Logica Lineare e semantica delle sue dimostrazioni: un'introduzione (a.a. 2022/2023)
L. Tortora de Falco

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Contents

Il testo che segue è largamente incompleto, molto schematico, scritto parzialmente in inglese e parzialmente in italiano. Si tratta di una traccia per fissare meglio il contenuto dei seminari di martedì 23 e giovedì 25 maggio 2023 ma sono solo appunti molto approssimativi. Vengono presentati alcuni risultati per il frammento moltiplicativo MLL della Logica Lineare:

1. l'interpretazione di una struttura di prova è invariante per eliminazione del taglio (Teorema 1.4, di cui si fornisce solo l'enunciato e lo schema della dimostrazione);

2. l'interpretazione delle strutture di prova è iniettiva (Teorema 1.6);

3. due esperimenti di una struttura di prova che sia aciclica nei grafi di correttezza hanno risultati coerenti tra loro: ne consegue che l'interpretazione di una tale struttura di prova è una clique dello spazio coerente che interpreta il suo sequente conclusione (Teorema 2.1);

4. se l'interpretazione di una struttura di prova è una clique dello spazio coerente che interpreta il suo sequente conclusione, allora la struttura di prova è aciclica nei grafi di correttezza (Teorema 2.3);

5. vale la separazione per le strutture di prova (Teorema 3.3);

6. non vale per le strutture di prova acicliche (per le reti di prova) lo stesso teorema di separazione valido per le strutture di prova (Proposizione 3.9).

I tre articoli principali cui si fa riferimento sono:

- Girard J.-Y., "Linear Logic", Theoretical Computer Science, vol. 50, 1987
- Retoré C. "On the relation between coherence semantics and multiplicative proof-nets", rapport INRIA n, 2430, dicembre 1994
- Pagani M., "Proofs, denotational semantics and observational equivalences in multiplicative Linear Logic", Mathematical Structures in Computer Science vol. 17(2), pp.341-361, Cambridge University Press, 2007

1 Proof structures and their interpretation, separation

A proof-net (or an AC-correct PS, or a PS satisfying AC) is a PS such that all of its correctness graphs are acyclic. We do not ask here for connectedness.

1.1 Coherent spaces

A coherent space is the data of

$$E = (|E|, \coh_E)$$

where $|E|$ is a set (which can be assumed to be at most countable) called *the web* of E and $\coh_E$ is a binary reflexive and symmetric relation on $|E|$ called *coherence*.

We use the following definitions and notations:

Strict coherence: $\frown_E = (\coh_E \cap \neq)$, that is $a \frown_E a'$ iff ($a \coh_E a'$ and $a \neq a'$);

Incoherence: $\asymp_E = \neg \frown_E$, that is $a \asymp_E a'$ iff ($a \not\coh_E a'$ or $a = a'$);

Strict incoherence: $\smile_E = \neg \coh_E$, that is $a \smile_E a'$ iff $a \not\coh_E a'$.

Note that any of these four relations characterises the three others.

A *clique* of E is a set of pairwise coherent points of $|E|$; we denote by $\mathrm{Cl}(E)$ the set of cliques:

$$\mathrm{Cl}(E) = \{u \subset |E|, \, \forall a, a' \in u, \, a \coh_E a'\}$$

Proposition 1.1 (Elementary properties of cliques)**.** *Let E be a coherent space. We have:*

1. *$\emptyset \sqsubset E$ so that $\mathrm{Cl}(E)$ is never empty (even when the web is empty).*
2. *Singletons are cliques: for any $a \in |E|$, $\{a\} \sqsubset E$.*
3. $\mathrm{Cl}(E)$ *is downward closed for inclusion: if $u \sqsubset E$ and $u' \subset u$ then $u' \sqsubset E$. For that reason we call u' a* subclique *of u.*
4. $\mathrm{Cl}(E)$ *is closed by compatible unions: if U is a family of cliques of E such that for every $u, v \in U$ we have $u \cup v \in \mathrm{Cl}(E)$, then $\bigcup U$ is a clique*
5. *Any clique u is the directed union of its finite subcliques:*

$$u = \bigcup \{u_0 \in \mathrm{Cl}(E), \, u_0 \subset_{\mathrm{fin}} u\}$$

6. *The space $E^\perp = (|E|, \asymp_E)$ is a coherent space. The cliques of $E^\perp$ are sets of pairwise incoherent points and are called the* anticliques *of E.*

   The dual of the dual $(E^\perp)^\perp$ is denoted $E^{\perp\perp}$. By definition of incoherence we have:

$$E^{\perp\perp} = E$$

Properties 3 and 4 actually yield an alternative (more "traditional" in the Scott's semantics sense) definition of coherent space.

1.2 Interpreting proof structures: experiments

We fix an infinite coherence space $\mathcal{A}t$ of atoms.

