ROMA TRE UNIVERSITÀ DEGLI STUDI
DIPARTIMENTO DI MATEMATICA E FISICA

# A proof of the Focusing Theorem via MALL proof nets

**(Focusing MALL Proof Nets)**

Roberto Maieli

Iaşi, 22 September 2022 – WoLLIC 2022

# introduction + outline

1. **Focusing** (Andreoli 1992) is "usually" presented as:
    - an efficient **proof search procedure**
    - a **proof normalization** result of the sequent system of Linear Logic (LL, Girard 1987).
2. We reformulate Focusing in terms of **Proof Nets** (PN) of LL:
    - PNs are a graphical syntax that allows abstract/canonical representations of proofs.
3. We show Focusing as a particular **sequentialization strategy** of PNs, in 2 steps:

    3.1 we recall the pure multiplicative case, **MLL** (Andreoli-Maieli)

    3.2 we analyse the pure multiplicative and additive case **MALL** (Maieli)

    In particular, we introduce **Canonical MALL PNs** satisfing the property that:

    - ∀ **asynchronous conclusion links** are always ready to sequentialization
    - ∃ a **synchronous conclusion link** (foc-link) that is hereditary ready to sequentialization.

## the pure (without units) multiplicative fragment of Linear Logic (MLL)

- Multiplicative linear logic (MLL) **formulas** are generated from a countable set $\mathcal{P} = \{P, Q, ...\}$ of propositional variables by the following grammar:

$$A, B ::= P \mid A^{\perp} \mid A \parr B \mid A \otimes B$$

- Formulas are considered modulo the **involutivity of the negation**

$$A^{\perp\perp} = A$$

and the **De Morgan laws**

$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}.$$

- A MLL **sequent** $\Gamma$ is a multiset of formulas $A_1, ..., A_n$.

- A MLL **proof** of $\Gamma$ is a tree built by the system $\Sigma_1$ of inference rules with root $\Gamma$:

$$\frac{}{A, A^{\perp}}\ \text{ax/id} \qquad \frac{\Gamma, A \qquad \Delta, A^{\perp}}{\Gamma, \Delta}\ \text{cut} \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr$$

Sequents are one-sided, so we omit turnstiles ($\vdash$).

# the MLL focusing dyadic system $\Sigma_2$ (Andreoli, 1992)

W.r.t. MLL proof search:

- **connectives** are split into two categories:
    - **asynchronous** (or **negative**) $\parr$, introduces a kind of "don't care non-determinism",
    - **synchronous** (or **positive**) $\otimes$, introduces a kind of "true non-determinism".
- conventionally, the class of **literals** is split into two dual, disjoint sub-classes:
    - the **positive** atoms $P, Q, \ldots$ and
    - their **negative** duals $P^\perp, Q^\perp, \ldots$
    - by involutivity of negation: $P^{\perp\perp} = P, \quad Q^{\perp\perp} = Q, \ldots$

# the MLL focusing dyadic system $\Sigma_2$ (Andreoli, 1992)

Focusing sequents are of two types:

$$\vdash \Gamma \Uparrow L \qquad \vdash \Gamma \Downarrow F$$

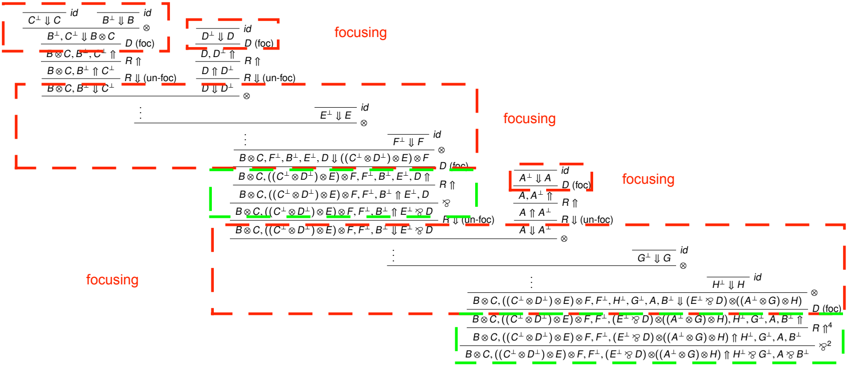
- $\Gamma$ is a multiset of non-asynchronous formulas
- $L$ is a list of formulas and $F$ is a single formula called the "focus" of the sequent.
- **Logical rules**: $⅋, \otimes$

negative $\dfrac{\Gamma \Uparrow L, F, G}{\Gamma \Uparrow L, F ⅋ G}\,⅋$ $\qquad$ $\dfrac{\Gamma \Downarrow F \qquad \Delta \Downarrow G}{\Gamma, \Delta \Downarrow F \otimes G}\,\otimes$ positive

