Interactive correctness criterion for multiplicative-additive proof-nets

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Proof-nets are special graphs (proof-structures) representing de-sequentialised proofs of the linear logic sequent calculus. Each proof-net stands for a class of sequent proofs which are equivalent modulo irrelevant permutations of logical rules.

Here we present an interactive characterisation of those cut-free proof-structures coming from proofs of the multiplicative-additive fragment of linear logic (MALL, see [1, 3]). This work is intended to extend to MALL proof-nets an original proposal in [2]: see Appendix E.7 of [2] for an interactive correction criterion for proof-nets of the multiplicative fragment of linear logic. Its natural consequence we will be the study of the question of *modularity* for additive proof-nets.

A MALL **proof** is a proof built with the rules of Figure 1.

$$\frac{}{X, X^{\perp}}\ \textit{logical axiom} \qquad \frac{\Gamma, A \qquad A^{\perp}, \Delta}{\Gamma, \Delta}\ \textit{cut}$$

$$\frac{\Gamma, A \qquad B, \Delta}{\Gamma, (A \otimes B), \Delta}\ \otimes \qquad \frac{\Gamma, A, B}{\Gamma, (A ⅋ B)}\ ⅋$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, (A \& B)}\ \& \qquad \frac{\Gamma, A}{\Gamma, (A \oplus B)}\ \oplus_1 \qquad \frac{\Gamma, B}{\Gamma, (A \oplus B)}\ \oplus_2$$

Figure 1. MALL sequent proofs calculus

A MALL **proof-structure** with a unique conclusion F (denoted π_F) is the graph obtained by gluing the formula tree of F together with a set of *axiom links* (a set of pairs of orthogonal *literals* A_i and $A_i^{\perp}$) via the border of F. Each &-vertex of F is equipped with a distinct variable, called **eigen-weight**, $x, y, z, ...$; each axiom link is then labelled with a **weight**, i.e., a term in the Boolean algebra generated by the set of eigen-weights of F (see an example in Figure 2).

Any proof in the sequent calculus is trivially mapped to a proof structure, but of course, not every proof-structure is correct in the sense that it arises from a sequent calculus proof. It is therefore crucial to be able to characterise

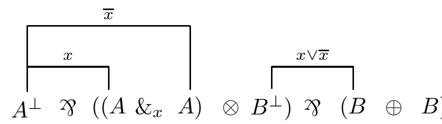

Figure 2. a MALL proof-structure

the correct proof-structures, if possible by simple geometrical means. It is moreover natural to look for an interactive characterisation, i.e., such that checking the correctness of a proof-structure π_F only amounts to play with a set of "para-proofs" (or "tests") $\sigma_{F^{\perp}}$ associated to the orthogonal formula $F^{\perp}$: an interaction between π_F and a $\sigma_{F^{\perp}}$ is then given by the cut-reduction between F and $F^{\perp}$, which may converge or not.

Formally, a MALL **para-proof** is a single-conclusion derivation, built bottom-up, by means of the rules of Figure 3. Each formula is possibly equipped with a *pseudo-weight*, i.e., a conjunction ($\wedge$) of variables $x, y, ...$ or negations of variables $\overline{x}, \overline{y}....$

$$\frac{}{L^1_{w_1}, ..., L^n_{w_n}}\ \textit{proper axiom} \qquad \frac{\Gamma, A_w \qquad A^{\perp}_{w'}, \Delta}{\Gamma, \Delta}\ \textit{cut}$$

(w_i, $1 \leq i \leq n$, is, possibly, a weight and w, w' may be equal)

$$\frac{\Gamma, A_w \qquad B_w, \Delta}{\Gamma, (A \otimes B)_w, \Delta}\ \otimes \qquad \frac{\Gamma, A_w, B_w}{\Gamma, (A ⅋ B)_w}\ ⅋$$

$$\frac{\Gamma, A_w}{\Gamma, (A \& B)_w}\ \&_1 \qquad \frac{\Gamma, B_w}{\Gamma, (A \& B)_w}\ \&_2$$

$$\frac{\Gamma, A_{w \wedge x}}{\Gamma, (A \oplus B)_w}\ \oplus_x \qquad \frac{\Gamma, B_{w \wedge \overline{x}}}{\Gamma, (A \oplus B)_w}\ \oplus_{\overline{x}}$$

(where x does not occur in w neither in any weight of Γ)

Figure 3. MALL para-proofs calculus

A para-proof π *reducts a (ready) cut in one step*, in symbols $\pi \mapsto \pi'$, in the following cases (*convergence*):

$$\frac{\dfrac{\Gamma, A \quad B, \Delta}{\Gamma, \Delta, A \otimes B} \quad \dfrac{A^\perp, B^\perp, \Sigma}{A^\perp \parr B^\perp, \Sigma}}{\Gamma, \Delta, \Sigma} \mapsto \frac{\dfrac{\Gamma, A \quad A^\perp, B^\perp, \Sigma}{\Gamma, \Sigma, B^\perp} \quad B, \Delta}{\Gamma, \Sigma, \Delta}$$

$$\frac{\dfrac{\Gamma, A}{\Gamma, A \& B} \quad \dfrac{A^\perp, \Delta}{A^\perp \oplus B^\perp, \Delta}}{\Gamma, \Delta} \mapsto \frac{\Gamma, A \quad A^\perp, \Delta}{\Gamma, \Delta}$$

$$\frac{\dfrac{\Gamma, B}{\Gamma, A \& B} \quad \dfrac{B^\perp, \Delta}{A^\perp \oplus B^\perp, \Delta}}{\Gamma, \Delta} \mapsto \frac{\Gamma, B \quad B^\perp, \Delta}{\Gamma, \Delta}$$

