theorem 1 establishes more: our purpose is to give information about the partition of the following sets

$$V^d := \left\{ v_{①}^{d,k} \,\middle|\, 0 \leqslant k \leqslant d \right\}, \quad W^d := \bigcup_{i=2}^{d} V^i \quad \text{for all } d \geqslant 2, \qquad \text{and} \qquad W := \bigcup_{d \geqslant 2} V^d, \tag{10}$$

in equivalence classes. We denote the equivalence class of $v_{①}^{d,k}$ in W, with respect the relation $\sim_{\text{ord}}$ or $\sim_{eq}$, by $\left[v_{①}^{d,k}\right]_{\text{ord}}$ and $\left[v_{①}^{d,k}\right]_{eq}$, respectively.
Before to state Theorem 1, we write in the following lemma an important consequence of the equations (8) and (9).

Lemma 1. *For all $d \geqslant 2$ and $0 \leqslant k \leqslant d$ we have* $\operatorname{ord}\left(v_{①}^{d,k}\right) = \operatorname{ord}\left(\left(\frac{d+1}{2^k}\right)^{①}\right)$.
Moreover

$$v_{①}^{t,h} \sim_{\text{ord}} v_{①}^{d,k} \qquad \textit{if and only if} \qquad t+1 = 2^{h-k} \cdot (d+1). \tag{11}$$

The proof of the previous lemma is an easy check; now we are ready to state the main theorem of the section.

Theorem 1.

(i) Let $d \geqslant 2$ and denote by ν_d the number of equivalence classes in the set W^d with respect the equivalence relation $\sim_{\text{ord}}$. Then

$$\nu_d = \begin{cases} 3 & \text{if } d = 2, \\ \dfrac{3d^2+9d+6}{8} + \dfrac{(-1)^d}{4} \cdot \left\lceil \dfrac{d+1}{2} \right\rceil & \text{if } d \geqslant 3. \end{cases} \tag{12}$$

Moreover, a system of minimal representatives of the classes in W is

$$\mathscr{R}_W := \bigcup \left\{ V^d \mid d \text{ even} \geqslant 2 \text{ or } d = 3 \right\} \cup \left\{ v_{①}^{t,h} \,\middle|\, t \text{ odd} \geqslant 5 \text{ and } h = 0 \text{ or } \frac{t+3}{2} \leqslant h \leqslant t \right\}, \tag{13}$$

and for every $v_{①}^{t,h} \in \mathscr{R}_W$, its equivalence class is

$$\left[v_{①}^{t,h}\right]_{\text{ord}} = \left\{ v_{①}^{2^{j-h}(t+1)-1,\, j} \,\middle|\, j \in \mathbb{N}_0,\ j \geqslant h \right\}. \tag{14}$$

(ii) Each equivalence class in W under the relation $\sim_{eq}$ consists of a single element.

For the proof of the theorem the interesting reader can see [34]. We remark that a trivial consequence of part (ii) of the preceding theorem, is the following

Corollary 0.1. *The elements $v_{①}^{d,k}$ are all distinct for every $d \geqslant 2$ and $0 \leqslant k \leqslant d$.*

Note moreover that W is clearly a totally ordered set, but the explicit relation is not obvious at first sight; an immediate consequence of Lemma 1 and of Theorem 1 (ii), is that

$$v_{①}^{t,h} > v_{①}^{d,k} \qquad \text{if and only if} \qquad \begin{cases} t > 2^{h-k} \cdot (d+1) - 1 & \text{or} \\ t = 2^{h-k} \cdot (d+1) - 1 & \text{and} \quad h > k \end{cases},$$

where "$>$" has the obvious meaning between the elements of W.