- **Identity** [*id*] : if $P$ is a positive atom $\dfrac{}{P^\perp \Downarrow P}\,id$
- **Decision** [$D$] : if $F$ is synchronous or a positive atom $\dfrac{\Gamma \Downarrow F}{\Gamma, F \Uparrow}\,D$ (**foc**)
- **Reaction** [$R\Downarrow$] : if $F$ is neither synchronous nor a positive atom $\dfrac{\Gamma \Uparrow F}{\Gamma \Downarrow F}\,R\Downarrow$ (**un-foc**)
- **Reaction** [$R\Uparrow$] : if $F$ is not asynchronous $\dfrac{\Gamma, F \Uparrow L}{\Gamma \Uparrow L, F}\,R\Uparrow$

# an example of focusing MLL proofs

focused proofs[1] "alternate" clusters of negative rules with focused clusters of positive rules:




[1] Focused proofs are in "normal form", i.e., cut-free

# The Focusing Theorem (Andreoli, 1992)

## Theorem

*If Γ is a multiset of non-asynchronous formulas and L an ordered list of formulas then:*

$$\vdash_{\Sigma_1} \Gamma, L \text{ if and only if } \vdash_{\Sigma_2} \Gamma \Uparrow L$$

Andreoli's Proof: by induction on the sequent proofs (via rules permutations).

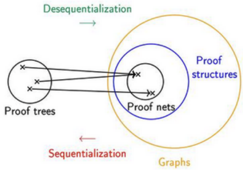
---

Alternative Proof: via sequentialization of PNs of linear logic.

$$\begin{array}{lccccl} \text{sequent proof :} & \Pi & \xrightarrow{\text{de-sequentialization}} & \pi & & \text{: proof net} \\ & ?\downarrow & & \downarrow & & \\ \text{focused proof :} & \Pi' & \xleftarrow{\text{foc-sequentialization}} & \pi' & & \text{: canonical proof net} \end{array}$$

---

**Proof nets**: graphical, more canonical representation of LL proofs




MALL Proof Net:

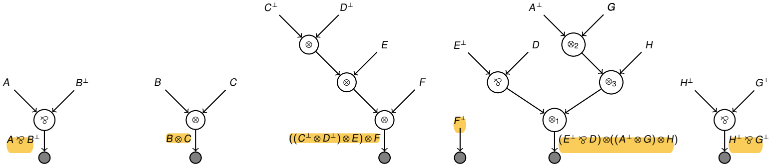
- Girard 1996
- Hughes & Van Glabbeek 2005

# focusing MLL proofs via proof nets: phase 1 de-sequentialization

Any MLL (standard) sequent proof can be focalized as follows:

$\Pi :\vdash^1_{MLL} A \parr B^\perp, B \otimes C, ((C^\perp \otimes D^\perp) \otimes E) \otimes F, F^\perp, (E^\perp \parr D) \otimes ((A^\perp \otimes G) \otimes H), H^\perp \parr G^\perp$

**phase 1**: de-sequentialization of sequent MLL proofs (by induction on the proof size).

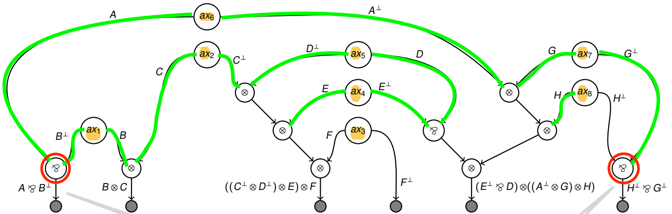



**step 1:** for each conclusion $X_i$ we build the the formula tree of $X_i$

# focusing MLL proofs via proof nets: phase 1 de-sequentialization

$\Pi :\vdash^1_{MLL} A \parr B^\perp, B \otimes C, ((C^\perp \otimes D^\perp) \otimes E) \otimes F, F^\perp, (E^\perp \parr D) \otimes ((A^\perp \otimes G) \otimes H), H^\perp \parr G^\perp$

**phase 1**: de-sequentialization of sequent MLL proofs (by induction on the proof size).




**step 2:** we match dual pairs of literals leaves according to the sequent axioms policy or, alternatively, put the **axiom links** in such a way that the resulting graph (**proof structure**) is **correct** (i.e, it is a **proof net**) that is, every **switching** (a mutilation of a premise of each $\parr$-link) is a **ACC graph**.

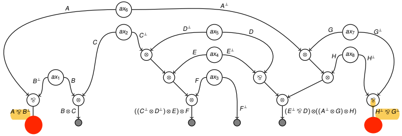
# focusing MLL proofs via proof nets: phase 2 sequentialization

Focusing sequentialization of a PN proceeds by induction on the size of the PN.