We associate with every multiplicative formula C a coherence space. The definition is by induction on (the complexity of the formula) A:

- with any atomic formula X we associate $\mathcal{A}t$;
- with $C = A \otimes B$, we associate the coherence space whose web is $|A| \times |B|$ and the coherence relation is defined as follows: $(a, b) \frown_{A \otimes B} (a', b')$ iff $a \frown_A a'$ and $b \frown_B b'$;
- with $C = A \parr B$, we associate the coherence space whose web is $|A| \times |B|$ and the coherence relation is defined as follows: $(a, b) \frown_{A \parr B} (a', b')$ iff $a \frown_A a'$ or $b \frown_B b'$.

By construction we have:

$$(X \otimes Y)^{\perp} = X^{\perp} \parr Y^{\perp}.$$

Definition 1.2. An **experiment** of a MELL proof-structure π is a function (= labelling) e s.t. $p \mapsto e(p) \in |A|$ for any edge $p : A$ of π, in such a way that the following conditions hold:

- If $a = a_1$ is the conclusion of an axiom node with conclusions the edges a_1 and a_2, then $e(a_1) = e(a_2)$.
- If a is the premise of a cut node with premises a and b, then $e(a) = e(b)$.
- If a is the conclusion of a $\parr$ (resp. $\otimes$) node with left premise a_1 and right premise a_2, then $e(a) = (x_1, x_2)$, where $e(a_1) = x_1$ and $e(a_2) = x_2$.

Let π be a MLL proof-structure with conclusions $p_1 : A_1, \dots, p_n : A_n$.

1. If e is an experiment of π, the *result* of e is $|e| = (e(p_1), \dots, e(p_n))$.
2. $[\pi] = \{|e| : e \text{ an experiment of } \pi\}$ is the interpretation of π.

Notice that we could define in the same way the interpretation of an *untyped* proof structure.

Remark 1.3. Notice that the interpretation is defined (also) on proof structures which are not AC-correct.

Theorem 1.4. *If R and R' are two PS such that $R \to R'$, then $[R] = [R']$.*

Proof. We need to prove that for every cut $x = (ax), (\parr/\otimes)$ such that $R \xrightarrow{[x]} R'$ one has:

- for every experiment e of R there exists an experiment e' of R' such that $|e| = |e'|$
- for every experiment e' of R' there exists an experiment e of R such that $|e| = |e'|$.

□

Remark 1.5. If $Ax_{A,A^\perp}$ is the proof net consisting of a unique ax-node with conclusion $A, A^\perp$ and $\eta_{A,A^\perp}$ is the so-called η-expansion of $Ax_{A,A^\perp}$, then $[Ax_{A,A^\perp}] = [\eta_{A,A^\perp}]$.

Provided we can arbitrarily choose the interpretation of the atoms, (in some sense) the converse of Theorem 1.4 holds too, meaning that two PS identified by the interpretation are equal "up to cut-elimination and η-expansion", namely:

Theorem 1.6 (Injectivity, semantic separation, faithfullness of the interpretation)**.** *If R and R' are two cut-free PS with atomic axioms such that, for every interpretaion of the atoms one has $[R] = [R']$, then $R = R'$.*

Proof. Say that an experiment of a proof-net is *injective* when it associates different labels with different axiom links. Take an injective experiment e of R (it exists since we can choose any interpretation for the atomic formulas): we have that $|e| \in [R] = [R']$. Then there exists an experiment e' of R' such that $|e'| = |e|$. One easily sees that e' has to be itself an injective experiment (in the result, we have pairwise distinct elements of the interpretation of the atoms).

Now notice that the only possible difference between R and R' (which are cut-free and have the same conclusion) might be in the axiom linking, but since e (resp. e') is an injective experiment of R (resp. R') and $|e| = |e'|$, this cannot be the case: we have $R = R'$ and $e = e'$. □

2 Correctness and coherence

Theorem 2.1. *If $\mathcal{R}$ is typed and (AC)-correct with conclusion Γ, then $[R] \in \parr\Gamma$ (here $\parr\Gamma$ is the coherent space interpreting the formula $\parr\Gamma$). That is: given any two experiments $e_1, e_2 : R$ one has $|e_1| \frown_{\parr\Gamma} |e_2|$.*

Proof. We are actually going to prove a stronger result, namely that if $|e_1| \asymp_{\parr\Gamma} |e_2|$ then $e_1 = e_2$ which immediately yields the result we look for, and Corollary 2.2. Notice that, by definition of experiment, $e_1 = e_2$ means that for every edge α of $\mathcal{R}$ we have $e_1(\alpha) = e_2(\alpha)$.

We define from R a directed graph R_0 as follows:

1. if $a :=$ erase the edge a, and erase also the links which have now become isolated;
2. for the "switching links" (contributing to correctness graphs) $\parr$ with premises a_1, a_2, if there is a premise a_i of l such that $a_i :\smile$ and for the other premise we have $a_j :\frown$, then we disconnect a_i from the link so that a_i becomes a new conclusion called "switched conclusion";

3. we direct all the edges b following the coherence relation on b between e_1 and e_2: if $b :\smile$ then $\uparrow b$ and if $b :\frown$ then $\downarrow b$.