A para-proof π *does not reduct a ready cut* ($\pi \mapsto \emptyset$, the proof is destroyed) in the following cases (*divergence*):

$$\frac{\dfrac{\Gamma, A}{\Gamma, A \& B} \quad \dfrac{\Delta, B^\perp}{\Delta, A^\perp \oplus B^\perp}}{\Gamma, \Delta} \mapsto \emptyset\,, \quad \frac{\dfrac{\Gamma, B}{\Gamma, A \& B} \quad \dfrac{\Delta, A^\perp}{\Delta, A^\perp \oplus B^\perp}}{\Gamma, \Delta} \mapsto \emptyset$$

A $\&$**-valuation** φ of a proof-structure π_F is a choice of one premise, left or right, for each $\&_x$-vertex in the formula tree of F.

Let φ be a valuation of π_F, σ_F be the sub-graph of π, called the φ**-slice**, which holds under φ, and $\tau_{F^\perp}$ be a para-proof of $F^\perp$, called a φ**-test**, built according to φ (choose an instance of $\oplus_x$-rule, resp. $\oplus_{\overline{x}}$-rule, if x, resp. $\overline{x}$, holds in φ). Then a φ**-interaction** of π is the graph, $\langle \sigma_F, \tau_{F^\perp} \rangle_\varphi$ obtained after (iteratively) reducing the cut $\frac{}{F \quad F^\perp}$ between σ_F and $\tau_{F^\perp}$.

A φ-interaction is said to be **complete** if it does not contain an axiom link with a *pending conclusion* (i.e., a conclusion which is not premise of any cut link).

An **interaction session** is obtained from a union of φ_i-interactions $\cup_{i \in I} \langle \sigma_F, \tau_{F^\perp} \rangle_{\varphi_i}$ by erasing each (proper) axiom link l that is common to all interactions $\langle \sigma_F, \tau_{F^\perp} \rangle_{\varphi_i}$, together with all those cut-links having l as a premise.

A pair of literals $L_w, L'_{w'}$ is said to be a **toggling pair** if its weights w and w' contain dual occurrences of a same variable x (e.g., $x \in w$ and $\overline{x} \in w'$).

A **critical cycle** is a cycle (in an interaction session) containing a unique toggling pair.

Interactive Correction Criterion : A proof-structure π_F with a single conclusion F is **correct**, or is a **proof-net**, when *(1)* any interaction $\langle \sigma_F, \tau_{F^\perp} \rangle_\varphi$, induced by a valuation φ is complete, acyclic and connected, and *(2)* any interaction session $\langle \sigma_F, \tau_{F^\perp} \rangle_{\varphi_i}, i \in I$, of at least 2 interactions, contains a toggling pair not occurring in any critical cycle.

Example. Assume we want interactively check the correctness of the proof-structure π of Figure 2. We first have to verify condition *(1)*: actually, there are only two possible interactions for π, one induced by $\varphi(x) = left$ (Figure 4) and the other one induced by $\varphi(x) = right$ (Figure 5); both are complete, connected and acyclic. Finally, in order to verify condition *(2)*, we only have to check that the unique

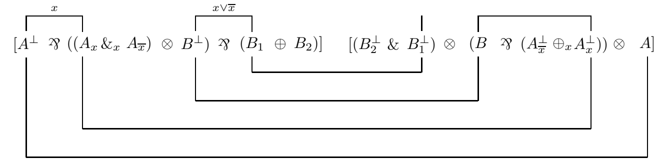

Figure 4. the unique $\varphi(x) = left$ interaction

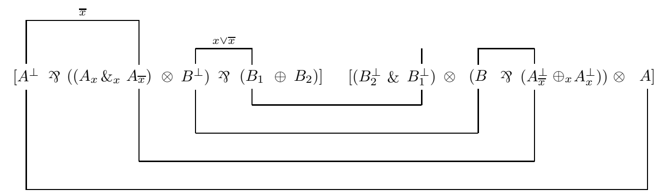

Figure 5. the unique $\varphi(x) = right$ interaction

interaction session of Figure 6, obtained by superposing the two interactions above contains a toggling pair not in a critical cycle, and that is the case.

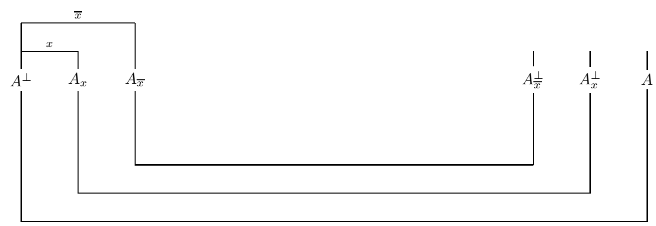

Figure 6. the unique interaction session of π

Theorem. Any MALL sequent proof of Γ can be desequentialised into a proof-net with the unique conclusion $\parr(\Gamma)$, and vice-versa.

We prove this by showing that a proof-structure is a proof-net iff it is so according to [3].

References

[1] J.-Y. Girard. *Proof-nets: the parallel syntax for proof theory*. Marcel Dekker, 1996.

[2] J.-Y. Girard. On the meaning of logical rules i: syntax vs. semantics. *Computational Logic*, eds. Berger and Schwictenberg, SV, Heidelberg:215–272, 1999.

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