At each induction steps, there are three cases to consider:

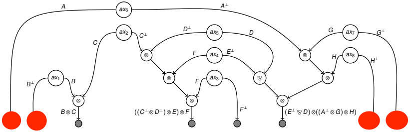
**Case 1**: If the PN contains (at least) an **asynchronous conclusion** then:

negative




negative

1. remove the corresponding **asynchronous ($\parr$) link**




2. **recursively** apply the focusing **sequentialization to the remaining sub-PN**

$$\frac{\vdots}{B \otimes C, ((C^\perp \otimes D^\perp) \otimes E) \otimes F, F^\perp, (E^\perp \parr D) \otimes ((A^\perp \otimes G) \otimes H) \Uparrow H^\perp, G^\perp, A, B^\perp}$$

3. complete the proof obtained with the corresponding **asynchronous inference**.

$$\dfrac{\dfrac{\vdots}{B \otimes C, ((C^\perp \otimes D^\perp) \otimes E) \otimes F, F^\perp, (E^\perp \parr D) \otimes ((A^\perp \otimes G) \otimes H) \Uparrow H^\perp, G^\perp, A, B^\perp}}{B \otimes C, ((C^\perp \otimes D^\perp) \otimes E) \otimes F, F^\perp, (E^\perp \parr D) \otimes ((A^\perp \otimes G) \otimes H) \Uparrow H^\perp \parr G^\perp, A \parr B^\perp} \parr^2$$

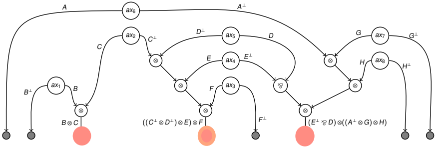
negative rule

# focusing MLL proofs via proof nets: phase 2 sequentialization

Focusing Sequentizilization of a PN proceeds by induction on the size of the PN.

At each induction steps, there are three cases to consider:

**Case 2**: the PN has no asynchronous conclusion but at least one **synchronous concl.**

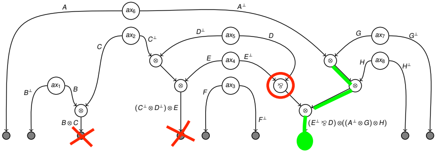



**positive**

then by **Splitting Theorem** the set of splitting links of $\pi'$ is not empty:

$$\{B \otimes C, \quad ((C^\perp \otimes D^\perp) \otimes E) \otimes F, \quad (E^\perp \invamp D) \otimes ((A^\perp \otimes G) \otimes H)\} = Split(\pi')$$

But not all of them are focusing links i.e., **hereditarily splitting**!
E.g. $((C^\perp \otimes D^\perp) \otimes E) \otimes F$ is not hereditary splitting: $(C^\perp \otimes D^\perp) \otimes E$ is blocked by a $\invamp$-link.




**focusing**

there is **only one focusing conclusion link** in $\pi'$: $(E^\perp \invamp D) \otimes ((A^\perp \otimes G) \otimes H)$

# focusing as hereditary splitting

## Definition (focusing conclusion)

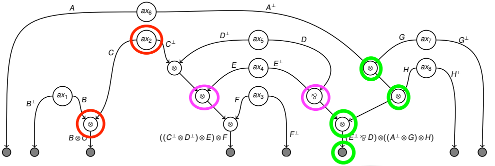
A conclusion $F$ is focusing for a PN $\pi$, $F \in Foc(\pi)$, iff one of these 2 conditions holds:

1. F is a positive atom and $\pi$ is reduced to an axiom link.
2. $F = \frac{A \quad B}{A \otimes B} \in Split(\pi)$ and $\pi$ is split at $F$ into two sub-PNs, $\pi_A$ and $\pi_B$, and
   - 2.1 $A$ is asynchronous or a negative atom or $A \in Foc(\pi_A)$,
   - 2.2 $B$ is asynchronous or a negative atom or $B \in Foc(\pi_B)$.

## Theorem (focusing)

*If $\pi$ is a PN with no asynchronous conclusion and at least a synchronous conclusion then,*

$$Foc(\pi) \neq \emptyset.$$




focusing

$B \otimes C \in Split(\pi')$ but $B \otimes C \notin Foc(\pi')$ since $ax_2$ is not a sub-PN

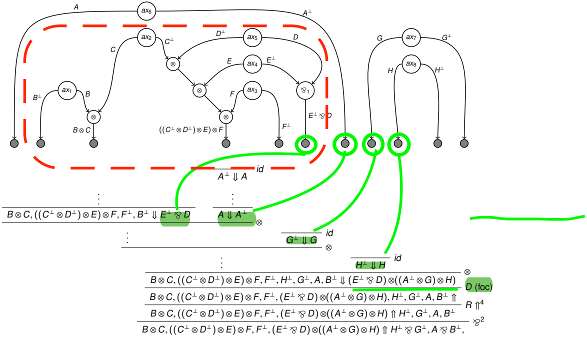
$((C^\perp \otimes D^\perp) \otimes E) \otimes F \in Split(\pi')$ but $((C^\perp \otimes D^\perp) \otimes E) \otimes F \notin Foc(\pi')$ since it is not hereditary spliting

$(E^\perp ⅋ D) \otimes ((A^\perp \otimes G) \otimes H) \in Foc(\pi')$

# focusing MLL proofs via proof nets: refinement of sequentialization

At each induction steps, there are three cases to consider:

1. PN contains an asynchronous conclusion, then we seqeuntializes it.
2. PN contains no asynchronous conclusion but at least one synchronous conclusion:
   - 2.1 by **Focusing Theorem** choose a focusing synchronous conclusion $A \otimes B$, and split the proof-net at this formula into two sub-proof-nets, $\pi_A$ and $\pi_B$;
   - 2.2 recursively apply focusing sequentialization to each of these sub-PNs $\pi_A$ and $\pi_B$;
   - 2.3 combine the resulting proofs with the corresponding synchronous inference of $\Sigma^2_{MLL}$.