We call R_0 the graph thus obtained. First notice that R_0 is a directed acyclic graph (dag):

- R_0 is still a graph, since we have only erased edges. And we have erased nodes only when they had become isolated: it is still the case that every edge connects two nodes. R_0 is oriented since e_1 and e_2 are still defined on every edge of R_0 (we added no new edge).

- since R is a proof net, the unique possibility for a path of $\mathcal{R}_0$ to be a cycle is to bounce in a switching node (i.e. for some $\parr$-node of $\mathcal{R}_0$ with premises b_1 and b_2 we have b_2^+ and b_1^- in $\mathcal{R}_0$) or to be a proper cycle of some $\parr$-node n of $\mathcal{R}_0$, with first edge a premise b_1^- of n and last edge the other premise b_2^+ of that same node n. Both cases are forbidden by items 2 and 3. Indeed, if b_1^- and b_2^+ are in $\mathcal{R}_0$ for two premises b_1 and b_2 of some $\parr$-node of R, then by Item 3 we have $b_1 :\smile$ and $b_2 :\frown$ and thus by Item 2 b_1 is one of the premises of some $\parr$-node of R that has become a conclusion of $\mathcal{R}_0$ (it is not, in $\mathcal{R}_0$, a premise of a $\parr$-node).

We want to prove that R_0 is empty. We are going to prove that whatever node n of R_0 one chooses, n is not a sink: there is always a way out from n. Since R_0 is a dag, if it is not empty there is a sink. Thus if we can prove that there is no sink in $\mathcal{R}_0$, then we can conclude that $\mathcal{R}_0$ is necessarily empty.

First notice that the conclusion nodes of $\mathcal{R}_0$ are not sinks:

1. the unique premise of a conclusion node of $\mathcal{R}_0$ that is not a conclusion of $\mathcal{R}$ is oriented bottom-up;

2. by hypothesis, for the unique premise α of a conclusion node c coming from a a conclusion node of $\mathcal{R}$ we have $\alpha :\asymp$. Then we cannot have $\alpha :\frown$ and c is not a sink.

To conclude that R_0 is empty, we show that every node n of R_0 which is not a conclusion node is not a sink:

- if n is an axiom or a cut (and it has not been erased) it is obvious: the labels of the two conclusions/premises are the same and the types of the conclusions/premises are dual

- $n = \otimes$: if for the conclusion a we have $a :\frown$ we can exit from there. If $a :=$, then $a_i :=$ for both the premises and n is not a node of R_0 by item 1. Finally, if $a :\smile$ then for one of the two premises a_i of l we have $a_i :\smile$ and we can exit from there

- $n = \parr$ i.e. a switching link: if for the conclusion a we have $a :\frown$ we can exit from there. Otherwise $a :\asymp$ and then for every premise a_i of n we have $a_i :\asymp$. If for every premise a_i of n we have $a_i :=$, then $a :=$ and n

is not a node of R_0 by item 1. Otherwise there is a premise a_i of n in R such that $a_i :\smile$ and by item 1 this premise is still present in R_0 (remember that for every premise a_i of n we have $a_i :\asymp$, so that no premise has been disconnected by item 1), and we can exit from there.

□

Corollary 2.2. *If R is typed and (AC)-correct and $e_1, e_2 : R$ are two experiments of R such that $|e_1| = |e_2|$, then $e_1 = e_2$.*

Proof. Since $|e_1| = |e_2|$ implies $|e_1| \asymp |e_2|$, it is an immediate consequence of the proof of Theorem 2.1. □

Path: By path we mean *simple path*: an edge is crossed at most once, and in particular $\downarrow d \uparrow d$ is forbidden so as $\uparrow d \downarrow d$. As a consequence, if c, c' are two conclusions and Φ is a path with starting edge $\uparrow c$ and terminal edge $\downarrow c'$, then for every conclusion $c'' \neq c, c'$ we have $\downarrow c'' \notin \Phi$ and $\uparrow c'' \notin \Phi$.

Theorem 2.3. *Let R be a typed MLL cut-free proof structure with conclusion $c_1 : C_1, \ldots, c_n : C_n$ and $\mathcal{X}$ a coherent space such that $x, y, z \in |\mathcal{X}|$ with $x \frown y$ and $x \smile z$.*

If for the interpretation of MLL obtained by associating with every atomic formula the coherent space $\mathcal{X}$ we have that $[R]$ is a clique of $\mathcal{C}_1 \parr \ldots \parr \mathcal{C}_n$, then R is AC-correct.

The proof of Theorem 2.3 follows from the following lemmas, where we call "switching path" a path of a correctness graph.

