Topological Correctness Criteria for Linear Logic Proof Structures

Roberto Maieli

Dipartimento di Matematica e Fisica
Università degli Studi "Roma Tre"
maieli@mat.uniroma3.it

PRIN Project: *Logical Methods of Information Management*

TACL 2015 – Ischia, 21-26 June 2015

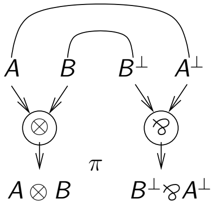
proofs vs proof nets

In his seminal article on linear logic (1987), Jean-Yves Girard develops two alternative notations for proofs:

- a **sequential syntax** where proofs are expressed as **derivation trees** in a sequent calculus,

$$\dfrac{\dfrac{\vdash A, A^{\perp} \qquad \vdash, B, B^{\perp}}{\vdash A \otimes B, B^{\perp}, A^{\perp}}\otimes}{\vdash A \otimes B, B^{\perp} \parr A^{\perp}}\parr$$

- a **parallel syntax** where proofs are expressed as **bipartite graphs** called **proof-nets**



the proof nets notation

- it exhibits more of the intrinsic structure of proofs than the derivation tree notation, and is closer to denotational semantics.
- while a derivation tree defines a unique proof-net, a proof-net may represent several derivation trees, each derivation tree witnessing a particular order of sequentialization of the proof-net.
- it requires to separate "real proofs" (proof-nets) from "proof alikes" (called proof-structures) using a correctness criteria
- correctness criteria reveal the "geometric" essence of the logic, beyond its "grammatical" (inductive) presentation as a sequent calculus.

MLL formulas and negation

- an MLL **formula** $A, B, C, \ldots$ is a tree with leaves $p, q, r, \ldots$ and $p^{\perp}, q^{\perp}, r^{\perp}, \ldots$ called atoms, and binary connectives $\otimes, \parr$.
- the **negation** $A^{\perp}$ of a formula A is the formula defined inductively by so-called **de Morgan laws**:

$$\begin{aligned} (p)^{\perp} &= p^{\perp} \\ (p^{\perp})^{\perp} &= p \\ (A \otimes B)^{\perp} &= A^{\perp} \parr B^{\perp} \\ (A \parr B)^{\perp} &= A^{\perp} \otimes B^{\perp} \end{aligned}$$

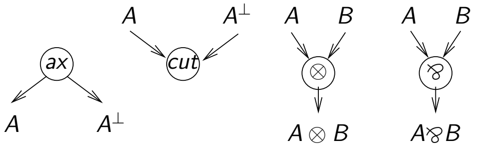
- it follows that $(A^{\perp})^{\perp} = A$ for every formula A.

MLL sequent calculus

- An MLL sequent is a finite sequence of formulas, $\vdash A_1, ..., A_n$.
- We usually write finite sequences of formulas as greek letters $\Gamma, \Delta, ...$
- A derivation is a tree with a sequent at each node, constructed inductively by the rules below (Exchange rule is implicit)
  - identity: $\dfrac{}{A, A^\perp}\,\text{ax} \qquad \dfrac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\,\text{cut}$
  - multiplicatives: $\dfrac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\,\otimes \qquad \dfrac{\Gamma, A, B}{\Gamma, A ⅋ B}\,⅋$

MLL links

An MLL link is a graph of the following form, whose edges (resp., vertexes) are labeled with MLL formulas (resp., connectives):



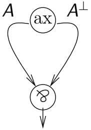
- Axiom link with two conclusions A and $A^{\perp}$, and no premise;
- Cut link with two premises A and $A^{\perp}$, and no conclusion;
- $\otimes$ and $\parr$ links where the formula A is the first premise, the formula B is the second premise, and $A \otimes B$ (or $A \parr B$) is the conclusion.

MLL proof-structure (PS)

A PS π is a graph built by links s.t. every (occurrence of) formula is the conclusion of one link, and the premise of at most one link.