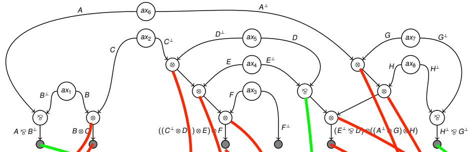



3. If the PN contains neither synchronous nor asynchronous conclusions then it is an instance of the identity link, so its sequentialization is reduced to the identity axiom *id*.

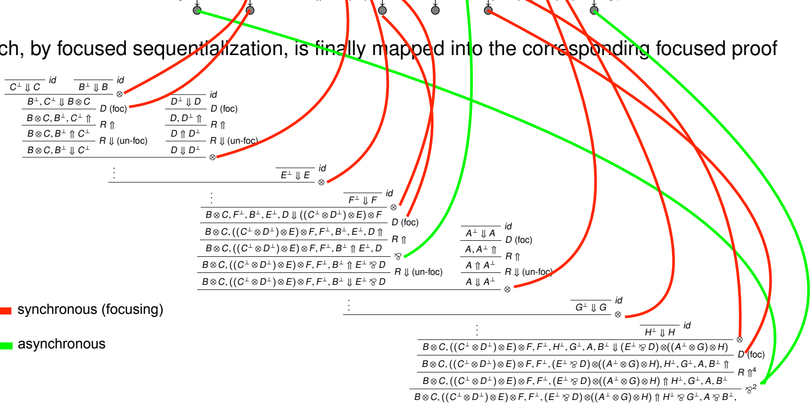
# let us assembly the result

$\Pi :\vdash^1_{MLL} A \parr B^\perp, B \otimes C, ((C^\perp \otimes D^\perp) \otimes E) \otimes F, F^\perp, (E^\perp \parr D) \otimes ((A^\perp \otimes G) \otimes H), H^\perp \parr G^\perp$

a MLL proof Π of Γis inductively (mapped) de-sequentialized into the MLL PN




which, by focused sequentialization, is finally mapped into the corresponding focused proof




synchronous (focusing)

asynchronous

# MALL focusing system

The monadic sequent system $\Sigma_1$ for MALL: $A, B := P \mid A^\perp \mid A \otimes B \mid A \parr B \mid A \,\&\, B \mid A \oplus B$

$$\frac{}{A, A^\perp}\ ax \qquad \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\ cut \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \& B}\ \& \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \oplus_i A_2}\ \oplus_{i=1,2}$$

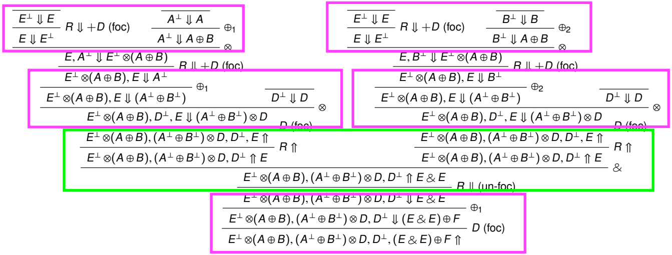
The dyadic (focused) sequent system $\Sigma_2$ for MALL:

- **Logical rules**: $\parr, \&$ (negative/asynchronous) and $\otimes, \oplus_{i=1,2}$ (positive/synchronous)

$$\frac{\Gamma \Uparrow L, F, G}{\Gamma \Uparrow L, F \parr G}\ \parr \qquad \frac{\Gamma \Uparrow L, F \qquad \Gamma \Uparrow L, G}{\Gamma \Uparrow L, F \,\&\, G}\ \& \qquad \frac{\Gamma \Downarrow F \qquad \Delta \Downarrow G}{\Gamma, \Delta \Downarrow F \otimes G}\ \otimes \qquad \frac{\Gamma \Downarrow F_i}{\Gamma \Downarrow F_1 \oplus F_2}\ \oplus_{i=1,2}$$

- **Identity** [*id*] : if $F$ is a positive atom $\quad \dfrac{}{F^\perp \Downarrow F}\ id$
- **Decision** [*D*] : if $F$ is synchronous or a positive atom $\quad \dfrac{\Gamma \Downarrow F}{\Gamma, F \Uparrow}\ D$ (**foc**)
- **Reaction** [$R \Downarrow$]: if $F$ is neither synchronous nor a pos. atom $\quad \dfrac{\Gamma \Uparrow F}{\Gamma \Downarrow F}\ R \Downarrow$ (**un- foc**)
- **Reaction** [$R \Uparrow$] : if $F$ is not asynchronous $\quad \dfrac{\Gamma, F \Uparrow L}{\Gamma \Uparrow L, F}\ R \Uparrow$

# an example of focused MALL proof




## Theorem (Andreoli, 1992)

*If $\Gamma$ is a multiset of non-asynchronous formulas and $L$ an ordered list of formulas then:*

$$\vdash^{\Sigma_1}_{MALL} \Gamma, L \quad \textit{iff} \quad \vdash^{\Sigma_2}_{MALL} \Gamma \Uparrow L$$

## Proof.

As for the MLL case, we show a proof via sequentialization of MALL PNs.

