Retractile Proof Nets of MALL

(Purely **M**ultiplicative and **A**dditive Fragment of **L**inear **L**ogic)

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- **Answer** : a correctness criterion formulated like an *algorithm* which implements simple *graph rewriting rules*.
- **Hint** : an *initial idea* of a retraction correctness criterion for proof nets of MLL, the purely multiplicative fragment of linear logic (Danos, 1990).

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  - additives: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\,\&$ $\qquad \dfrac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1$ $\qquad \dfrac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2$

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- This idea leads to the notion of proof structure (*graph*).
- In particular, some proof structures (proof nets) can be seen as quotients of classes of sequent proofs that are *equivalent modulo irrelevant permutation of sequent rules*.

Proof Structures (PS)

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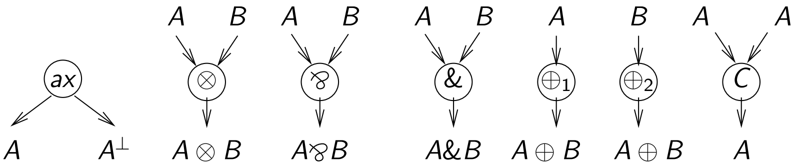

Figure: MALL Links

- entering (resp., exiting) edges are *premises* (resp., *conclusions*)
- pending edges are called *conclusions* of PS

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- For doing that we need to go trough some more abstract objects (*Abstract Proof Structures*) which allow us to get rid of some concrete matters of proof structures

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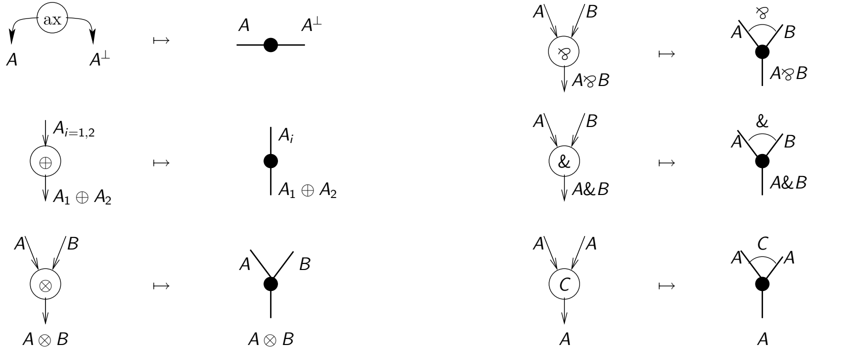
– a PN is mapped into an APS as follows:



The Correctness Criterion: Proof Nets

A PS π with conclusions $A_1, ..., A_n$, with $n \geq 1$, is *correct* (i.e., it is a proof net) if its corresponding APS π^* *retracts* to a single node $\bullet$, by iterating the following retraction rules $(R_1, ..., R_5)$

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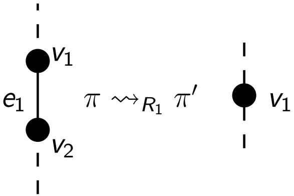
$$\text{a PS } \pi \text{ is a PN iff its corresponding APS } \pi^* \rightsquigarrow^* \bullet$$

Multiplicative Retraction Rules: R_1 and R_2



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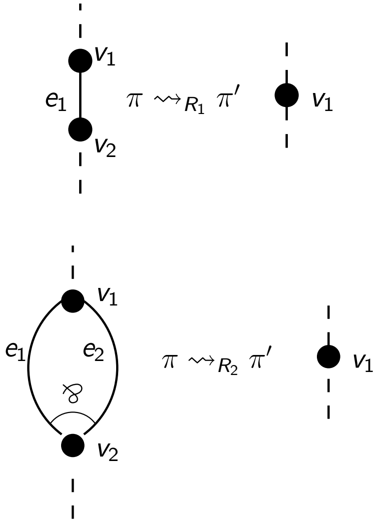
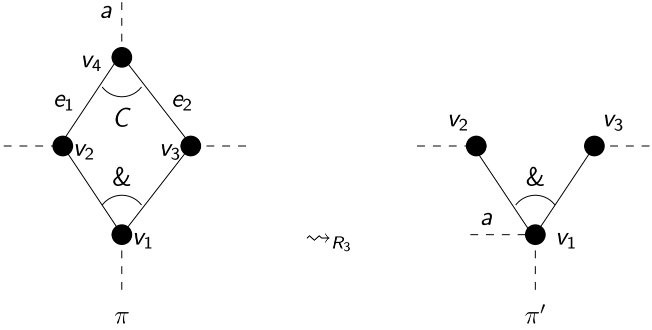