Remark 0.1. To prove that all the elements $v_{①}^{d,k}$ of the new computational system are effectively distinct, is not a trivial issue; for instance, when $k, h \geqslant 1$, it is equivalent to show that the following system

$$\begin{cases} \dfrac{\sqrt{k+1}}{k!\sqrt{2^k}} \cdot \dbinom{d+1}{k+1} = \dfrac{\sqrt{h+1}}{h!\sqrt{2^h}} \cdot \dbinom{t+1}{h+1} \\ \dfrac{d+1}{2^k} = \dfrac{t+1}{2^h} \end{cases} \tag{15}$$

has no integer solutions $2 \leqslant d < t$, $1 \leqslant k \leqslant d$ and $1 \leqslant h \leqslant t$. But to prove the nonexistence of such solutions of a Diophantine system as (15), is a non trivial problem; for example, by using the most powerful computer algebra systems or scientific computational software available today, like *Mathematica*® 11.0 by Wolfram Research Inc., or many others, it is not possible to obtain any answer except for very small values of t cause the complexity of (15). Instead, as consequence of our results, we can give a full response to this problem and to similar ones arising from the other cases. In particular, Corollary 0.1, which follows from Theorem 1 (ii), guarantees that all the numbers in W are effectively distinct and a Diophantine systems like (15) has no non-trivial solutions.

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On relations between systems of numerations and fractal properties of subsets of non-normal numbers

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The report is devoted to analysis of the dependence of fractal and topological properties of subsets of non-normal numbers on systems of numerations.

Till 1994 the set of numbers, which are non-normal w.r.t. s-adic expansion, was considered as a ”rather small” one in the sense of Lebesgue measure as well as in the sense of the Hausdorff-Besicovitch dimension. After the proof of the superfractality of sets of non-normal and essentially non-normal numbers for s-adic and some other expansions [2] and construction of such systems of representation for which the set of essentially non-normal numbers was of full Lebesgue measure, the conjecture about the superfractality and topological massivity of the set of essentially non-normal numbers (independently of the choice of a system of numeration) became dominating.

Probabilistic approach is shown to be very useful to prove the superfractality of the set of essentially non-normal numbers for Q-expansions [3], Q_∞-expansions [4] and Q^*-expansions (under additional assumptions $\inf\limits_{i,k} q_{ik} > 0$ on the matrix Q^*).

We shall answer the following problems, which are well motivated by the above arguments:

1. Is the condition $\inf\limits_{i,k} q_{ik} > 0$ necessary for the superfractality of the set of Q^*-essential non-normal numbers?

2. Are there systems of numeration, for which the corresponding set of essentially non-normal numbers is not a superfractal?

To answer the first question we show, in particular, that there are Q^*-expansions of real numbers, for which the corresponding set of essentially non-normal numbers has zero Hausdorff-Besicovitch dimension: $dim_H(L(Q^*)) = 0$.

Theorem 1 ([1]). *Let*

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$$Q^* = \|q_{ik}\|, \;\; q_{2k} = \frac{1}{(k+1)^{k+1}}, \;\; q_{0k} = q_{1k} = \frac{1-q_{2k}}{2}, \;\; i \in \{0,1,2\}.$$

Then

$$\dim_H(L(Q^*)) = 0.$$

So, we also refuted the conjecture about the superfractality of the set of essentially non-normal numbers independently of the choice of a system of numeration.

Moreover, we prove the existence of such Q^*-expansions of real numbers, for which even the whole set $D(Q^*)$ of all non-normal numbers has zero Hausdorff-Besicovitch dimension: $dim_H(D(Q^*)) = 0$.

Theorem 2 ([1]). *Let $Q^* = \|q_{ik}\|$, $i \in \{0,1,2,\dots,s-1\}$.*
Let $A = \{n : n = 10^k, k \in N\}$, $B = \{n : n \neq 10^k, k \in N, n \in N\}$, and let

$$q_{1k} = q_{2k} = \dots = q_{s-1,k} = \begin{cases} \frac{1}{(s-1)(k+1)^{k+1}}, & \text{if} \quad k \in B; \\ \frac{1}{s}, & \text{if} \quad k \in A. \end{cases}$$

$$q_{0k} = \begin{cases} 1 - \frac{1}{(k+1)^{k+1}}, & \text{if} \quad k \in B; \\ \frac{1}{s}, & \text{if} \quad k \in A. \end{cases}$$

Then

$$\dim_H(D(Q^*)) = 0.$$

We also give the sufficient conditions that the set $L(Q^*)$ of essentially non-normal numbers is of full and zero Hausdorff–Besicovitch dimension.