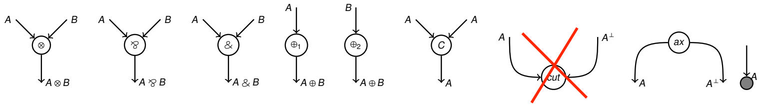
$$\begin{array}{lccccl} \text{sequent proof :} & \Pi & \xrightarrow{de-sequentialization} & \pi & & : \text{proof net} \\ & \vdots\downarrow & & \downarrow & & \\ \text{focused proof :} & \Pi' & \xleftarrow{foc-sequentialization} & \pi' & & : \text{canonical MALL proof net} \end{array}$$

□

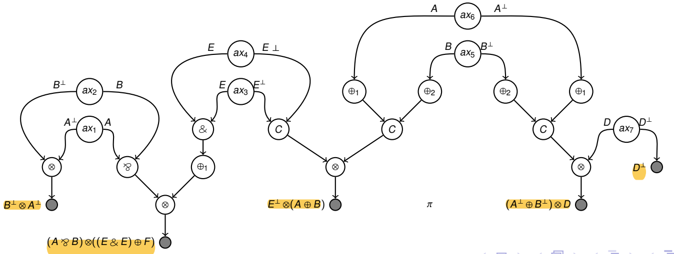
# MALL proof structures

## Definition (proof structure, PS)

A MALL PS $\pi$ is a directed graph s.t. edges are labelled by MALL formulas, each node has a type in $\{ax, cut, \otimes, \parr, \&, \oplus_1, \oplus_2, C, \bullet\}$ and built according to the following typing constraints, called **links**:




1. entering (resp., exiting) edges of a node $L$ are **premises** (resp., *conclusions*) of $L$;
2. each edge must be conclusion of exactly one node and premise of at most one node;
3. "•" denotes a dummy node whose unique premise is also called **conclusion of** $\pi$;
4. a node $L$ whose conclusion (resp., conclusions) is (resp., are) conclusion (resp., conclusions) of $\pi$ is called **terminal** or **conclusion nodes** of $\pi$.



# MALL proof structures

## Definition (proof structure, PS (... continues))

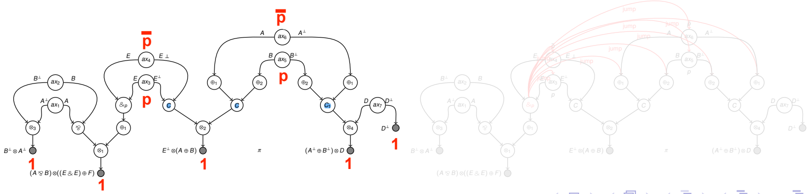
All nodes of a MALL PS are equipped with **monomial weights**:

1. we associate a (different) **Boolean variable** ($p, q, ...$), **eigen weight**, to each &-node of $\pi$
2. we associate a **weight** $w$, i.e., a product (conjunction) of eigen weights or negations of eigen weights of $\pi$ ($p, \overline{p}, q, \overline{q}$...), to each node with the constraint that two nodes have the same weight if they have a common edge, except when the edge is the premise of a & or $C$-node (take the sum):
3. every node that is **conclusion** of $\pi$ has **weight** 1 (dummy nodes • have weight 1);

## Definition (canonical proof net)

A MALL PS $\pi$ is **correct**, it is a **proof net** (PN), iff :

1. $\pi$ is in **canonical form** (CPN) i.e., axiom are atomic and contraction are allowed only immediately below literals or immediately below two different instances of $\oplus$-link (i.e., $\oplus_1$ and $\oplus_2$); ;
2. every (additive) **switching** $S(\pi)$ is an **ACC** graph.



# MALL proof structures

## Definition (proof structure, PS (... continues))

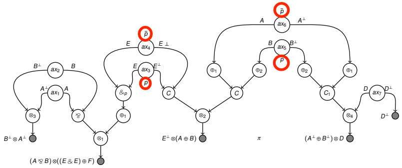
All nodes of a MALL PS are equipped with **monomial weights**:

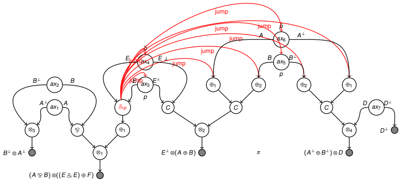
1. we associate a (different) **Boolean variable** ($p, q, ...$), **eigen weight**, to each &-node of $\pi$
2. we associate a **weight** $w$, i.e., a product (conjunction) of eigen weights or negations of eigen weights of $\pi$ ($p, \overline{p}, q, \overline{q}$...), to each node with the constraint that two nodes have the same weight if they have a common edge, except when the edge is the premise of a & or $C$-node (take the sum):
3. every node that is **conclusion** of $\pi$ has **weight** 1 (dummy nodes • have weight 1);

## Definition (canonical proof net)

A MALL PS $\pi$ is **correct**, it is a **proof net** (PN), iff :

1. $\pi$ is in **canonical form** (CPN) i.e., axiom are atomic and contraction are allowed only immediately below literals or immediately below two different instances of $\oplus$-link (i.e., $\oplus_1$ and $\oplus_2$); ;
2. every (additive) **switching** $S(\pi)$ is an **ACC** graph.