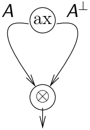
Every derivation tree (inductively) defines a PS:

$$\frac{\vdash A, A^{\perp}}{\vdash A \parr A^{\perp}} \parr$$ de-sequentializes into



but conversely, not every PS is deduced from a derivation tree.

none derivation de-sequentializes into



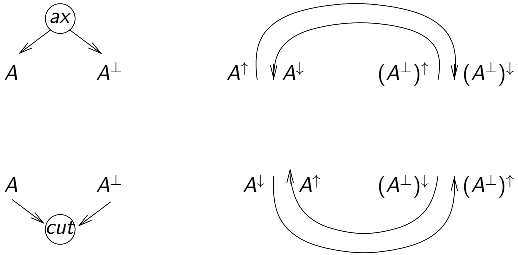
So, which proof-structures exactly are obtained from derivation trees?

ribbon diagrams

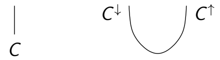
idea: we associate to each link (at least) one **ribbon diagram** (or *switching position*) with a **directed border** labeled by

decorated formulas $(A)^{\downarrow}, (A)^{\uparrow}$

– **axiom** and **cut links** are replaced by simple **ribbon diagrams**

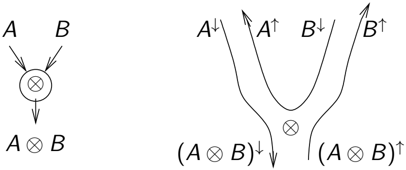


– each **conclusion** C is replaced by a 2-dimensional "*cul-de-sac*":

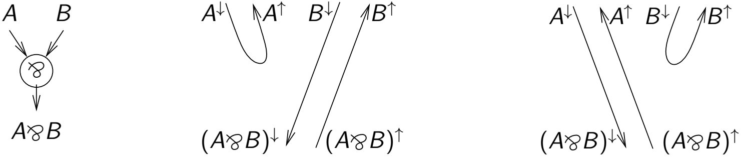


ribbon diagrams

– each $\otimes$-**link** is replaced by the following ribbon diagram



– each $⅋$-**link** is replaced by the choice of one of the two ribbon diagrams



$⅋_R$-switching position or $⅋_L$-switching position

the border orientation defines a trajectory for a particle visiting the proof

topological correction criterion (Mellies, 2003)

Definition (switching or test)

Given a proof-structure π, a **switching or test** $S(\pi)$ is the ribbon surface obtained by replacing every link and conclusion by (the choice of one of) the associated ribbon diagrams and pasting all diagrams together.

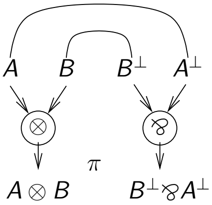
Definition (topological criterion)

A **proof-net** (PN) is a proof-structure π such that **each switching** $S(\pi)$ (ribbon surface) is **homeomorphic to the disk**.

Remarks

1. *intuitively, an "homeomorphism" is a map between topological spaces modeling a "deformation without tearing"*
2. *this criterion is equivalent to the original Girard's one (1987) based on* **long trips**: *a proof-structure is a proof-net when, for every switching, the particle visits every part of the proof net without being captured into a (proper) cycle.*

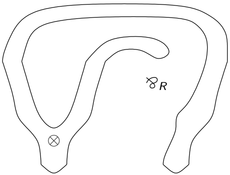
an instance of correct proof structure



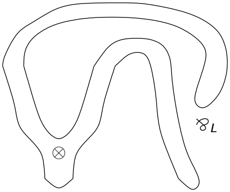
sequentializes into:

$$\dfrac{\dfrac{\vdash A, A^\perp \qquad \vdash, B, B^\perp}{\vdash A \otimes B, B^\perp, A^\perp}\otimes}{\vdash A \otimes B, B^\perp \invamp A^\perp}\invamp$$

there are only two switchings: both of them homeomorphic to the disk:

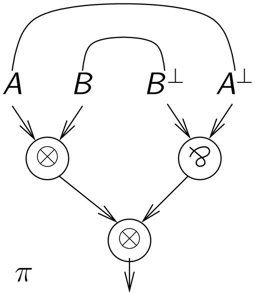


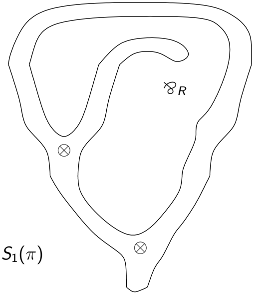
$S_1(\pi)$



$S_2(\pi)$

an instance of un-correct proof structure





π is not correct (i.e., it is not image of any derivation) since there exists a switching $S_R(\pi)$ that is not homeomorphic to the disk (symmetrically, the switching $S_L(\pi)$ with position $⅋_L$)

a larger fragment: MALL

Since its inception (1987) the problem of finding a "good notion" of MALL proof nets has remained open. Some works on MALL