Theorem 3 ([1]). *If all entries of a matrix Q^* are bounded from below by some positive constant q_*, i.e.,*

$$\inf_{i,k} q_{ik} \geq q_* > 0,$$

then the set $L(Q^*)$ of essentially non-normal numbers is of full Hausdorff dimension, i.e.,

$$\dim_H L(Q^*) = 1.$$

Theorem 4 ([1]). *Let $Q^* = \|q_{ik}\|$, $i \in A = \{0,1,\dots,s-1\}$, and let there exists a digit $i_0 \in A$, such that $\forall \alpha > 0$ the condition: $\lim\limits_{k\to\infty} s^k \cdot q_{i_0,k}^{\alpha} = 0$ is satisfied.*

Then $\dim_H(L(Q^*)) = 0$.

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On Littlewood and Newman polynomial multiples of Borwein polynomials

Paulius Drungilas, Jonas Šiurys, Jonas Jankauskas

Abstract A Newman polynomial has all the coefficients in $\{0,1\}$ and constant term 1, whereas a Littlewood polynomial has all coefficients in $\{-1,1\}$. We call $P(X) \in \mathbb{Z}[X]$ a *Borwein* polynomial if all its coefficients belong to $\{-1,0,1\}$ and $P(0) \neq 0$. We exploit an algorithm developed previously by Lau and Stankov in their research on *the spectra of numbers* and independently by Akiyama, Thuswaldner and Zaïmi in their study of *Height Reducing Property*. The algorithm decides whether a given monic integer polynomial with no roots on the unit circle $|z| = 1$ has a non-zero multiple in $\mathbb{Z}[X]$ with coefficients in a finite set $\mathscr{D} \subset \mathbb{Z}$. Our results are as follows. For every Borwein polynomial of degree ≤ 9 we determine whether it divides any Littlewood or Newman polynomial. We show that every Borwein polynomial of degree ≤ 8 which divides some Newman polynomial divides some Littlewood polynomial as well. For every Newman polynomial of degree ≤ 11, we check whether it has a Littlewood multiple, extending the previous results of Borwein, Hare, Mossinghoff. We find examples of polynomials whose products and squares have no Littlewood or Newman multiples, while the original polynomials possess such multiples. Described results were presented in the paper "On Littlewood and Newman polynomial multiples of Borwein Polynomials" (to appear in AMS *Mathematics of Computation*).

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An introduction to p-adic systems: A new kind of number system

Mario Weitzer

In this talk a new kind of number system on the p-adic integers $\mathbb{Z}_p$ will be introduced which generalizes and provides a common framework for several different notions, such as positional notation systems, the $3n+1$ problem, integer fibred systems, or permutation polynomials. This framework allows notions and methods of one discipline to be translated to the other, admitting new points of views in both areas. Furthermore, it provides interesting examples of new discrete dynamical systems showing unexpected behavior.

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Salem numbers as unusual Mahler measures

Artūras Dubickas

Recall that for a polynomial

$$f(x) = a(x - \alpha_1)\dots(x - \alpha_n) \in \mathbb{C}[x], \quad \text{where} \quad a \neq 0,$$

its Mahler measure is defined by $M(f) := |a| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}$. The polynomial f is called *reciprocal* if the set of its roots $\{\alpha_1, \dots, \alpha_n\}$ is equal to $\{\alpha_1^{-1}, \dots, \alpha_n^{-1}\}$, i.e. $f(x) = \pm x^n f(x^{-1})$, and *nonreciprocal* otherwise. A root $\alpha > 1$ of a monic irreducible polynomial f in $\mathbb{Z}[x]$ of degree $2n \geq 4$ is called a *Salem number* if f is reciprocal and has $2n - 2$ roots on the unit circle $|z| = 1$.

Let L_0 be the set of all possible Mahler measures of nonreciprocal (but not necessarily irreducible) polynomials in $\mathbb{Z}[x]$. Various aspects of the set of Mahler measures

$$L := \{M(f) : f \in \mathbb{Z}[x]\}$$

and of its subset of nonreciprocal measures

$$L_0 := \{M(f) : f \in \mathbb{Z}[x], f - \text{nonreciprocal}\}$$

have been investigated in the papers of Adler and Marcus [1], Boyd [2], [3], [4], Dixon and the author [5], the author [7], Schinzel [9]. One of the problems from the BIRS workshop “The Geometry, Algebra and Analysis of Algebraic Numbers” held in 2015 in Banff (Canada) suggested by David Boyd $7(c)$ was the following:

- Does L_0 contain any Salem numbers?