Lemma 2.4. *Let R be a MLL cut-free proof structure, and let c, c' be two different conclusions of R and Φ be a switching path of R with starting edge $\uparrow c$ and terminal edge $\downarrow c'$. Then there exists a switching path Φ' of R with starting edge $\uparrow c$ and terminal edge $\downarrow c'$ such that for every $d \geq c$[1] edge of R, one has $\downarrow d \notin \Phi$).*

Proof. Per andare da c a c' ad un certo punto bisognerà per forza (essendo $c \neq c'$) che Φ "abbandoni" l'insieme degli archi d tale che $d \geq c$, e bisognerà che lo faccia "definitivamente", cioè esiste $\uparrow d$, con $d \geq c$ tale che in Φ dopo $\uparrow d$ passo solo da archi b che non sono sopra c. Prendo allora $\Phi' = \overline{\Psi}\Phi''$, where Φ'' is the subpath of Φ from $\uparrow d$ to $\downarrow c'$ and Ψ is the descent path from $\downarrow d$ to $\downarrow c$: the paths $\overline{\Psi}$ and Φ'' are both switching and disjoint and they meet in an axiom node, thus $\Phi' = \overline{\Psi}\Phi''$ is switching. Furthermore every edge crossed by Φ' above c is crossed upwards and thus Φ' satisfies the conclusion of the lemma. □

[1] For a, b edges of R one can define the partial order $a \leq b$ iff b is "above" a. Notice that antisimmetry is a consequence of the absence of vicious cycles: if a has type A and b has type B and $a \leq b$, then B is a subformula of A (no vicious cycle can ever occur in a typed framework).

Lemma 2.5. *Let $\mathcal{X}$ be a coherent space such that $x, y, z \in |\mathcal{X}|$ with $x \frown y$ and $x \smile z$, and consider the interpretation of MLL obtained by associating with every atomic formula the coherent space $\mathcal{X}$. Let R be a MLL cut-free proof structure, let c, c' be two conclusions of R, and let Φ be a switching path of R with starting edge $\uparrow c$ and terminal edge $\downarrow c'$ such that, for every $d \geq c$ edge of R, one has $\downarrow d \notin \Phi$.*

There exists two experiments e_1 and e_2 of R s.t. $c :\frown$, $c' :\smile$ and for every c'' conclusion of R such that $c'' \neq c, c'$ we have $c'' :\asymp$.

Proof. Since we are cut-free and in MLL, an experiment is entirely determined by its values on the axiom edges. Notice that whatever formula A of MLL we consider, there exist $\alpha, \beta, \gamma \in |\mathcal{A}|$ such that $\alpha \frown \beta$ and $\alpha \smile \gamma$[2]. Now we define e_1 and e_2 by declaring their values on the conclusions of the generic axiom node whose conclusions are labelled by $A, A^\perp$: if a is the conclusion of type A, we set $e_1(a) = \alpha$ (we associate the element of $|\mathcal{A}|$ inheriting the property we assume is satisfied by the space $\mathcal{X}$ interpreting every propositional variable). The values of e_2 depend on the way Φ crosses the axiom edges. Since Φ is a (simple) path, an axiom-node is crossed at most once by Φ, so that we can define e_2 on the conclusions of the axiom nodes of R as follows:

- if $\uparrow a \in \Phi$, then we set $e_2(a) = \beta$ (thus $a :\frown$)
- if $\downarrow a \in \Phi$, then we set $e_2(a) = \gamma$ (thus $a :\smile$)
- if $\uparrow a \notin \Phi$ and $\downarrow a \notin \Phi$, then we set $e_2(a) = \alpha = e_1(a)$.

Now, for every edge d of R, we prove that:

1. if for some $d' \geq d$ we have $\uparrow d' \in \Phi$ or $\downarrow d' \in \Phi$, then $d :\neq$
2. if $\uparrow d \notin \Phi$, then $d :\asymp$
3. if for every $d' \geq d$ we have $\downarrow d' \notin \Phi$, then $d :\frown$

Property 1 is a consequence of the fact that whenever $\uparrow a \in \Phi$ or $\downarrow a \in \Phi$, there exists an axiom edge $b \geq a$ such that $\uparrow b \in \Phi$ or $\downarrow b \in \Phi$ (in which case we have by definition $b :\neq$ and thus $a :\neq$): if $\uparrow a \in \Phi$ I need to go up until I can and if $\downarrow a \in \Phi$ then either a is conclusion of an axiom and we are done or it is the conclusion of a $\otimes$ or $⅋$ node such that one of its premises a_1 satisfies $\downarrow a_1 \in \Phi$ (intuitively $\downarrow a$ comes from an axiom conclusion $\downarrow b$).

Let's now prove properties 2 and 3 by induction on the number of nodes above d:

- if d is the conclusion of an axiom node which is not crossed by Φ (i.e. $\downarrow d \notin \Phi$ and $\uparrow d \notin \Phi$) then by definition $d :=$ and the properties 2 and 3 hold for d;

[2]Proof: exercise by induction on A. If $A = A_1 \otimes A_2$ or $A = A_1 ⅋ A_2$, then by IH $\alpha_i \frown \beta_i$ and $\alpha_i \smile \gamma_i$, thus $(\alpha_1, \alpha_2) \frown (\beta_1, \beta_2)$ and $(\alpha_1, \alpha_2) \smile (\gamma_1, \gamma_2)$.