# focusing links

## Definition (focusing conclusions)

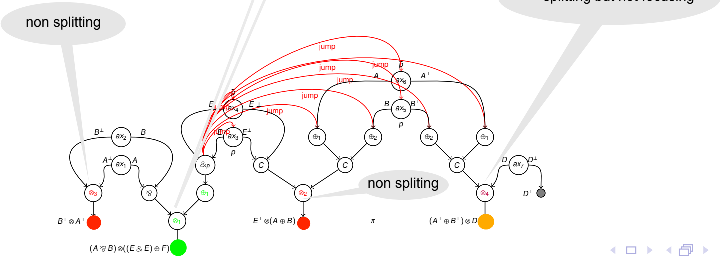
A conclusion $F$ is **focusing** for $\pi$ i.e., $F \in \mathbf{Foc}(\pi)$, iff one of the following conditions holds:

1. $F$ is a positive atom and $\pi$ is an axiom link.
2. $F = (A_1 \oplus A_2)$ is the conclusion of a terminal $\oplus_i$-link $L$ of type $\frac{A_i}{A_1 \oplus A_2} \oplus_i$ and $A_i$ is asynchronous OR a negative atom OR $A_i \in \mathrm{Foc}(\pi_{A_i})$, for $1 \leq i \leq 2$.
3. $F = (A \otimes B) \in \mathrm{Split}(\pi)$ and $\pi$ is split at $F$ into two sub-PNs, $\pi_A$ and $\pi_B$, and
   - 3.1 $A$ is asynchronous or a negative atom or $A \in \mathrm{Foc}(\pi_A)$ and
   - 3.2 $B$ is asynchronous or a negative atom or $B \in \mathrm{Foc}(\pi_B)$;

## Theorem (focusing)

*If $\pi$ is a PN with no asynchronous conclusion but at least a synchronous conclusion then,* $\mathbf{Foc}(\pi) \neq \emptyset$.

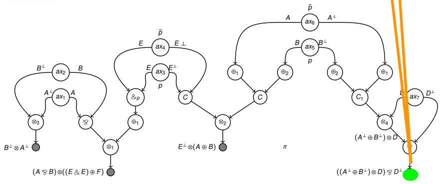
EXAMPLE. $\mathbf{Foc}(\pi) = \{(A \parr B) \otimes ((E \,\&\, E) \oplus F)\}$



# focusing sequentialization of CPN

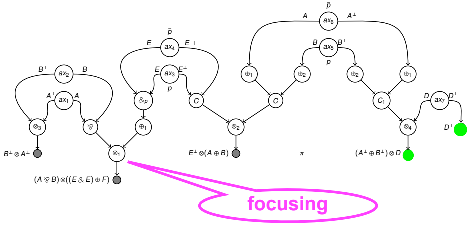
At each induction steps, there are three cases to consider (as in the MLL case):

**Case 1**: If the PN contains an **asynchronous conclusion**, $A \parr B$ or $A \& B$, then



$$\frac{??}{E^{\perp}\otimes(A\oplus B), (B^{\perp}\otimes A^{\perp}), (A\parr B)\otimes((E\& E)\oplus F) \Uparrow ((A^{\perp}\oplus B^{\perp})\otimes D)\parr D^{\perp}}$$

1. remove the corresponding link (together with all contraction links $C$ depending on $\&_p$);
2. recursively apply sequentialization to the remaining proof-net;
3. complete the sequent proof with the corresponding **asynchronous inference**

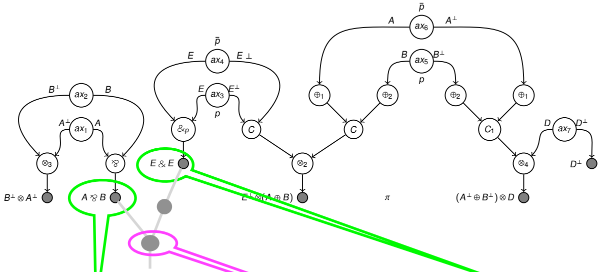



$$\dfrac{\dfrac{\dfrac{???}{E^{\perp}\otimes(A\oplus B), (B^{\perp}\otimes A^{\perp}), (A\parr B)\otimes((E\& E)\oplus F), (A^{\perp}\oplus B^{\perp})\otimes D, D^{\perp} \Uparrow}}{E^{\perp}\otimes(A\oplus B), (B^{\perp}\otimes A^{\perp}), (A\parr B)\otimes((E\& E)\oplus F) \Uparrow (A^{\perp}\oplus B^{\perp})\otimes D, D^{\perp}}\ R\Uparrow^2}{E^{\perp}\otimes(A\oplus B), (B^{\perp}\otimes A^{\perp}), (A\parr B)\otimes((E\& E)\oplus F) \Uparrow ((A^{\perp}\oplus B^{\perp})\otimes D)\parr D^{\perp}}\ \parr$$