The problem, as stated, was actually solved in [5] by Dixon and the author (see also [6]). Selecting, for instance, the nonreciprocal quartic polynomial $x^4 - x + 1$ whose Galois group is isomorphic to S_4 and Mahler measure is equal to the product

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$$|\beta|^2 = \beta\beta' = 1.40126\dots \in L_0$$

of two complex conjugate roots β and $\beta' = \overline{\beta}$ of $x^4 - x + 1$ that are outside the unit circle, we see that $\alpha = \beta\beta'$ must be of degree 6 over $\mathbb{Q}$ and, thus it is a Salem number (in this case, with minimal polynomial $x^6 - x^4 - x^3 - x^2 + 1$). This is true for any totally complex nonreciprocal quartic unit β whose Galois group is doubly transitive: each such Mahler measure $M(\beta)$ belongs to the set L_0 and at the same time it is a Salem number of degree 6.

This construction seems, however, an accidental one. So one may ask a more general question:

- Are there Salem numbers of other degrees in the set L_0?

It turns out that

Theorem 1. *The set of nonreciprocal Mahler measures L_0 contains infinitely many Salem numbers of degree $d = 4$ and also of each degree $d = 4\ell + 2$, where $\ell \in \mathbb{N}$.*

This theorem and some other related results have been proved in [8].

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On multiplicative independent bases for canonical number systems in cyclotomic number fields

Manfred Madritsch

In the present talk we focus on number systems in the ring of integers of cyclotomic number fields. Our goal is to show a variant of Cobham's theorem [2] stating that the only sets that are recognizable with respect to two multiplicative independent bases are finite unions of arithmetic progressions.

We start with the cases of fourth and third roots of unity, which correspond to the ring of Gaussian and Eisenstein integers, respectively. In the case of quadratic number fields the polynomials giving rise to a base have been completely characterized. In particular, the possible bases are of the form $-m \pm \zeta_3$ and $-m \pm \zeta_4$ with $m \geq 1$ a positive integer and ζ_k a primitive kth root of unity.

Moreover we clearly have that if $-m + \zeta_k$ is a base of a numeration system, then it is also an integral power base of the ring of integers. Győry [3] showed that there are only finitely many integral power bases and a conjecture of Bremner [1] and Robertson [6] states that the only possible power bases for $\mathbb{Z}[\zeta_k]$ are ζ_k, $\eta_k := (1+\zeta_k)^{-1}$ and $\theta_k := (1-\zeta_k)^{-1}$. This has been verified for several small values of k.

Let $\beta_1, \ldots, \beta_r$ be the integral power bases of $\mathbb{Z}[\zeta_k]$ then a result of Kovacs and Pethő [4] states that there exists M_i such that $-m + \beta_i$ is a base for $m \geq M_i$. Our first result state that $-m + \zeta_k$, $-m + \eta_k$ and $-m + \theta_k$ are bases of numeration systems in $\mathbb{Z}[\zeta_k]$ provided that $m \geq \varphi(k) + 1$ where φ is Euler's totient function.

Now we turn to the second ingredient: the multiplicative independence. We call two bases α and β multiplicative independent if the only solution (m,n) of the equation $\alpha^m = \beta^n$ over the integers is the trivial solution $(m,n) = (0,0)$. Our second result shows that $-m + \zeta_k$ and $-n + \zeta_k$ are multiplicative independent provided that $m \neq n$ and $k \neq 2,3,4,6$. The excluded cases were already treated by Madritsch and Ziegler [5].

The last part considers the variant of Cobham's theorem [2]. The four exponents conjecture states that for any two pairwise $\mathbb{Q}$-independent pairs of complex numbers

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(x_1, x_2) and (y_1, y_2) at least one of the numbers

$$e^{x_1 y_1}, \quad e^{x_1 y_2}, \quad e^{x_2 y_1} \quad \text{and} \quad e^{x_2 y_2}$$

is transcendent. We can show that under the four exponents conjecture every subset $S \subset \mathbb{Z}[\zeta_k]$ which is α- and β-recognizable for multiplicative independent α and β has to be syndetic. In this context a subset $S \subset \mathbb{Z}[\zeta_k] \subset \mathbb{C}$ is called syndetic if there exists $r > 0$ such that for each $\gamma \in \mathbb{Z}[\zeta_k]$ the intersection $S \cap B(\gamma, r)$ is non-empty (here $B(\gamma, r)$ is the closed disk with center γ and radius r.