- if d is the conclusion of an axiom node such that $\downarrow d \in \Phi$ or $\uparrow d \in \Phi$, then if we call a and b the conclusions of the axiom we necessarily have $\downarrow a, \uparrow b \in \Phi$ or $\downarrow b, \uparrow a \in \Phi$: suppose we have for example $\downarrow a, \uparrow b \in \Phi$, then by definition of e_1, e_2 we have $a : \smile$ and $b : \frown$ and the properties 2 and 3 hold for d;

- if d is the conclusion of a $\otimes$ node with left (resp. right) premise a (resp. b):

property 2: suppose $\uparrow d \notin \Phi$. If $\uparrow a \in \Phi$ (resp. $\uparrow b \in \Phi$), then since $\uparrow d \notin \Phi$ and $\uparrow a \in \Phi$ (resp. $\uparrow b \in \Phi$) cannot be the first edge of Φ (which has a conclusion as starting node) we necessarily have $\downarrow b \in \Phi$ (resp. $\downarrow a \in \Phi$): then $\uparrow b \notin \Phi$ (resp. $\uparrow a \notin \Phi$) and by induction hypothesis $b : \asymp$ (resp. $a : \asymp$) which by property 1 implies $b : \smile$ (resp. $a : \smile$) and thus $d : \smile$. If on the contrary $\uparrow a, \uparrow b \notin \Phi$, then by induction hypothesis $a : \asymp$ and $b : \asymp$, which implies $d : \asymp$.

property 3: suppose for every $d' \geq d$ we have $\downarrow d' \notin \Phi$. Then obviously for every $d' \geq a$ (resp. for every $d' \geq b$) we have $\downarrow d' \notin \Phi$, and thus by IH $a : \overset{\frown}{\smile}$ (resp. $b : \overset{\frown}{\smile}$), which implies $d : \overset{\frown}{\smile}$.

- if d is the conclusion of a ⅋ node with left (resp. right) premise a (resp. b):

property 2: suppose $\uparrow d \notin \Phi$. Since Φ is a switching path, necessarily $\uparrow a \notin \Phi$ and $\uparrow b \notin \Phi$. Indeed, if $\uparrow a \in \Phi$ (resp. $\uparrow b \in \Phi$) and $\uparrow d \notin \Phi$, since $\uparrow a \in \Phi$ (resp. $\uparrow b \in \Phi$) cannot be the first edge of Φ (which has a conclusion as starting node), we should have $\downarrow b \in \Phi$ (resp. $\downarrow a \in \Phi$), but in this case Φ would not be switching. Then by IH $a : \asymp$ and $b : \asymp$ which implies $d : \asymp$;

property 3: suppose for every $d' \geq d$ we have $\downarrow d' \notin \Phi$. Then obviously for every $d' \geq a$ (resp. for every $d' \geq b$) we have $\downarrow d' \notin \Phi$, and thus by IH $a : \overset{\frown}{\smile}$ (resp. $b : \overset{\frown}{\smile}$), which implies $d : \overset{\frown}{\smile}$.

From the properties 1, 2 and 3 we can conclude that $c : \frown$: indeed, since $\uparrow c \in \Phi$ and (by the hypothesis of the lemma) for every $d \geq c$ we have $\downarrow d \notin \Phi$, by property 3 this yields $c : \overset{\frown}{\smile}$, and by property 1 this means that $c : \frown$. On the other hand, since $\downarrow c' \in \Phi$ we have $\uparrow c' \notin \Phi$ and hence by property 2 $c' : \asymp$, which implies by property 1 that $c' : \smile$. Finally, if c'' is a conclusion of R and $c'' \neq c, c'$, then $\uparrow c'' \notin \Phi$ (otherwise $\downarrow c'', \uparrow c'' \in \Phi$ and Φ would not be a path) and thus by property 2 we have $c'' : \asymp$. □

Let's now prove Theorem 2.3:

Proof. We prove, by induction on the number of nodes of R, that if R does not satisfy AC, then $[R]$ is not a clique. If R does not satisfy AC, then it necessarily has a terminal $\otimes$-node or a terminal ⅋-node:

- if R has a terminal $\parr$-node, then we can remove it thus obtaining R' which is still a proof structure which does not satisfy AC. Then by IH $[R']$ is not a clique, and $[R]$ neither (from two experiments e'_1 and e'_2 of R' such that $|e'_1| \smile |e'_2|$ it is easy to build two experiments e_1 and e_2 of R such that $|e_1| \smile |e_2|$[3])
- if R has a terminal $\otimes$-node l, then we can remove l thus obtaining R', for which there are two possibilities:
  - R' is still a proof structure which does not satisfy AC, in which case by IH $[R']$ is not a clique and $[R]$ neither (proceeding like in the $\parr$case)
  - R' has become a proof structure satisfying AC: then there exists a switching path Φ of R' with starting edge a premise $\uparrow a$ of l and terminal edge the other premise $\downarrow b$ of l (since R has a switching cycle and R' hasn't, every switching cycle of R crosses l: we just select one of them). By Lemma 2.4, we can suppose that, for every $g \geq a$ edge of R', one has $\downarrow g \notin \Phi$. We can now apply Lemma 2.5 and find two experiments e'_1, e'_2 of R' such that for every conclusion $c \neq a, b$ of R' we have $c :\asymp$ while $a :\frown$ and $b :\smile$: e'_1 and e'_2 immediately yield two experiments e_1, e_2 of R such that $e_1(d) \smile e_2(d)$ (where d is the conclusion of l), and for every conclusion $c \neq d$ we have $e_1(c) \asymp e_2(c)$: thus $|e_1| \smile |e_2|$ and $[R]$ is not a clique.