Additive Retraction Rules: R_3 and R_4



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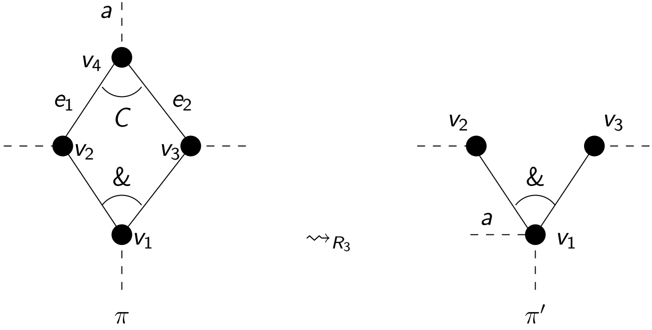
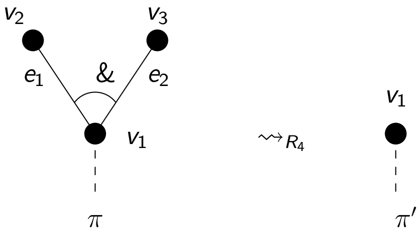
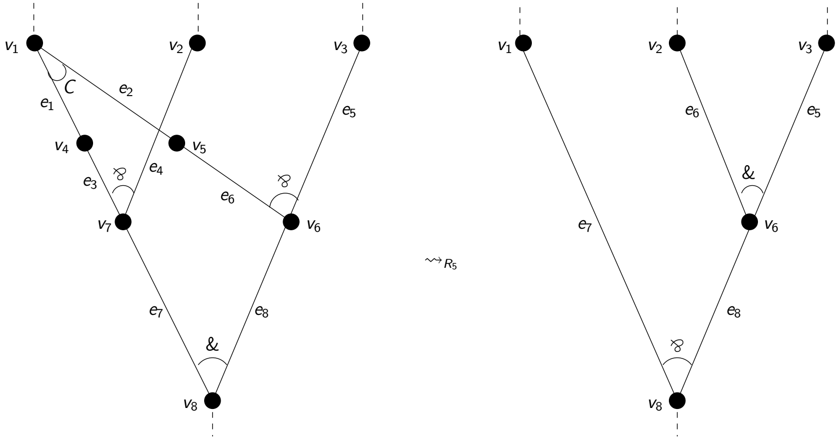


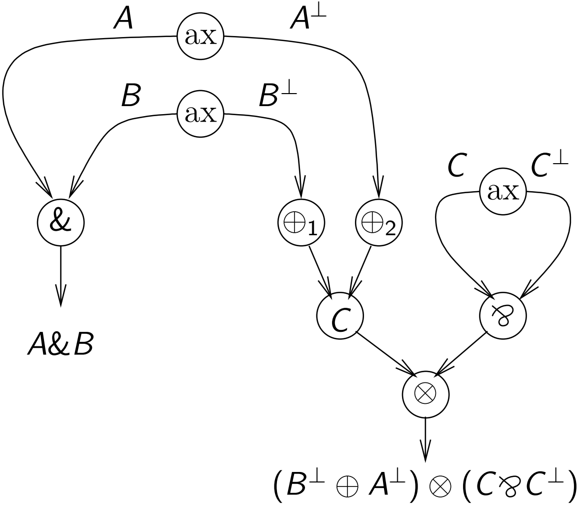


Distributive Retraction Rule: R_5



$$(b ⅋ c) \& (b ⅋ d) \vdash b ⅋ (c \& d)$$

An example of Proof Net π (1/2)



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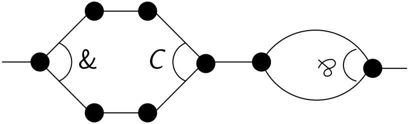

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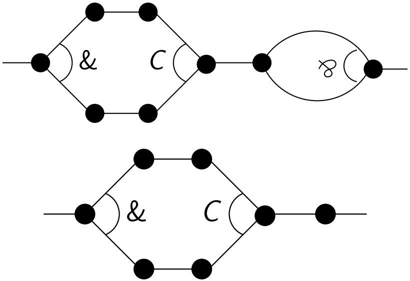

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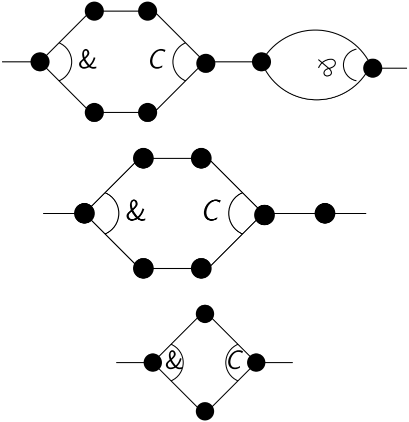