# focusing sequentialization of CPN

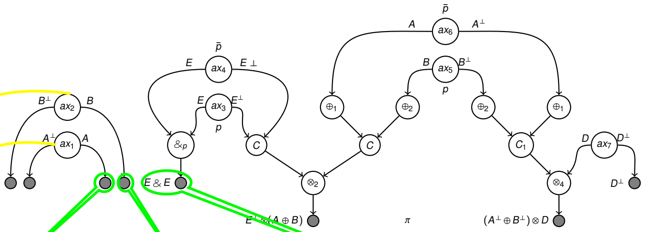
**Case 2**: If the PN has no asynchronous concl. but at least one **synchronous concl.** then:

1. by **Focusing Theorem** choose a focusing synchronous conclusion $A \otimes B$ (resp., $A \oplus B$) and split the PN at this formula into possibly two sub-PNs, $\pi_A$ and $\pi_B$ (resp, only $\pi_A$ or $\pi_B$, in case $A \oplus B$ is conclusion of a unary $\oplus$-link);
2. recursively apply the focusing sequentialization to each of these sub-PNs $\pi_A$ and $\pi_B$;
3. combine the resulting proofs with the corresponding synchronous inference of $\Sigma^2_{MALL}$.




$$\dfrac{\dfrac{B^\perp \otimes A^\perp \Uparrow A \parr B}{B^\perp \otimes A^\perp \Downarrow A \parr B}\ R\Downarrow\text{(un-foc)} \qquad \dfrac{\dfrac{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Uparrow E \,\&\, E}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Downarrow E \,\&\, E}\ R\Downarrow\text{(un-foc)}}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Downarrow (E \,\&\, E) \oplus F}\ \oplus_1}{\dfrac{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, (B^\perp \otimes A^\perp) \Downarrow (A \parr B) \otimes ((E \,\&\, E) \oplus F)}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, (B^\perp \otimes A^\perp), (A \parr B) \otimes ((E \,\&\, E) \oplus F) \Uparrow}\ D\text{ (foc)}}\ \otimes$$

we continue by induction on the l.h.s. sub-PN (it is an MLL sub-PN)

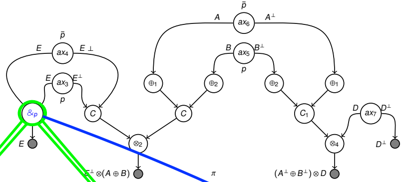



$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{A^\perp \Downarrow A}\ id}{A \Downarrow A^\perp}\ R\Downarrow + D\ (\text{foc})
\qquad
\dfrac{\dfrac{}{B^\perp \Downarrow B}\ id}{B \Downarrow B^\perp}\ R\Downarrow + D\ (\text{foc})
}{A, B \Downarrow B^\perp \otimes A^\perp}\ \otimes
}{B^\perp \otimes A^\perp, A, B \Uparrow}\ D\ (\text{foc})
}{B^\perp \otimes A^\perp \Uparrow A \parr B}\ \parr + R \Uparrow^2
}{B^\perp \otimes A^\perp \Downarrow A \parr B}\ R\Downarrow\ (\text{un-foc})
\qquad
\dfrac{
\dfrac{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Uparrow E \& E}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Downarrow E \& E}\ R\Downarrow\ (\text{un-foc})
}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Downarrow (E \& E) \oplus F}\ \oplus_1
}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, (B^\perp \otimes A^\perp) \Downarrow (A \parr B) \otimes ((E \& E) \oplus F)}\ \otimes
$$

**Case 3**: if the PN contains neither synchronous nor asynchronous conclusions, i.e. it must be an instance of the identity link, so it sequentializes in to an identity axiom *id*.

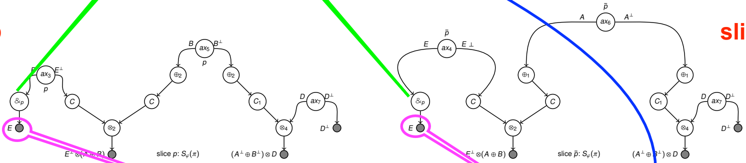
# focusing sequentialization of CPN

we continue by induction on the r.h.s. sub-PN




we can apply Case 1: there is an asynchronous conclusion $E \mathbin{\&} E$
the sequentialization produces **two slices**: $\varphi(p) = 1$ and $\varphi(\bar{p}) = 1$

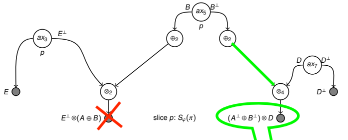
**slice p** **slice $\overline{\text{p}}$**