□

Remark 2.6. The absence of cuts is not really a limit (substitute with $\otimes$).

3 Syntactic separation in MLL

Coming back to separation, after Theorem 1.6 we now move to another possible approach to separation: observational equivalence. We will discuss observational equivalence in the restricted MLL case, where the situation is a bit different for general proof structure and for proof nets. We are dealing here with the general question of separation in Logic. The general aim is to distinguish proofs (or programs) that are not computationally equivalent. This can be done either by semantic means (injectivity/faithfulness of the model) or by syntactic/interactive means with respect to some given values (observational equivalence). In this last case we distinguish proof structures[4] through their interactive behaviour wrt the contexts of a given type. In MLL, consider the two cut-free proof-structures (actually proof-nets) with conclusion $\mathbb{B} = (X^\perp \otimes X^\perp) \parr (X \parr X)$, where Ω (resp. $\mho$) is the planar (resp. non planar) proof-net: in $\mho$ (resp. Ω) axiom links do not cross (resp. cross). The two proof-nets Ω and $\mho$ are taken here as *values* or *observables*.

[3] Do it! Distinguish the case $e'_1(a) \smile e'_2(a)$ above the $\parr$ and the other case.

[4] Notice that in the λ-calculus, the question of separation for "λ-structures" cannot even be stated since everything is correct...

Definition 3.1 (observational equivalence in MLL)**.** For $C_1, \ldots, C_n$ formulas of MLL, a context $C[\ ]$ of type $C_1, \ldots, C_n$ is a PS with conclusion $\mathbb{B}$ where proper axioms of type $C_1, \ldots, C_n$[5] can occur. If π is a PS with conclusion $C_1, \ldots, C_n$ and $C[\ ]$ is a context of type $C_1, \ldots, C_n$, we denote by $C[\pi]$ the proof structure with conclusion $\mathbb{B}$ obtained by substituting every proper axiom of type $C_1, \ldots, C_n$ with the PS π.

Two proof structures π_1 and π_2 with conclusions $C_1, \ldots, C_n$ are *observationally equivalent* ($\pi_1 \sim_{\mathbb{B}} \pi_2$) when, for every context $C[\ ]$ of type $C_1, \ldots, C_n$, we have $C[\pi_1] =_\beta C[\pi_2]$. (In MLL we write $\pi =_\beta \pi'$ when π and π' have the same normal form.)

Remark 3.2. Observational equivalence, so as semantic equivalence, is a congruence: if π_1 and π_2 are equivalent and if π_1' (resp. π_2') is obtained from π_1 (resp. π_2) by adding a given link, then π_1' and π_2' are equivalent too.

Theorem 3.3 (Separation for MLL)**.** *Let π_1 and π_2 be two proof structures with the same conclusion D. If $\pi_1 \neq_\beta \pi_2$, then there exists a PS σ with conclusion $D^\perp, \mathbb{B}$ such that $cut(\pi_1, \sigma) \twoheadrightarrow \Omega$ and $cut(\pi_2, \sigma) \twoheadrightarrow \mho$*[6]*.*

Proof. We just have to build the proof net σ. □

Corollary 3.4. *Let π_1 and π_2 be two PS with the same conclusion. We have $\pi_1 \sim_{\mathbb{B}} \pi_2 \iff \pi_1 =_\beta \pi_2$.*

Proof. Let π_1^0 (resp. π_2^0) be the normal form of π_1 (resp. π_2): for every context $C[\ ]$ we have $C[\pi_1] \twoheadrightarrow C[\pi_1^0]$ (resp. $C[\pi_2] \twoheadrightarrow C[\pi_2^0]$). Now, from $\pi_1 =_\beta \pi_2$ we deduce $\pi_1^0 = \pi_2^0$, and thus $C[\pi_1^0] = C[\pi_2^0]$ which entails $C[\pi_1] =_\beta C[\pi_2]$. Then $\pi_1 \sim_{\mathbb{B}} \pi_2$.

Conversely, if $\pi_1 \neq_\beta \pi_2$, by Theorem 3.3 there exists a context $C[\ ]$ such that $C[\pi_1] = \Omega$ and $C[\pi_2] = \mho$, which means in particular that $C[\pi_1] \neq_\beta C[\pi_2]$ and thus $\pi_1 \not\sim_{\mathbb{B}} \pi_2$. □

Corollary 3.5 (maximality of $=_\beta$)**.** *Let $\equiv$ be a congruence containing $=_\beta$*[7]*. Then either $\equiv$ and $=_\beta$ coincide or $\equiv$ is degenerated, meaning that whenever π_1 and π_2 are two PS with the same conclusion one has $\pi_1 \equiv \pi_2$.*

Proof. (**Sketch of**) One proves that if there exists π_1 and π_2 two PS with the same conclusion such that $\pi_1 \equiv \pi_2$ and $\pi_1 \neq_\beta \pi_2$, then for every two PS τ_1 and τ_2 with the same conclusion we have $\tau_1 \equiv \tau_2$.