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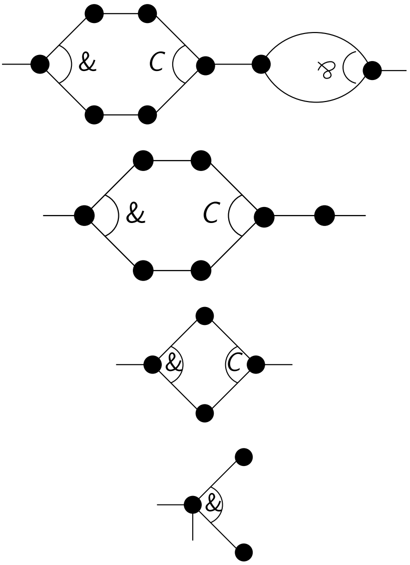

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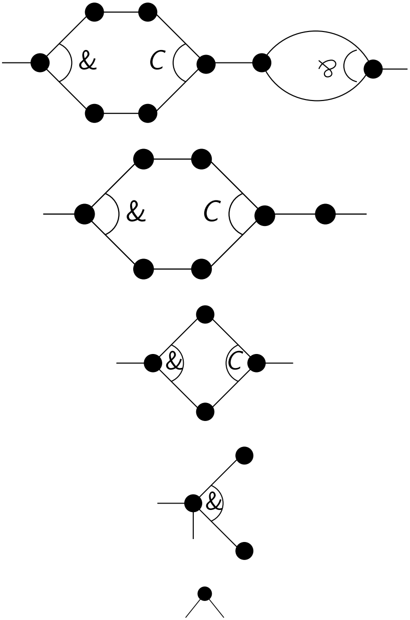

De-squentialization Theorem

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Proof.

We associate a link to each derivation rule, then we proceed by induction on π. □

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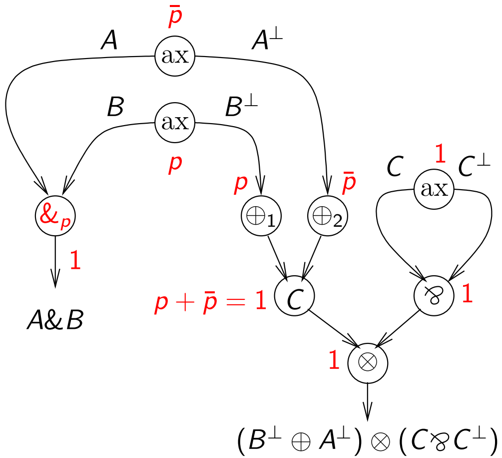

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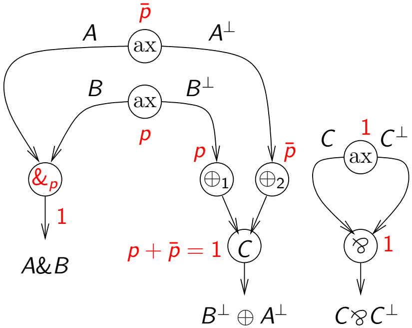

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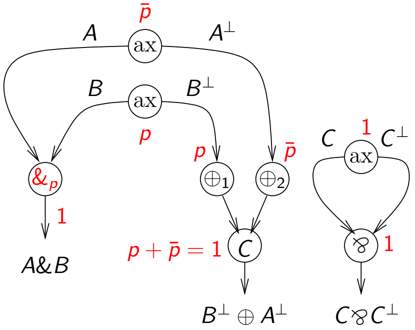

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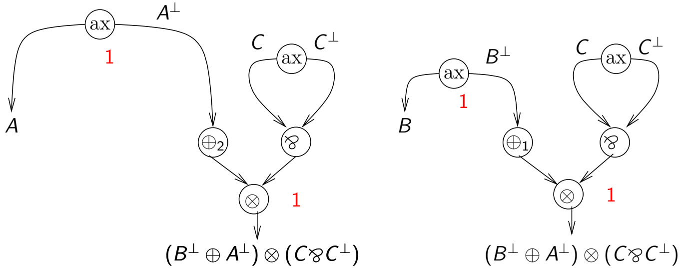

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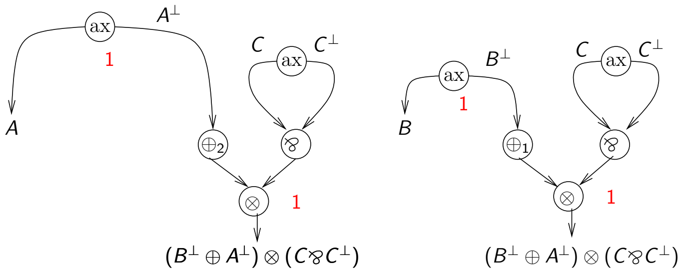

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Confluence

Theorem

If a PN π retracts to $\bullet$, then all retraction sequences start with π^* and terminate with $\bullet$.

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- lead to possible applications like Transactional Systems, navigation of Formal Ontologies ...