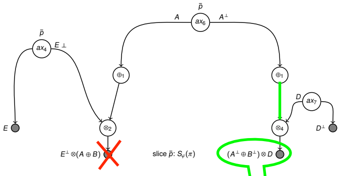



$$\dfrac{\dfrac{\dfrac{??}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, E \Uparrow}}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Uparrow E}\, R\Uparrow \quad \dfrac{\dfrac{??}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, E \Uparrow}}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Uparrow E}\, R\Uparrow}{\dfrac{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Uparrow E \mathbin{\&} E}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp \Downarrow E \mathbin{\&} E}\, R\Downarrow \text{(un-foc)}}\, \&$$

# focusing sequentialization of CPN

by induction on the two sub-PNs, we can apply Case 2 (select the focusing conclusion):




Observe that:

– in the l.h.s. PN, $\langle \otimes_4, \oplus_2 \rangle$, is a **focusing** section while $\langle \otimes_2, \oplus_2 \rangle$ is not a focusing section

– in the r.h.s. PN, $\langle \otimes_4, \oplus_1 \rangle$, is a **focusing** section while $\langle \otimes_2, \oplus_1 \rangle$ is not a focusing section

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{E^\perp \Downarrow E}}{E \Downarrow E^\perp}\, R \Downarrow +D\ (\text{foc}) \qquad \dfrac{\dfrac{}{B^\perp \Downarrow B}}{A^\perp \Downarrow A \oplus B}\, \oplus_2}{E, B^\perp \Downarrow E^\perp \otimes (A \oplus B)}\, \otimes}{E^\perp \otimes (A \oplus B), E \Downarrow B^\perp}\, R \Downarrow +D\ (\text{foc})}{E^\perp \otimes (A \oplus B), E \Downarrow (A^\perp \oplus B^\perp)}\, \oplus_2 \qquad \dfrac{}{D^\perp \Downarrow D}}{E^\perp \otimes (A \oplus B), D^\perp, E \Downarrow (A^\perp \oplus B^\perp) \otimes D}\, \otimes}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, E \Uparrow}\, D\ (\text{foc})$$

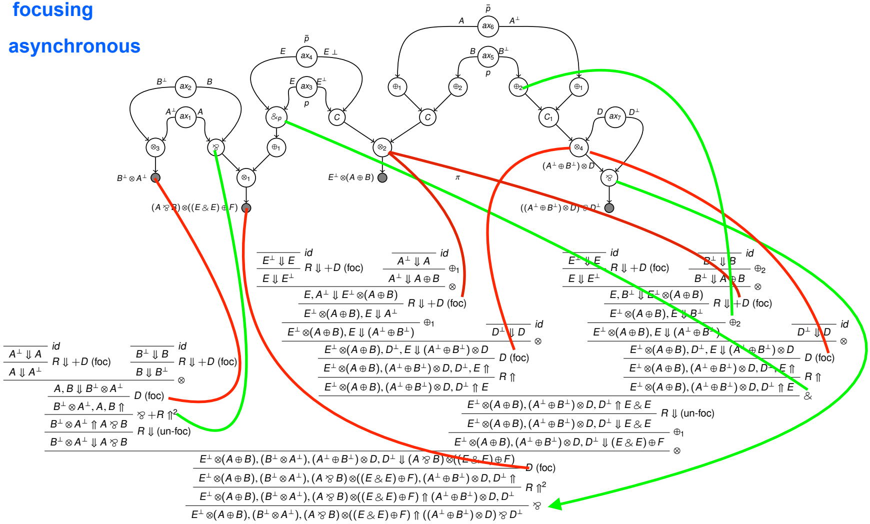
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{E^\perp \Downarrow E}}{E \Downarrow E^\perp}\, R \Downarrow +D\ (\text{foc}) \qquad \dfrac{\dfrac{}{A^\perp \Downarrow A}}{A^\perp \Downarrow A \oplus B}\, \oplus_1}{E, B^\perp \Downarrow E^\perp \otimes (A \oplus B)}\, \otimes}{E^\perp \otimes (A \oplus B), E \Downarrow A^\perp}\, R \Downarrow +D\ (\text{foc})}{E^\perp \otimes (A \oplus B), E \Downarrow (A^\perp \oplus B^\perp)}\, \oplus_1 \qquad \dfrac{}{D^\perp \Downarrow D}}{E^\perp \otimes (A \oplus B), D^\perp, E \Downarrow (A^\perp \oplus B^\perp) \otimes D}\, \otimes}{E^\perp \otimes (A \oplus B), (A^\perp \oplus B^\perp) \otimes D, D^\perp, E \Uparrow}\, D\ (\text{foc})$$

polarities on literals play a role, reducing the number of possible focusing links

# let us assembly the final result

**focusing**

**asynchronous**



# conclusions

**question**: so, finally, what did we gain from this alternative proof of the Focusing Theorem?

**answer**: we gained a new, finer, syntax for MALL proof nets in canonical form.

# conclusions

**question**: so, finally, what did we gain from this alternative proof of the Focusing Theorem?

**answer**: we gained a new, finer, syntax for MALL proof nets in canonical form.

i.e. **Focusing MALL Proof Nets**