This is joint work with Paul Surer from the Universität für Bodenkultur and Volker Ziegler from the University of Salzbourg.

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Topology of a class of $p2$-crystallographic replication tiles

Benoît Loridant and Shu-qin Zhang

Abstract We study the topological properties of a class of planar crystallographic replication tiles. Let $M \in \mathbb{Z}^{2\times 2}$ be an expanding matrix with characteristic polynomial x^2+Ax+B ($A, B \in \mathbb{Z}$, $B \geq 2$) and $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, M\mathbf{v})$ are linearly independent. Then the equation

$$MT + \frac{B-1}{2}\mathbf{v} = T \cup (T+\mathbf{v}) \cup (T+2\mathbf{v}) \cup \cdots \cup (T+(B-2)\mathbf{v}) \cup (-T) \qquad (1)$$

defines a unique nonempty compact set T satisfying $\overline{T^o} = T$. Moreover, T tiles the plane by the crystallographic group $p2$ generated by the π-rotation and the translations by integer vectors. Leung and Lau [2] proved in the context of self-affine lattice tiles with collinear digit set that $T \cup (-T)$ is homeomorphic to a closed disk if and only if $2|A| < B+3$. However, this characterization does not hold anymore for T itself (see Figure 1, for instance). We prove that

The crystile T is disk-like if and only if $-2 \leq A \leq 1$ **and** $B \geq 2$ **or** $A = B = 2$.

To achieve this, we find that the crystallographic replication tiles as (1) are homeomorphic to the tiles defined as follows,

$$g(T) = T \cup \left(T + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \cup \cdots \cup \left(T + \begin{pmatrix} B-2 \\ 0 \end{pmatrix}\right) \cup (-T) \qquad (2)$$

where

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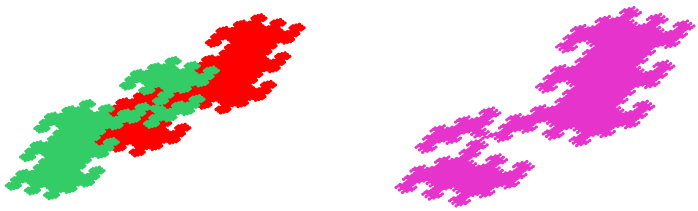
$$g(x,y) = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix} \tag{3}$$

is an expanding mapping on $\mathbb{R}^2$. Thus to prove the result, it is sufficient to prove it for T given by (2).

As a result of [3], we notice that the lattice tile T^ℓ given by

$$\begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} T^\ell = T^\ell \cup \left(T^\ell + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cup \cdots \cup \left(T^\ell + \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right). \tag{4}$$

is a translation of $T \cup (-T)$ for fixed A and B. For the case $2|A| - B < 3$, the associated lattice T^ℓ is disk-like by the result of Leung and Lau [2] and a result of Akiyama and Thuswaldner [1] on CNS tiles allow us to estimate the set of neighbors of T. Then finding out the disk-like tiles for this case will rely on the construction of the associated neighbor graphs for the whole class. Then using a criterion of Loridant and Luo [4], we can check the disk-likeness of a crystile. For the case $2|A| - B \geq 3$, we use a purely topological argument to prove that the associated tiles are not disk-like.



(a) The lattice tile T^ℓ (b) The crystallographic tile T

Fig. 1: Lattice tile and Crystile for $A = 2, B = 3$.

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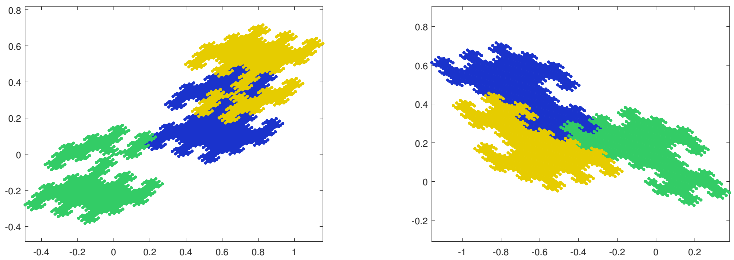

Fig. 2: $B = 3$. For $A = 2$ on the left, T is not disk-like and for $A = -2$ on the right, T is disk-like