The key point is to show that if τ_1 and τ_2 with conclusion D differ exactly because for o, o' and e, e' atomic edges we have $\tau_1(o) = e$ (resp. $\tau_2(o) = e'$) and $\tau_1(o') = e'$ (resp. $\tau_2(o') = e$), there exists a PS $\sigma (= \sigma_{o,o',e,e'})$ with conclusion $D^\perp, D, \mathbb{B}^\perp$ such that $cut(\tau_1, \sigma, \Omega) \twoheadrightarrow \tau_2$ and $cut(\tau_1, \sigma, \mho) \twoheadrightarrow \tau_1$.

[5] A proper axiom with conclusion $C_1, \ldots, C_n$ is just a link with n conclusions of type $C_1, \ldots, C_n$.

[6] In particular, this means that there exists a context $C[\ ]$ (of type D) such that $C[\pi_1] = \Omega$ and $C[\pi_2] = \mho$.

[7] Namely: if $\pi_1 =_\beta \pi_2$ we also have $\pi_1 \equiv \pi_2$.

Once this construction has been performed one can argue as follows. Since $\pi_1 \neq_\beta \pi_2$, by Theorem 3.3 there exists a context $C[\,]$ such that $C[\pi_1] =_\beta \Omega$ and $C[\pi_2] =_\beta \mho$; and on the other hand since $\pi_1 \equiv \pi_2$ we have $C[\pi_1] \equiv C[\pi_2]$.

Applying the key point and the fact that $\equiv$ is a congruence containing $=\beta$, we obtain that $\tau_1 \equiv cut(\tau_1, \sigma, \mho) \equiv cut(\tau_1, \sigma, C[\pi_2]) \equiv cut(\tau_1, \sigma, C[\pi_1]) \equiv cut(\tau_1, \sigma, \Omega) \equiv \tau_2$. □

Remark 3.6. Corollary 3.5 yields another proof of Theorem 1.6: the relation $\equiv$ between proof-structures with the same conclusion given by $\pi_1 \equiv \pi_2 \iff [\pi_1] = [\pi_2]$ is a congruence which extends $=_\beta$, and we know that $[\Omega] \neq [\mho]$ so that $\equiv$ is not degenerated. By Corollary 3.5 we deduce that $\equiv$ and $=_\beta$ coincide.

We can then turn to correct proof structures (proof nets). Observational equivalence *is not* the same as for proof structures:

Definition 3.7 (observational equivalence in MLL)**.** Two proof-nets π_1 and π_2 with conclusions $C_1, \ldots, C_n$ are *observationally weak equivalent* ($\pi_1 \sim^w_{\mathbb{B}} \pi_2$) when, for every *correct* context[8] $C[\,]$ of type $C_1, \ldots, C_n$, we have $C[\pi_1] =_\beta C[\pi_2]$.

Lemma 3.8 (context lemma)**.** *Let π_1 and π_2 be two proof-nets with conclusion $C_1, \ldots, C_n$, we denote by $\pi_1^\star$ (resp. $\pi_2^\star$) π_1's (resp. π_2's) closure with conclusion $C_1 \parr \ldots \parr C_n$. Then $\pi_1 \not\sim^w_{\mathbb{B}} \pi_2$ iff there exists a proof-net σ with conclusion $C_1^\perp \otimes \ldots \otimes C_n^\perp, \mathbb{B}$ such that $cut(\pi_1^\star, \sigma) \neq_\beta cut(\pi_2^\star, \sigma)$.*

Proposition 3.9. *The equivalence relation $\sim^w_{\mathbb{B}}$ on the set of proof-nets with the same conclusion strictly contains $=_\beta$: there exist two proof-nets π_1 and π_2 such that $\pi_1 \sim^w_{\mathbb{B}} \pi_2$ and $\pi_1 \neq_\beta \pi_2$.*

Proof. (**Sketch of**) Let $C = ((X \otimes X) \parr X) \parr ((X^\perp \parr X^\perp) \parr X^\perp)$ and let π_1 and π_2 be any two different cut free proof nets with conclusion C: we have $\pi_1 \neq_\beta \pi_2$ and we want to prove that $\pi_1 \sim^w_{\mathbb{B}} \pi_2$. By contradiction, suppose $\pi_1 \not\sim^w_{\mathbb{B}} \pi_2$; applying Lemma 3.8 we can find a proof-net σ with conclusion $C^\perp, \mathbb{B}$ such that $cut(\pi_1, \sigma) \neq_\beta cut(\pi_2, \sigma)$. Since both $cut(\pi_1, \sigma)$ and $cut(\pi_2, \sigma)$ have conclusion $\mathbb{B}$ and Ω and $\mho$ are the only two cut free proof nets of type $\mathbb{B}$ that are not in the same equivalence class wrt $=_\beta$, we have that $cut(\pi_1, \sigma) \twoheadrightarrow \mho$ and $cut(\pi_2, \sigma) \twoheadrightarrow \Omega$ (or $cut(\pi_2, \sigma) \twoheadrightarrow \mho$ and $cut(\pi_1, \sigma) \twoheadrightarrow \Omega$ which is perfectly symmetric).