- 1996, Girard, Monomial Nets;
- 2003, Hughes-van Glabbeek, Linkings Nets;
- 2005, Curien-Faggian, Ludics-nets;
- 2007, Maieli, Contractible MALL PN;
- 2008, Maieli-Laurent, strong normalization for Monomial PN;
- 2008, Mogbil-de Naurois. Correctness of MALL is PS NL-Complete;
- 2008, Tortora de Falco, GoI for MALL;
- 2011, Heijltjes, MALL proof nets with units;
- 2015, Bagnol, MALL proof equivalence is Logspace-complete.
- ...

MALL Sequent Calculus

Formulas $A, B, \ldots$ are built from *literals* by the binary connectives $\otimes$ (*tensor*), $\parr$ (*par*), $\&$ (*with*) and $\oplus$ (*plus*).

Negation $(.)^\perp$ extends to any formula by de Morgan laws:

$$(A \otimes B)^\perp = (B^\perp \parr A^\perp) \qquad (A \parr B)^\perp = (B^\perp \otimes A^\perp)$$
$$(A \& B)^\perp = (B^\perp \oplus A^\perp) \qquad (A \oplus B)^\perp = (B^\perp \& A^\perp)$$

MALL *Sequents* Γ, Δ are proved using the following rules:

- identity: $\dfrac{}{A, A^\perp}\,\text{ax} \qquad \dfrac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\,\text{cut}$
- multiplicatives: $\dfrac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\,\otimes \qquad \dfrac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr$
- **additives**: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\,\& \qquad \dfrac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \dfrac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2$

MALL Proof Structures (PSs)

Concerning PSs, the **problem** is to cope with the &-rule for which a **superposition** of two proof structures $\pi_{\Gamma,A}$ and $\pi_{\Gamma,B}$ must be made.

A **solution** is to introduce for each &-link a **boolean variable**

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A\&B}\,\&_p \qquad p \text{ is called } \textit{eigen-wight}$$

which distinguishes between two **slices** of the superposition:

$$\bar{p}\ \textit{slice} \qquad\qquad p\ \textit{slice}$$

$$\frac{\qquad \Gamma, A}{\Gamma, A\&B}\,\&_p \qquad\qquad \frac{\Gamma, B \qquad}{\Gamma, A\&B}\,\&_p$$

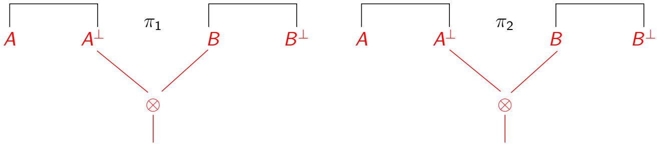
This immediately opens to the **problem of which kind of superposition** can be performed over already de-sequentialized PSs?

Example

Assume a sequential proof as follows:

$$\Pi : \dfrac{\Pi_1 : \dfrac{\dfrac{\overline{A, A^\perp} \quad \overline{B, B^\perp}}{A, A^\perp \otimes B, B^\perp}\otimes}{A \oplus A, A^\perp \otimes B, B^\perp}\oplus_1 \quad \Pi_2 : \dfrac{\dfrac{\overline{A, A^\perp} \quad \overline{B, B^\perp}}{A, A^\perp \otimes B, B^\perp}\otimes}{A \oplus A, A^\perp \otimes B, B^\perp}\oplus_2}{A \oplus A, A^\perp \otimes B, B^\perp \&_p B^\perp}\&_p$$