Now we want to show that the existence of this proof-net σ yields a contradiction: we prove that any such PS σ is not a proof-net by semantic means. We show that we can appropriately choose an interpretation of the atoms of C in such a way that $[\sigma]$ is not a clique: there exist two experiments e and e' of σ such that $|e| \smile |e'|$.

So, take a coherent space $\mathcal{X}$ such that $x, y, z \in |\mathcal{X}|$ with $x \frown y$, $x \smile z$, $y \smile z$, and consider the interpretation of MLL obtained by associating with every atomic formula the coherent space $\mathcal{X}$. Since $<< x, z >, < x, z >> \in \mho$ and $<< x, z >, < z, x >> \in \Omega$, from $[cut(\pi_1, \sigma)] = [\mho]$ (resp. $[cut(\pi_2, \sigma)] = [\Omega]$) it follows that there exist $u \in [\pi_1]$ and $v \in [\pi_1]$ such that $< u, << x, z >, < x, z >>> \in [\sigma]$ and $< v, << x, z >, < z, x >>> \in [\sigma]$. Now if in the experiment e_1 (resp. e_2)

[8] A context is correct when all its correctness graphs are acyclic.

of π_1 (resp. π_2) such that $|e_1| = u$ resp. $|e_2| = v$ we substitute $x \in |\mathcal{X}|$ for every atom different from z (and x) occurring in u (resp. v), we obtain an experiment of π_1 (resp. π_2) with result u' (resp. v'). And by performing the same operation in the experiment with result $< u, << x, z >, < x, z >>>$ (resp. $< v, << x, z >, < z, x >>>$) of σ, we obtain an experiment of σ with result $< u', << x, z >, < x, z >>>$ (resp. $< v', << x, z >, < z, x >>>$).

Notice that now the only atoms occurring in u' and v' are x and z. We look at the atom a (resp. b) of u' (resp. v') corresponding to the occurrence of X in bold in the formula $C^\perp = ((X^\perp \parr X^\perp) \otimes X^\perp) \otimes ((X \otimes X) \otimes \mathbf{X})$. We have two possibilities:

1. $a = b$: in this case $a = b = x$ or $a = b = z$;

2. $a \neq b$: in this case $a = x$ and $b = z$ or $a = z$ and $b = x$.

In Case 2 we have $a \smile b$ ($mod\ \mathcal{X}$) and thus $u' \smile v'$ ($mod\ \mathcal{C}^\perp$). From $x \smile z$ ($mod\ \mathcal{X}$) we also deduce $<< x, z >, < x, z >> \smile << x, z >, < z, x >>$ ($mod\ \mathbb{B}$): then $< u', << x, z >, < x, z >>> \smile < v', << x, z >, < z, x >>>$ ($mod\ \mathcal{C}^\perp \parr \mathbb{B}$) and $[\sigma]$ is not a clique.

In Case 1 we suppose $a = b = x$ (the case $a = b = z$ can be treated in a similar way). In the experiment of π_1 with result u', so as in the experiment of σ with result $< u', << x, z >, < x, z >>>$, we substitute y for z: we thus obtain that $u'' \in [\pi_1]$ and $< u'', << x, y >, < x, y >>> \in [\sigma]$, where $u'' = u'[y/z]$. In the experiment of π_2 with result v', so as in the experiment of σ with result $< v', << x, z >, < z, x >>>$, we substitute x for z and z for x: we thus obtain that $v'' \in [\pi_2]$ and $< v'', << z, x >, < x, z >>> \in [\sigma]$, where $v'' = v'[x/z, z/x]$. Notice that the atom corresponding to the occurrence of X in bold in the formula $C^\perp = ((X^\perp \parr X^\perp) \otimes X^\perp) \otimes ((X \otimes X) \otimes \mathbf{X})$ is x for u'' and z for v'', so that from $x \smile z$ ($mod\ \mathcal{X}$) we can deduce that $u'' \smile v''$ ($mod\ \mathcal{C}^\perp$). Moreover, recalling that $\mathbb{B} = (X^\perp \otimes X^\perp) \parr (X \parr X)$, from $x \smile y$ ($mod\ \mathcal{X}^\perp$) and $y \smile z$ ($mod\ \mathcal{X}$) we deduce that $<< x, y >, < x, y >> \smile << z, x >, < x, z >>$ ($mod\ \mathbb{B}$). Then $< u'', << x, y >, < x, y >>> \smile < v'', << z, x >, < x, z >>>$ ($mod\ \mathcal{C}^\perp \parr \mathbb{B}$) and $[\sigma]$ is not a clique. □

Remark 3.10. Proposition 3.9 shows that there exist proof-nets that are not in the same $=_\beta$-class and that nevertheless cannot be separated by any *correct* context. This does not contradict Corollary 3.5: those proof-nets can certainly be separated by some context, but none of these context is *correct*.