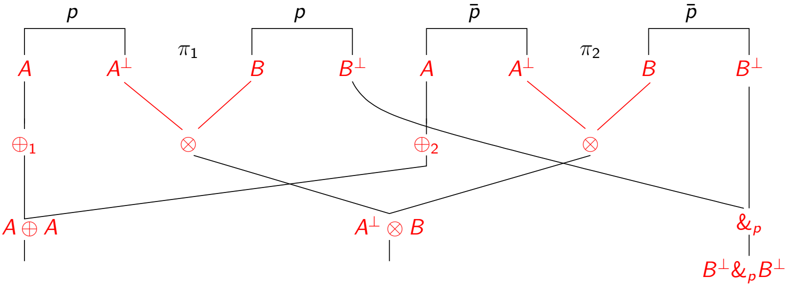
By hypothesis of induction (MLL case):
Π_1 desequentializes in to π_1 and Π_2 desequentializes in to π_2 as below



But, then there are different possibilities of superposing π_1 and π_2 in order to get a proof structure π that is a desequentialization of Π.

solution 1: minimal superposition

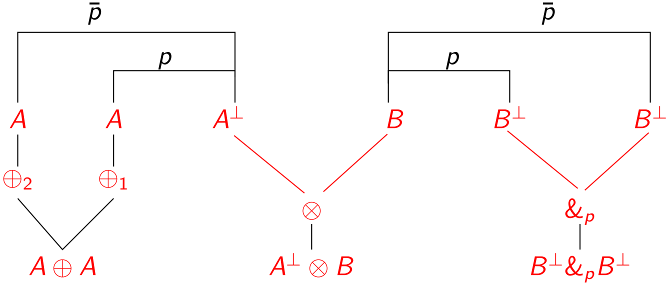
only the conclusions superpose



this solution allows to **preserve the monomiality of boolean weights** associated to a proof structure [Girard, 1996]

solution 2: “moderate” superposition

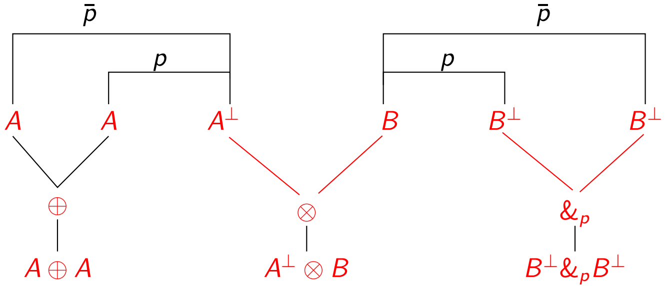
only some links (other than conclusions) superpose:



different choices of unary $\oplus$-links cannot be superposed!

solution 3: maximal superposition

all formula-tree conclusions superpose (like in the MLL case).



Warning! this solution:

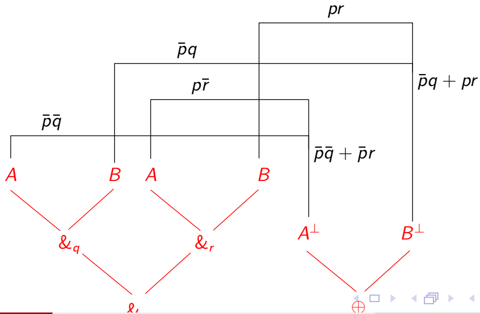
– requires binary $\oplus$-links;

– disrupts the monomiality of weights [Hughes-Van Glabbeek, 2003]

an example of PS with non-monomial weights

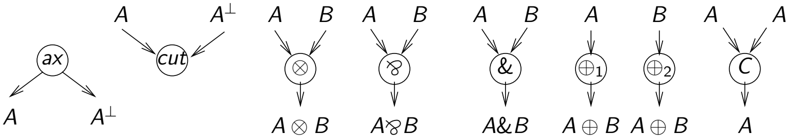
$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, A^\perp}\,\bar{p}\bar{q}\ \mathrm{ax}}{A, A^\perp \oplus B^\perp}\oplus_1 \quad \dfrac{\dfrac{}{B, B^\perp}\,\bar{p}q\ \mathrm{ax}}{B, A^\perp \oplus B^\perp}\oplus_2}{A \&_q B, A^\perp \oplus B^\perp}\&_q \quad \dfrac{\dfrac{\dfrac{}{A, A^\perp}\,p\bar{r}\ \mathrm{ax}}{A, A^\perp \oplus B^\perp}\oplus_1 \quad \dfrac{\dfrac{}{B, B^\perp}\,pr\ \mathrm{ax}}{B, A^\perp \oplus B^\perp}\oplus_2}{A \&_r B, A^\perp \oplus B^\perp}\&_r}{(A \&_q B) \&_p (A \&_r B), A^\perp \oplus B^\perp}\&_p$$

this proof de-sequentializes into a PS with some links weighted by "non-monomial" weights.
Still, terminal (conclusion) links are labeled by the monomial weight 1.



MALL Proof Structures: *links*

An MALL link is a graph of the following form, whose edges (resp., vertexes) are labeled with MALL formulas (resp., connectives):



Entering (resp. emerging) edges are **premises** (resp. **conclusions**)

Pending edges are called **conclusions of** π.

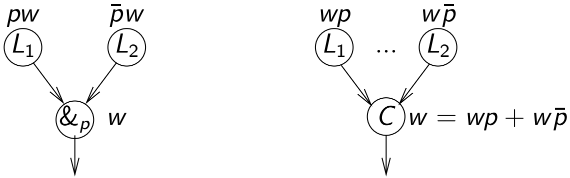
MALL Proof Structures: *weights*

- a set of Boolean variables denoted by $p, q, \ldots$
- a **monomial weight** $w, v, \ldots$ is a product "." (conjunction) of variables or negation of variables
- 1, for the empty product
- 0, for a product where both p and $\bar{p}$ appear
- two weights, v and w, are **disjoint** when $v.w = 0$
- a weight w **depends** on a variable p when p or $\bar{p}$ appears in w

MALL Proof Structures: definition

A PS π is a graph built on links with assigned **weights** as follows:

1. we assign a (different) *eigen weight* p, to each & node of π (notation $\&_p$):
2. we assign a weight $w \neq 0$ to each node; two nodes have the same weight if they have a common edge, except when:

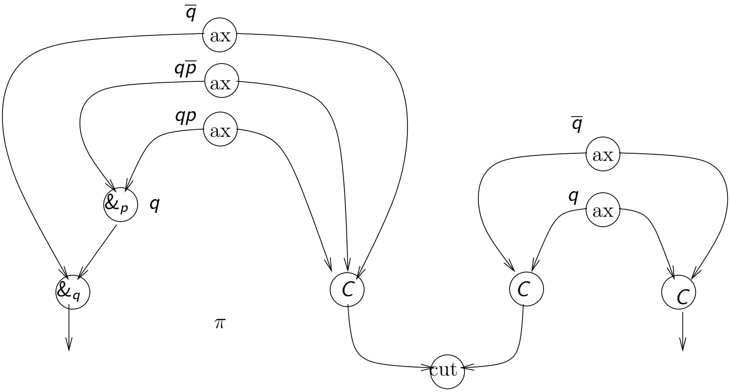


neither p nor $\bar{p}$ occurs in w $\qquad$ $wp.w\bar{p} = 0$

3. a conclusion node has weight 1;
4. if w in π depends on p, then $w \leq v$, where v is the weight of the $\&_p$ node (**monomial condition**).

MALL Proof Structures: example 1

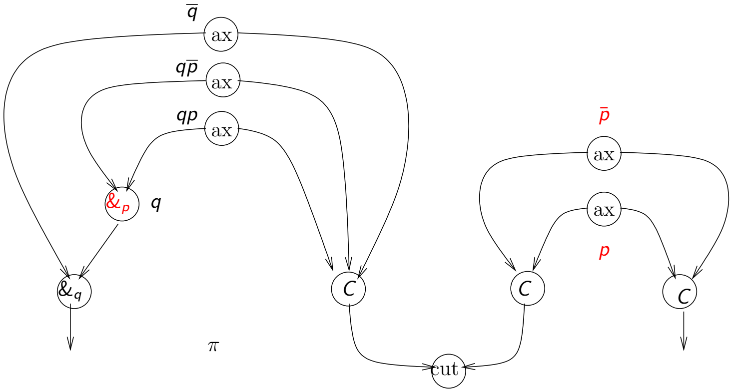
The following is a PS:



Observe that $q, \bar{q} \leq 1$

MALL Proof Structures: (counter-)example 2

The following is not a PS:



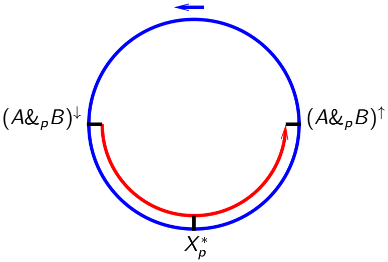
Remark. it violates the *monomial condition* on PS: there exists a axiom (with weight p) depending on $\&_p$ (with weight q) but $p \not\leq q$

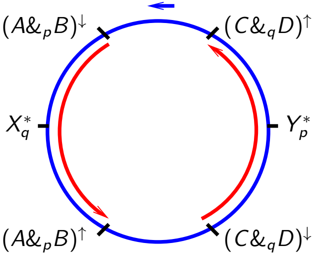
Proof Nets: valuation, slices, switchings, criterion

- a **valuation** for π is a function $\varphi : p \mapsto \{0,1\}$ ($\varphi : w \mapsto \{0,1\}$)
- fixed a φ, a **slice** $\varphi(\pi)$ is the graph obtained from π by keeping only those nodes (together its emerging edges) whose weights are $\neq 0$.

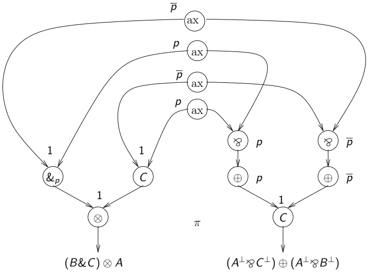
Definition (topological criterion)

A PS π is correct if, for each slice, every switching is homeomorphic to the disk with the (oriented) **border not containing** any **red trip** as below



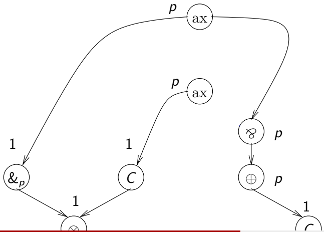


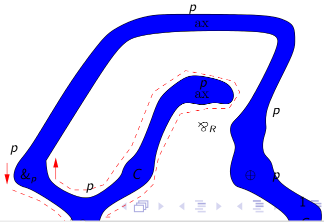
an instance of un-correct PS



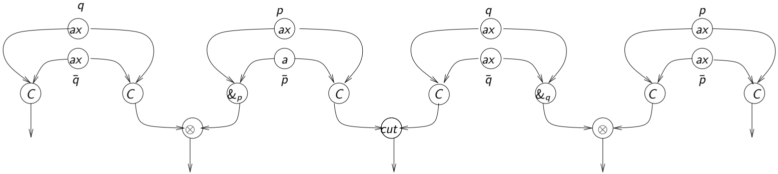
$\exists$ a ribbon $S(\pi)$ whose oriented border contains a "bad trip"

$\ldots, (\&_p)^{\downarrow}, \ldots, A_p^{\uparrow}, A_p^{\downarrow}, \ldots, (\&_p)^{\uparrow}, \ldots$



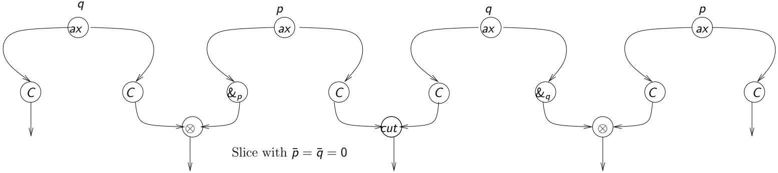


an instance of un-correct PS

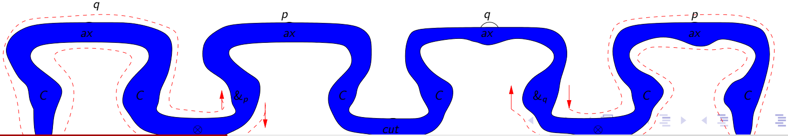


∃ a ribbon $S(\pi)$ whose border contains a “bad trip”

$\ldots, (\&_p)^{\downarrow}, \ldots, ax_q^{\uparrow}, \ldots ax_q^{\downarrow}, \ldots, (\&_p)^{\uparrow}, \ldots, (\&_q)^{\downarrow}, \ldots, ax_p^{\uparrow}, \ldots ax_p^{\downarrow}, \ldots, (\&_q)^{\uparrow}, \ldots$



Slice with $\bar{p} = \bar{q} = 0$



thank you for your kind attention!